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STUDY OF BIFURCATION AND HYPERBOLICITY IN DISCRETE DYNAMICAL SYSTEMS[*]

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Abstract – Bifurcations leading to chaos have been investigated in a number of one dimensional dynamical systems by varying the parameters incorporated within the systems. The property hyperbolicity has been studied in detail in each case which has significant characteristic behaviours for regular and chaotic evolutions. In the process, the calculations for invariant set have also been carried out. A broad analysis of bifurcations and hyperbolicity provide some interesting results. The fractal property, self-similarity, has also been observed for chaotic regions within the bifurcation diagram. The results of numerical calculations assume significant values.

Keywords – Hyperbolicity, invariant set, chaos, nonlinearity

1. INTRODUCTION

While studying regular and chaotic evolutions of a dynamical system, we regularly vary the system parameters. This can be observed clearly through bifurcation diagrams. The chaotic evolutions observed in certain critical values of the systems parameters are interesting phenomena to study. Many research articles appearing in recent times discuss the cause of chaotic evolution, controlling chaos, identifying chaotic and regular motions etc. [1-7] and many others. In the process of detailed analysis, we come across the property of hyperbolicity which is an important ingredient to discuss evolutionary motions of the system concerned. This hyperbolicity induces invariant sets for different systems, as suggested by Devaney [8]. The invariant set is nothing but a set of invariant measures within the chaotic domain (Strange Attractor) of the system. After certain steps of regular bifurcations, we reach a state where the bifurcation diagram shows self-similar or fractal properties. Within the chaotic regions, we also have periodic windows, where again we can observe similar fractal properties.

The objective of the present study is mainly confined to one or two dimensional discrete maps, and the calculations followed here similar to one initiated by Jonassen [9], to study hyperbolicity. Also, we wish to observe clearly the bifurcation phenomena of the systems conserned. A close range of bifurcation shows some interesting results. The numerical results for every case reveal very significant hyperbolic sets. The study can be extended to other maps which may produce important results.

2. BASIC CONCEPTS AND DEFINITIONS

Consider nonlinear time-invariant system or recursive system $x_{t+1} = f(x_t)$

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Definition 2.1. A set $S \subseteq R^n$ is said to be invariant with respect to the system if for every trajectory *x*,
$x_t \in S \Rightarrow \; x_\tau \in S$ for all $\tau \geq t$

i.e., trajectories can enter, but cannot leave set *S*.

In other words, *S* is an invariant set if every trajectory which starts from a point in *S* remains in *S* for all time.

Definition 2.2. Let $f: R \to R$ be a C^1 function and suppose that *K* is a compact invariant set for *f* (*i.e. f(K)=K*). Then K is a hyperbolic set for *f* if there are constants *C > 0* and *r > 1* such that
$\left|(f^n)'(x)\right| \geq C\, r^n \quad \forall\; x \in K$ and all $n \geq 1$
The *C* in the definition takes care of the fact that f^{-1} may stretch some intervals (*i.e.*, $\left|f'(x)\right| \leq 1$ for some $x \in K$), in which case *C<1*, but *r>1* implies the shrinking under f^{-n} eventually dominates any stretching when $C\, r^n > 1$.

3. CALCULATIONS OF HYPERBOLIC SET AND INVARIANT SET FOR SOME DISCRETE MODELS

Consider the quadratic discrete map $x_{n+1} = r + x_n - x_n^2$, where $x \in I = [-1.5, 2.5] \subset \measuredangle$, from [10]. Here we have examined the various hyperbolic properties of this map by obtaining hyperbolic fixed points and discussing the stability of such points.

It can easily be checked out that this map is not a diffeomorphism, so the backward orbits do not exist. The figure below, Fig. 1, shows the graphs of f_r for different values of r. Here we have used *r=0.5, r=1.0, r=1.5, r=2.0, r=2.25, r=2.5,* with different colours in Fig. 1, from the lower to the upper curve respectively.

