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CYCLIC SURFACES IN E_1^5 GENERATED BY HOMOTHETIC MOTIONS[*]

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Abstract – In this paper, we study cyclic surfaces in E_1^5 generated by homothetic motions of a Lorentzian circle. The properties of these cyclic surfaces up to first order are investigated. We show that, as it is shown in E^5, cyclic 2-surfaces in E_1^5, in general, are contained in canal hypersurfaces. Finally, we give an example.

Keywords – Minkowski space, cyclic surfaces, homothetic motions

1. INTRODUCTION

Homothetic motions are a general form of Euclidean motions. It is crucial that homothetic motions are regular motions. These motions have been studied in kinematic and differential geometry in recent years. Under these motions, the point paths of the circle c_0 generate a surface which is called the cyclic surface [1, 2]. In papers written in recent years, these surfaces seem to be the leading surfaces in geometric modeling [1, 3, 4].

Abdel-All and Hamdoon gave some first order properties of cyclic surfaces generated by homothetic motions in five dimensional Euclidean space and obtained some new theorems [5]. In Minkowski (semi-Euclidean) space, hyperbolas (Lorentzian circles) play the role of circle in Euclidean space [6]. We generalize the cyclic surfaces by studying the surfaces generated by motions of hyperbolas.

2. NORMAL HYPERPLANES

A cyclic surface in E_1^5 is given by

$$X(t,\phi) = \rho(t)A(t)x(\phi) + d(t)\,,\ t,\phi \in R$$

where $A(t) = (a_{ij}(t)),\ i, j = 1,2,\ldots,5$ is an semi orthogonal matrix and $x(\phi) = (ch\phi, sh\phi, 0,0,0)^T$ represents the unit Lorentzian circle c_0. $d(t) = (b_1(t), b_2(t), b_3(t), b_4(t), b_5(t))^T$ is the translational part of the motion. We also assume that all involved functions are a class of C^1. Let $a_i(t)\,, i = 1,\ldots,5$ be the column vectors of the matrix $A(t)$, hence the cyclic surface can be written as follows

$$X(t,\phi) = \rho(t)[a_1(t)ch\phi + a_2(t)sh\phi] + d(t) \tag{1}$$

where $d(t)$ is the center of the moving Lorentzian circle and $a_1(t), a_2(t)$ are two orthogonal vectors in the plane of the Lorentzian circle. The velocity vectors of the points of the Lorentzian circle are given by

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$$X'(t,\phi) = \rho'(t)[a_1(t)ch\phi + a_2(t)sh\phi] + \rho(t)[a_1'(t)ch\phi + a_2'(t)sh\phi] + d'(t) \tag{2}$$

where $'$ denotes the derivative with respect to the time t.
The equation of the hyperplanes orthogonal to such a path is

$$Y^T X'(t,\phi) = X^T(t,\phi)X'(t,\phi)$$

where $Y = (y_1, y_2, y_3, y_4, y_5)^T$ is the position vector of an arbitrary *Y* in the hyperplane. The scalar product in the above equation is Lorentzian metric. According to the inner product this equation is

$$Y^T \varepsilon X'(t,\phi) = X^T(t,\phi)\varepsilon X'(t,\phi) \tag{3}$$

where $\varepsilon = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is the sign matrix.

Substituting (1) and (2) into (3), we have

$$\rho'(t)[Y^T\varepsilon a_1(t)ch\phi + Y^T\varepsilon a_2(t)sh\phi] + \rho(t)[Y^T\varepsilon a_1'(t)ch\phi + Y^T\varepsilon a_2'(t)sh\phi] + Y^T\varepsilon d'(t) =$$
$$[\rho(t)[a_1^T(t)ch\phi + a_2^T(t)sh\phi] + d^T(t)]\varepsilon.$$
$$[\rho'(t)[a_1(t)ch\phi + a_2(t)sh\phi] + \rho(t)[a_1'(t)ch\phi + a_2'(t)sh\phi] + d'(t)]. \tag{4}$$

The left hand side of (4) gives

(*) $L.h.s. = ch\phi[\rho'(t)Y^T\varepsilon a_1(t) + \rho(t)Y^T\varepsilon a_1'(t)] + sh\phi[\rho'(t)Y^T\varepsilon a_2(t) + \rho(t)Y^T\varepsilon a_2'(t)] + Y^T\varepsilon d'(t)$.

Since $A^T\varepsilon A = \varepsilon$ and $A^T\varepsilon A'$ is a skew symmetric matrix, the right hand side of (4) is

(**) $R.h.s. = -\rho(t)\rho'(t) + d^T(t)\varepsilon\, d'(t) + ch\phi[\rho(t)(a_1^T(t)\varepsilon d'(t) + d^T(t)\varepsilon a_1'(t)) + \rho'(t)d^T(t)\varepsilon\, a_1(t)]$
$+ sh\phi[\rho(t)(a_2^T(t)\varepsilon d'(t) + d^T(t)\varepsilon a_2'(t)) + \rho'(t)d^T(t)\varepsilon\, a_2(t)]$.

Let $e_k(t) = a_k^T(t)\varepsilon\, d'(t),\ h_k(t) = d^T(t)\varepsilon\, a_k'(t)$ and $l_k(t) = d^T(t)\varepsilon\, a_k(t), k = 1,2$. A point *Y* now will be situated in all the normal hyperplanes for all $\phi \in R$ (*t* constant), if (*) and (**) hold for all ϕ. By comparing the coefficients of $\{1, ch\phi, sh\phi\}$ in (*) and (**), we obtain

$$\sum_{i=1}^{5} \varepsilon_i y_i b_i'(t) = \sum_{i=1}^{5} \varepsilon_i b_i(t) b_i'(t) - \rho(t)\rho'(t)$$

$$\rho'(t)\sum_{i=1}^{5} \varepsilon_i y_i a_{i1}(t) + \rho(t)\sum_{i=1}^{5} \varepsilon_i y_i a_{i1}'(t) = \rho(t)(e_1(t) + h_1(t)) + \rho'(t)l_1(t) \tag{5}$$

$$\rho'(t)\sum_{i=1}^{5} \varepsilon_i y_i a_{i2}(t) + \rho(t)\sum_{i=1}^{5} \varepsilon_i y_i(t) a_{i2}'(t) = \rho(t)(e_2(t) + h_2(t)) + \rho'(t)l_2(t)$$

where $\varepsilon_1 = -1, \varepsilon_j = 1, j = 2,3,4,5$. Moreover, all these normal hyperplanes for fixed t, in general, intersect in a plane.

3. LOKAL STUDY IN CANONICAL FRAMES

By using Taylor's expansion, up to the first order the representation of motion is given by

$$X(t,\phi)=\left[\rho(0)A(0)+(\rho'(0)A(0)+\rho(0)A'(0))t\right]x(\phi)+d(0)+d'(0)t$$

We assume the moving frame E_1^5 and fixed frame Σ coinciding at the zero position (t=0), then we have

$$A(0)=I,\quad \rho(0)=1 \text{ and } d(0)=0.$$

Thus we have

$$X(t,\phi)=\left[I_5+(\rho'(0)I_5+A'(0))t\right]x(\phi)+d'(0)t$$

where $A'(0)=(w_k),\ k=1,2,\ldots,10$ is a semi skew symmetric matrix. For simplicty we write ρ' and b_i' instead of $\rho'(0)$ and $b_i'(0)$ respectively.
In these frames, the representation of the motion up to the first order is given by

$$\begin{pmatrix} X_1\\ X_2\\ X_3\\ X_4\\ X_5 \end{pmatrix}=\begin{pmatrix} 1+t\rho' & tw_1 & tw_2 & tw_3 & tw_4\\ tw_1 & 1+t\rho' & tw_5 & tw_6 & tw_7\\ tw_2 & -tw_5 & 1+t\rho' & tw_8 & tw_9\\ tw_3 & -tw_6 & -tw_8 & 1+t\rho' & tw_{10}\\ tw_4 & -tw_7 & -tw_9 & -tw_{10} & 1+t\rho' \end{pmatrix}\begin{pmatrix} ch\phi\\ sh\phi\\ 0\\ 0\\ 0 \end{pmatrix}+t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5' \end{pmatrix} \tag{6}$$

$$=t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5' \end{pmatrix}+ch\phi\begin{pmatrix} 1+t\rho'\\ tw_1\\ tw_2\\ tw_3\\ tw_4 \end{pmatrix}+sh\phi\begin{pmatrix} tw_1\\ 1+t\rho'\\ -tw_5\\ -tw_6\\ -tw_7 \end{pmatrix}$$