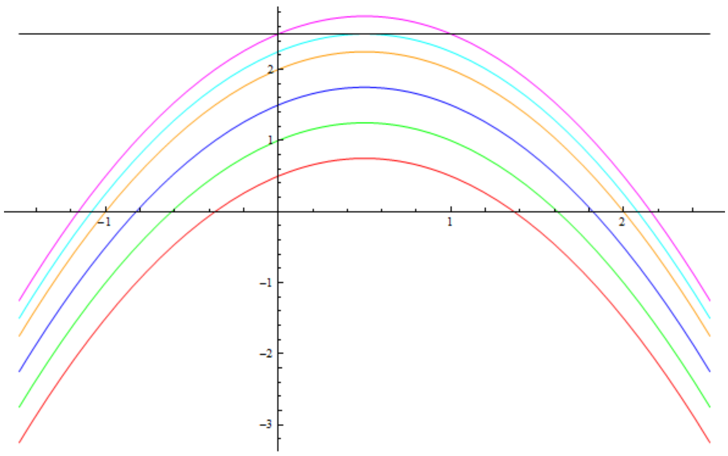

Fig. 1. Graph of f_r for different values of *r*

It can be easily seen that if $r \leq 2.25$ then $f_r(I) \subset (-\infty, 2.5]$, but for *r* > 2.25, $f_r(I) \supset [-1.5, 2.5]$. Particularly, at $r = 2.25$, $f_r(I) \subset [-1.5, 2.5]$. Further, if we take $r = \frac{5}{2}$, we get an interval $J = (0,1)$ which has the property that $f_{\frac{5}{2}}(J) \cap I = \phi$ and $f_{\frac{5}{2}}^n(x) \to -\infty$ as $n \to \infty$.

Now let $I = I_0 \cup J \cup I_1$, where $I_0 = [-1.5, 0]^2$ and $I_1 = [1, 2.5]$ for which $f_{\frac{5}{2}}(I_0) \subset [-1.5, 2.5]$ and $f_{\frac{5}{2}}(I_1) \subset [-1.5, 2.5]$. Further there exist open intervals in I_0 and I_1 such that the second iterate is mapped outside $[-1.5, 2.5]$. These intervals can be calculated and given by

$$J_1=\left(\frac{1}{2}\left(1-\sqrt{11}\right),\frac{1}{2}\left(1-\sqrt{7}\right)\right),\ J_2=\left(\frac{1}{2}\left(1+\sqrt{7}\right),\frac{1}{2}\left(1+\sqrt{11}\right)\right).$$

Thus, there exist four intervals $I_{00}=\left[-1.5,\frac{1}{2}\left(1-\sqrt{11}\right)\right]$, $I_{01}=\left[\frac{1}{2}\left(1-\sqrt{7}\right),0\right]$, $I_{10}=\left[1,\frac{1}{2}\left(1+\sqrt{7}\right)\right]$, $I_{11}=\left[\frac{1}{2}\left(1+\sqrt{11}\right),2.5\right]$ such that $f_{\frac{5}{2}}^2(I_{ij})\subset[-1.5,2.5]$, $ij=00,01,10,11$. Note that $I_{00},I_{01}\subset I_0$; $I_{10},I_{11}\subset I_1$. While continuing this process, we observe that there exists exactly one open interval inside each interval I_i such that these intervals are mapped to $[-1.5,2.5]$ on the second iterate and mapped outside $[-1.5,2.5]$ on the third iterate.

So, we find that there exists eight intervals $I_{ijk}, ijk=000,001,010,011,100,101,110,111$ such that $f_{\frac{5}{2}}^3(I_{ijk})\subset[-1.5,2.5]$. This discussion can easily be seen graphically from the lower figure of Fig. 2, where f_r is plotted in red, f_r^2 is plotted in green and f_r^3 is plotted in blue.

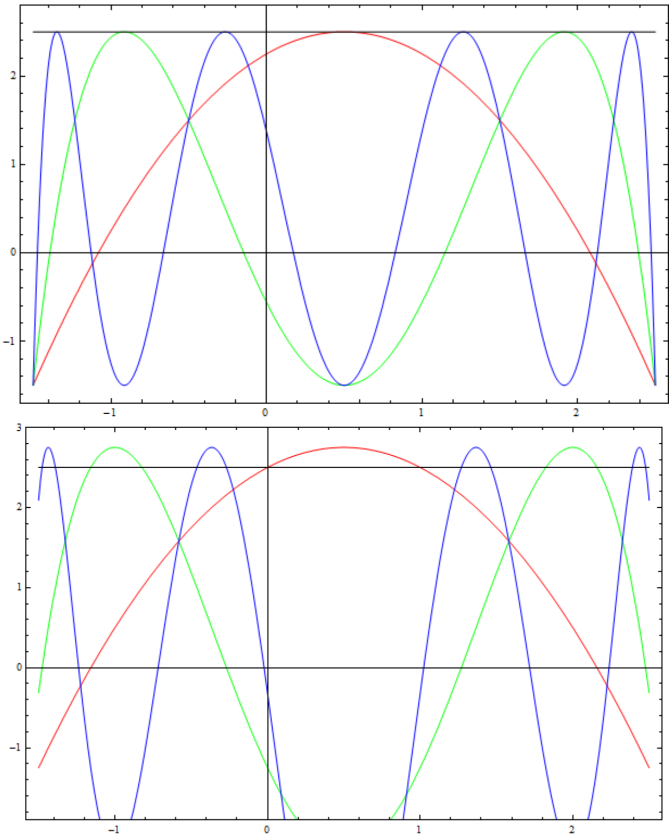

Fig. 2. Shows two diagrams; three iterations of f_r with *r = 2.25* in the upper one and three iterations of f_r with *r = 2.5* in the lower one

In the two graphs given above, one can easily observe that $f_r^n(x)\not\to-\infty$ as $n\to\infty$ till $r\le 2.25$, but $f_r^n(x)\to-\infty$ as $n\to\infty$ for r > 2.25. Further, we will show that a hyperbolic set exists in this function for r > 2.25.