For any fixed *t* in equation (6), we gain a curve for $\phi\in R$. The orthogonal projection of these curves (*t* constant) on the $[x_1x_2]$ plane is

$$\begin{pmatrix} X_1\\ X_2 \end{pmatrix}=t\begin{pmatrix} b_1'\\ b_2' \end{pmatrix}+ch\phi\begin{pmatrix} 1+t\rho'\\ tw_1 \end{pmatrix}+sh\phi\begin{pmatrix} tw_1\\ 1+t\rho' \end{pmatrix} \tag{7}$$

Thus, we have the following.

Corollary 3.1. The orthogonal projection of these curves (*t* constant in (6)) on the $[x_1x_2]$ plane are Lorentzian circles whose centers are given by (tb_1',tb_2') and radii by $r=\sqrt{\left|t^2w_1^2-(1+t\rho')^2\right|}$. Degenerate cases are given for $w_1=\left|\dfrac{1+t\rho'}{t}\right|$.

For constant ϕ and varying $t\in R$, formula (7) presents the parametrisation of a straight line $g(\phi)$. Its equation is given by

$$(x_1\rho'-x_2w_1+b_1')sh\phi+(x_1w_1-x_2\rho'-b_2')ch\phi=x_2b_1'-x_1b_2'+w_1 \tag{8}$$

The equation of the envelope of all these lines $g(\phi)$ then is gained by eliminating ϕ from the equation (8) and its derivative with respect to ϕ, which is given by

$$\left(x_1 w_1 - x_2 \rho' - b_2'\right) sh\phi + \left(x_1 \rho' - x_2 w_1 + b_1'\right) ch\phi = 0 . \quad (9)$$

Thus the equation of this envelope is

$$-x_1^2(\rho'^2 - w_1^2 + b_2'^2) + x_2^2(\rho'^2 - w_1^2 - b_1'^2) + 2x_1x_2b_1'b_2' - 2x_1\rho' b_1' + 2x_2\rho' b_2' = w_1^2 + b_1'^2 - b_2'^2 . \quad (10)$$

Therefore, we have the following.

Lemma 3.1. The envelope of the lines (8) in general is a conic section, which is given by (10).
We can classify the conic section (10) according to the relation between the two values $m = w_1^2 - \rho'^2$ and $n = b_2'^2 - b_1'^2$. We exclude the case $m = 0$ since it is equivalent to $w_1 = |\rho'|$.

Lemma 3.2. The envelope of the lines (8) in general will be a conic section of the following type:
(i) ellipse if $m < n$,
(ii) hyperbola if $m > n$,
(iii) parabola if $m = n$.

4. THE TANGENT LORENTZIAN SPHERE OF CYCLIC SURFACE IN E_1^5

In this section, we will show at any instant *t* there exist a Lorentzian sphere $\kappa(t)$, which is tangent to a given cyclic 2-surface (1) in all points of the instantaneous position $c(t)$ of the Lorentzian circle c_0. Without loss of generality we investigate the situation at the zero position. Any Lorentzian sphere κ_0 which is tangent to a given cyclic 2-surface (1) along c_0 has to contain c_0; hence the center of κ_0 has coordinates $(0,0,m_3,m_4,m_5)$ with $m_3, m_4, m_5 \in R$. On the other hand, since κ_0 has to be tangent to all velocity vectors of the motion, the center of κ_0 has to lie in each of the hyperplanes through the points of $c(t)$, orthogonal to these velocity vectors. This gives us the additional condition

$$\begin{aligned} m_3(b_3' + w_2 ch\phi - w_5 sh\phi) + m_4(b_4' + w_3 ch\phi - w_6 sh\phi) + m_5(b_5' + w_4 ch\phi - w_7 sh\phi) \\ = -b_1' ch\phi + b_2' sh\phi - \rho' . \end{aligned} \quad (11)$$