We can proceed with the above process an infinite number of times, and label the intervals according to the following rule:

$$I_{i_0i_1i_2\ldots i_n}=I_{i_0i_1i_2\ldots i_{n-1}}\cap f_r^{-n}(I_{i_n})\subset I_{i_0i_1i_2\ldots i_{n-1}}\ \text{ where } i_j=0,1.$$

It will give us a nested sequence of closed intervals, $I_{i_0} \supset I_{i_0 i_1} \supset I_{i_0 i_1 i_2} \supset \ldots \supset I_{i_0 i_1 i_2 \ldots i_n}$. Finally, we come to the conclusion that

$$\bigcap_{n=1}^{\infty} I_{i_0 i_1 i_2 \ldots i_n}$$

consists of a single point (*x*). It is clear that the orbit of such a point will stay in *I* forever. So now we define the itinerary of a point *x* as

$$h(x) = i_0 i_1 i_2 \ldots i_n \ldots \text{ where } i_j = 0 \text{ if } f_r^j(x) \in I_0 \text{ and } i_j = 1 \text{ if } f_r^j(x) \in I_1 .$$

Now, let $K \subset I$ denote the set of points staying in *I* forever. The above argument shows that this set will be non-empty and consists of all intersections of nested intervals which we have given above. Further, we see that $h : K \to \sum_2^+$ is a homeomorphism, and $f = h \circ \sigma \circ h^{-1}$.
Then the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{f} & K \\ \downarrow h & & \downarrow h \\ \sum_2^+ & \xrightarrow{\sigma} & \sum_2^+ \end{array}$$

Finally, we find that the set K is an invariant set and f_r has a dense orbit in K for *r > 2*.
This same set K is the hyperbolic set for our given map. It is to be noted that the map under discussion evolves chaotically for *r > 2.25* and can also be observed through the bifurcation diagram shown in a later section. For the Logistic map $x_{n+1} = \lambda x(1-x)$ ([11-13]). It has been also proved that the hyperbolic set exists for λ *> 4.*

Next, we consider another discrete map: $x_{n+1} = -(1+r)x_n + x_n^3$, where $I = [-2, 2] \subset R$, as in [10]. The parameter r stands for a certain rate of change of population during the evolution. The figure, given below, Fig. 3, shows the graph of f_r for $r = 0.1, r = 0.5, r = 1.0, r = 1.5, r = 2.0, r = 2.2$, with different colours from the inner to outer curve respectively.

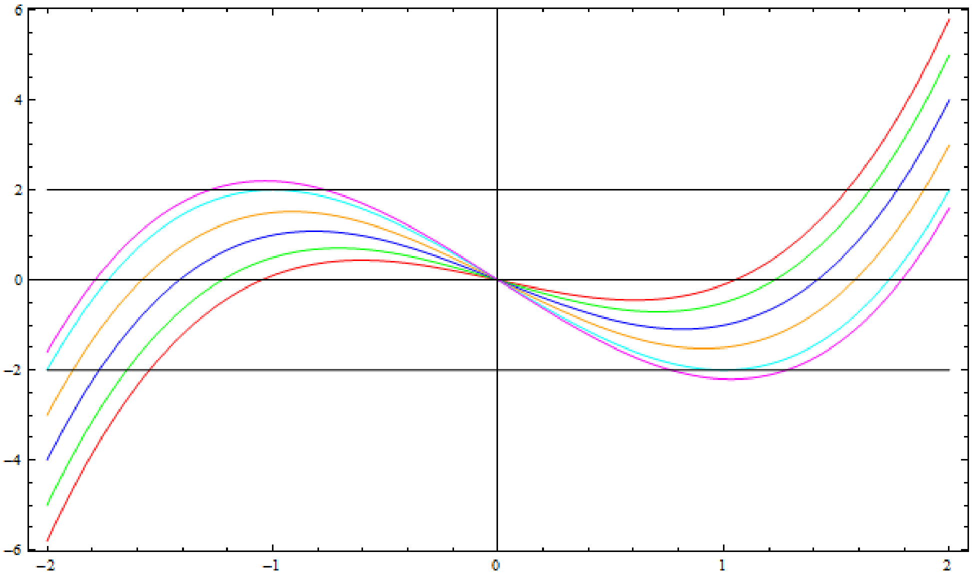

Fig. 3. Six curves of $x_{n+1} = -(1+r)x_n + x_n^3$ map for different parameter values r = *0.1* to r = *2.2* from inner to outer