By comparing the coefficients of $\{1, ch\phi, sh\phi\}$ in (11), we have the system of linear equations

$$BM = H \quad (12)$$

where should be lowercase.

$$B = \begin{pmatrix} b_3' & b_4' & b_5' \\ w_2 & w_3 & w_4 \\ w_5 & w_6 & w_7 \end{pmatrix}, \; M = \begin{pmatrix} m_3 \\ m_4 \\ m_5 \end{pmatrix} \; and \; H = \begin{pmatrix} -\rho' \\ -b_1' \\ -b_2' \end{pmatrix}.$$

By solving (12), we get

$$\begin{aligned} m_3 &= \frac{1}{\Delta}\left[\rho'(w_4w_6 - w_3w_7) + b_1'(b_4'w_7 - b_5'w_6) + b_2'(b_5'w_3 - b_4'w_4)\right] \\ m_4 &= \frac{1}{\Delta}\left[\rho'(w_2w_7 - w_4w_5) + b_1'(b_5'w_5 - b_3'w_7) + b_2'(b_3'w_4 - b_5'w_2)\right] \\ m_5 &= \frac{1}{\Delta}\left[\rho'(w_3w_5 - w_2w_6) + b_1'(b_3'w_6 - b_4'w_5) + b_2'(b_4'w_2 - b_3'w_3)\right] \end{aligned} \quad (13)$$

with should be lowercase.

$$\Delta = b_3'(w_3 w_7 - w_4 w_6) + b_4'(w_4 w_5 - w_2 w_7) + b_5'(w_2 w_6 - w_3 w_5) \neq 0 .$$

Therefore, if the system (12) is not singular ($\Delta \neq 0$), we have the following theorem

Theorem 4.1. In general there is a 4-dimensional Lorentzian sphere with a center $(0,0,m_3,m_4,m_5)$ which contains the Lorentzian circle c_0, which is tangent to all tangent planes $\tau(\phi)$ of the given cyclic surface (1). This Lorentzian sphere is given by

$$-x_1^2 + x_2^2 + (x_3 - m_3)^2 + (x_4 - m_4)^2 + (x_5 - m_5)^2 = \left| -1 + m_3^2 + m_4^2 + m_5^2 \right|$$

where m_3, m_4, m_5 given by (13).

Definition 4.1. Canal hypersurfaces in E_1^n are envelope hypersurfaces of one-parametric sets of Lorentzian spheres.

Theorem 4.2. Any cyclic 2-surface in E_1^n in general is contained in a canal hypersurface, which is gained as an envelope of a one-parametric set of 4-dimensional Lorentzian spheres.

4.1. The Singular Cases

If the system of equations (12) is singular ($\Delta = 0$), we have many cases:

Case 1. $rank(B) = rank(B \backslash H) = 2$. In this case, we have a one-parametric set of Lorentzian spheres whose centers fulfill a straight line in the $x_1 x_2 x_3$ − space

$$M = \left(0,\ 0,\ m_3,\ \frac{-w_4(\rho' + b_3' m_3) + b_5'(b_1' + w_2 m_3)}{b_4' w_4 - b_5' w_3},\ \frac{w_3(\rho' + b_3' m_3) - b_4'(b_1' + w_2 m_3)}{b_4' w_4 - b_5' w_3} \right)$$

with arbitrary $m_3 \in R$. Thus, we gain a straight line of possible centers.

Case 2. $rank(B) = rank(B \backslash H) = 1$. *In* this case, we have a plane of possible centers.

Case 3. $rank(B) = 2 \neq rank(B \backslash H)$. In this case, we assume

$$\frac{w_2}{w_5} = \frac{w_3}{w_6} = \frac{w_4}{w_7} = \lambda,\ \frac{b_1'}{b_2'} \neq \lambda .$$

By using the homogenous coordinates

$$m_0 = \Delta = 0,\ m_1 = 0,\ m_2 = 0,$$

$$m_3 = (b_1' - \lambda b_2')(b_4' w_7 - b_5' w_6),\ m_4 = (b_1' - \lambda b_2')(b_5' w_5 - b_3' w_7)$$

$$m_4 = (b_1' - \lambda b_2')(b_3' w_6 - b_4' w_5) .$$

the centers of the Lorentzian spheres are an ideal point (point at infinity) and its coordinates are given as above. The corresponding Lorentzian sphere degenerates into a hyperplane.

Case 4. $rank(B) = 1 \neq rank(B \backslash H)$. In this case the centers of the possible Lorentzian spheres become a straight line at infinity. The corresponding Lorentzian spheres degenerate and form a pencil of the hyperplane.

5. CURVE OF CENTERS OF THE LORENTZIAN SPHERES

Now, we consider t as varying and in this section, we will determine the centers of the Lorentzian sphere which contain the Lorentzian circle $c(t)$ and are tangent to all tangent planes $\tau(t,\phi)$ of the cyclic surface (1). We know from the inital position, that the hyperplanes of the cyclic surfaces contain a point $m(t)$ for any t,ϕ such that $m(t)$ is the center of this Lorentzian sphere. In the moving space of the Lorentzian circle the center of the Lorentzian sphere at the moment t is given by $m(t)=(0,0,m_3(t),m_4(t),m_5(t))$. Then from (5), one can find

$$\begin{pmatrix} b_3'(t) & b_4'(t) & b_5'(t) \\ T_{31}(t) & T_{41}(t) & T_{51}(t) \\ T_{32}(t) & T_{42}(t) & T_{52}(t) \end{pmatrix}\begin{pmatrix} m_3(t) \\ m_4(t) \\ m_5(t) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{5} \varepsilon_i b_i(t) b_i'(t) - \rho(t)\rho'(t) \\ \rho(t)(e_1(t)+h_1(t)) + \rho'(t) l_1(t) \\ \rho(t)(e_2(t)+h_2(t)) + \rho'(t) l_2(t) \end{pmatrix} \tag{14}$$

where $T_{kr}(t)=\rho'(t)a_{kr}(t)+\rho(t)a'_{kr}(t), \quad k=3,4,5\ , r=1,2$. By solving (14), we obtain

$$\begin{aligned} m_3(t) = \frac{1}{\Delta_1}\Bigg[&\left(\sum_{i=1}^{5}\varepsilon_i b_i(t)b_i'(t) - \rho(t)\rho'(t)\right)\left(T_{41}(t)T_{52}(t)-T_{51}(t)T_{42}(t)\right) \\ &-(\rho(t)(e_1(t)+h_1(t))+\rho'(t)l_1(t))(b_4'(t)T_{52}(t)-b_5'(t)T_{42}(t)) \\ &-(\rho(t)(e_2(t)+h_2(t))+\rho'(t)l_2(t))(b_5'(t)T_{41}(t)-b_4'(t)T_{51}(t))\Bigg] \end{aligned} \tag{15}$$

$$\begin{aligned} m_4(t) = \frac{1}{\Delta_1}\Bigg[&\left(\sum_{i=1}^{5}\varepsilon_i b_i(t)b_i'(t) - \rho(t)\rho'(t)\right)\left(T_{51}(t)T_{32}(t)-T_{31}(t)T_{52}(t)\right) \\ &-(\rho(t)(e_1(t)+h_1(t))+\rho'(t)l_1(t))(b_5'(t)T_{32}(t)-b_3'(t)T_{52}(t)) \\ &-(\rho(t)(e_2(t)+h_2(t))+\rho'(t)l_2(t))(b_3'(t)T_{51}(t)-b_5'(t)T_{31}(t))\Bigg] \end{aligned} \tag{16}$$

$$\begin{aligned} m_5(t) = \frac{1}{\Delta_1}\Bigg[&\left(\sum_{i=1}^{5}\varepsilon_i b_i(t)b_i'(t) - \rho(t)\rho'(t)\right)\left(T_{31}(t)T_{42}(t)-T_{41}(t)T_{32}(t)\right) \\ &-(\rho(t)(e_1(t)+h_1(t))+\rho'(t)l_1(t))(b_3'(t)T_{42}(t)-b_4'(t)T_{32}(t)) \\ &-(\rho(t)(e_2(t)+h_2(t))+\rho'(t)l_2(t))(b_4'(t)T_{31}(t)-b_3'(t)T_{41}(t))\Bigg] \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Delta_1 = {} & b_3'(t)\left(T_{41}(t)T_{52}(t)-T_{51}(t)T_{42}(t)\right)+b_4'(t)\left(T_{51}(t)T_{32}(t)-T_{31}(t)T_{52}(t)\right) \\ & +b_5'(t)\left(T_{31}(t)T_{42}(t)-T_{41}(t)T_{32}(t)\right) \neq 0 . \end{aligned}$$

Therefore, the coordinates of the centers of the Lorentz spheres in the fixed frame at any instant t are given by

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} = \rho(t)A(t)\begin{pmatrix} 0 \\ 0 \\ m_3(t) \\ m_4(t) \\ m_5(t) \end{pmatrix} + d(t) \tag{18}$$

Theorem 5.1. At any instant t, there is a Lorentzian spheres $\kappa(t)$ with centers given by $(0,0,m_3(t),m_4(t),m_5(t))$ which contains the Lorentzian circle $c(t)$, which is tangent to all tangent planes

$\tau(t,\phi)$ of the given cyclic surface (1). The curve of the centers of these Lorentzian spheres in the moving frame is given by $m(t) = (0,0,m_3(t),m_4(t),m_5(t))$, where $m_3(t), m_4(t), m_5(t)$ are given by equations (15)-(17) and in the fixed frame it is given by (18).

Example 5.1. We consider cyclic surfaces generated by the motion given by

$$A(t) = \begin{pmatrix} ch\lambda t & 0 & 0 & \sin t\ sh\lambda t & -\cos t\ sh\lambda t \\ 0 & \cos\lambda t & -\sin\lambda t & 0 & 0 \\ 0 & \sin\lambda t & \cos\lambda t & 0 & 0 \\ 0 & 0 & 0 & \cos t & \sin t \\ -\sin\lambda t & 0 & 0 & -\sin t\ ch\lambda t & \cos t\ ch\lambda t \end{pmatrix} \tag{19}$$

such that $\lambda \in R-\{0\}$. We assume $\rho(t) = e^{qt}$ and $d(t) = (0,0,0,vt,0)^T$, where $q \neq 0$ and $v \neq 0$. We compute by differentianting $A(t)$ and put $t = 0$, one can find

$$w_4 = w_5 = -\lambda \text{ , } w_{10} = 1 \text{ and } w_k = 0, \quad k = 1,2,3,6,7,8,9 .$$

Substutiting into (13), we have

$$m_3 = 0,\ m_4 = -\frac{q}{v},\ m_5 = 0 .$$

Then, the Lorentzian sphere which contains a Lorentzian circle c_0 and is tangent to all tangent planes of the corresponding cyclic surface is given by

$$-x_1^2 + x_2^2 + x_3^2 + (x_4 + \frac{q}{v})^2 + x_5^2 = \left| -1 + \frac{q^2}{v^2} \right| .$$

After differentiation of (19), and substitution into (14), we get

$$\begin{pmatrix} 0 & v & 0 \\ 0 & 0 & -e^{qt}(qsh\lambda t + \lambda ch\lambda t) \\ e^{qt}(q\sin\lambda t + \lambda\cos\lambda t) & 0 & 0 \end{pmatrix} \begin{pmatrix} m_3(t) \\ m_4(t) \\ m_5(t) \end{pmatrix} = \begin{pmatrix} v^2 t - qe^{2qt} \\ 0 \\ 0 \end{pmatrix} .$$

Then, for the general centers of the tangent Lorentzian spheres

$$m_3(t) = 0,\ m_4(t) = \frac{v^2 t - qe^{2qt}}{v},\ m_5(t) = 0 .$$

Therefore, the parametric representation of the curve of centers of the Lorentzian spheres in the moving frame is given by

$$m(t) = \left(0,0,0, \frac{v^2 t - qe^{2qt}}{v}, 0 \right) .$$

From (19) and (1) its parametrization in the fixed frame is

$$M(t) = e^{qt}\begin{pmatrix} m_4(t)\sin t \;\; sh\lambda t \\ 0 \\ 0 \\ m_4(t)\cos t \; + vt \\ -m_4(t)\sin t \;\; ch\lambda t \end{pmatrix}^T, \quad m_4(t) = \frac{v^2 t - qe^{2qt}}{v}.$$

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