As we have obtained the hyperbolic set in the earlier considered map, here, for this map we are giving a brief idea about the hyperbolic set. As the parameter is increasing from *0.1* to *2.2* in the graphs of the considered map shown above, the graphs are lying in the interval till the parameter $r \le 2$. But if we take $r = 2.2$, the graph of the map lies outside the interval I. Further, we have observed that for the parameter $r = 2$, the higher iterations lies in the interval [-2, 2] as shown in the upper graph of Fig. 4, but for $r = 2.2$, the higher iterations goes to $\pm\infty$ as shown in the lower graph of Fig. 4 below.

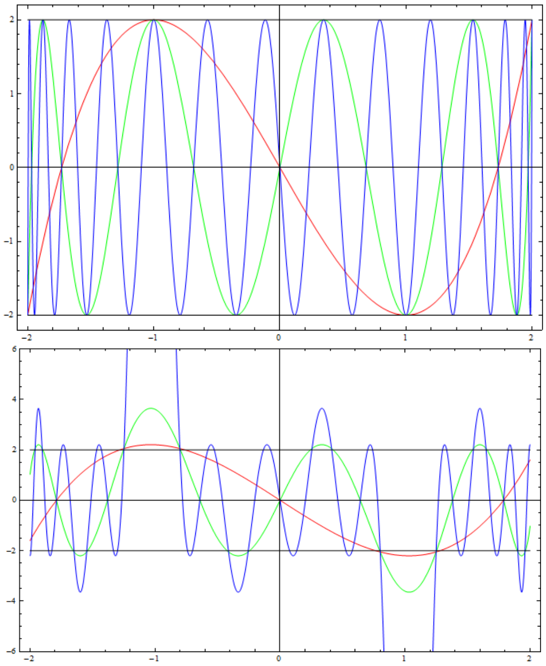

Fig. 4. Shows two diagrams; three iterations of f_r with $r = 2$ in the upper one and three iterations of f_r with $r = 2.2$ in the lower one

And as shown in the earlier model, in this map the hyperbolic set will also be the intersection of all the iterations. Hyperbolic set is nothing but the invariant set with the condition that the intersection of all the iterations should go to ∞ or $-\infty$ or both.

Consider an another discrete map named Cubic map: $x_{n+1} = r\, x_n(1 - x_n^2)$, from [14]. We have shown below, in Fig. 5, the graph of this Cubic map for different values of the parameter *r*. Here we have taken *r=1, r=1.5, r=2, r=2.5, r=3, r=3.5*. The interval of *x* is taken as [-1, 1]. But the interval of the function is [-1.155, 1.155]. Till $r \le 3$, $f_r(x) \subset [-1.155, 1.155]$ and for $r > 3$, $f_r(x) \supset [-1.155, 1.155]$

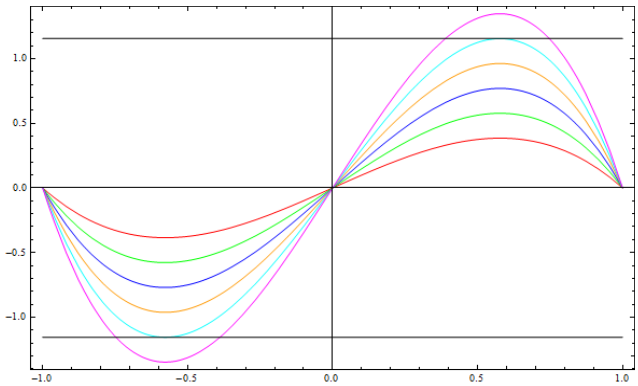

Fig. 5. Six curves of $x_{n+1} = r\, x_n(1 - x_n^2)$ map for different parameter values $r = 1$ to $r = 3.5$ from inner to outer

Now the question arises:
How can one decide an appropriate interval for the function concerned? Whether the interval for x and $f(x)$ is same in all kinds of discrete maps?
The answer to the question above can be obtained from the next topic 'Hyperbolicity and Bifurcation'.

Further, we have shown below the iteration diagrams of the above discrete cubic map with parameter $r = 2$ and $r = 2.2$ and we have again observed that for $r = 2$ the further iterations are lying in a certain limit. But, as we exceed this parameter from two, the iterations are moving towards ∞ and $-\infty$, particularly for $r = 2.2$, which is shown in the lower diagram of Fig. 6 below.

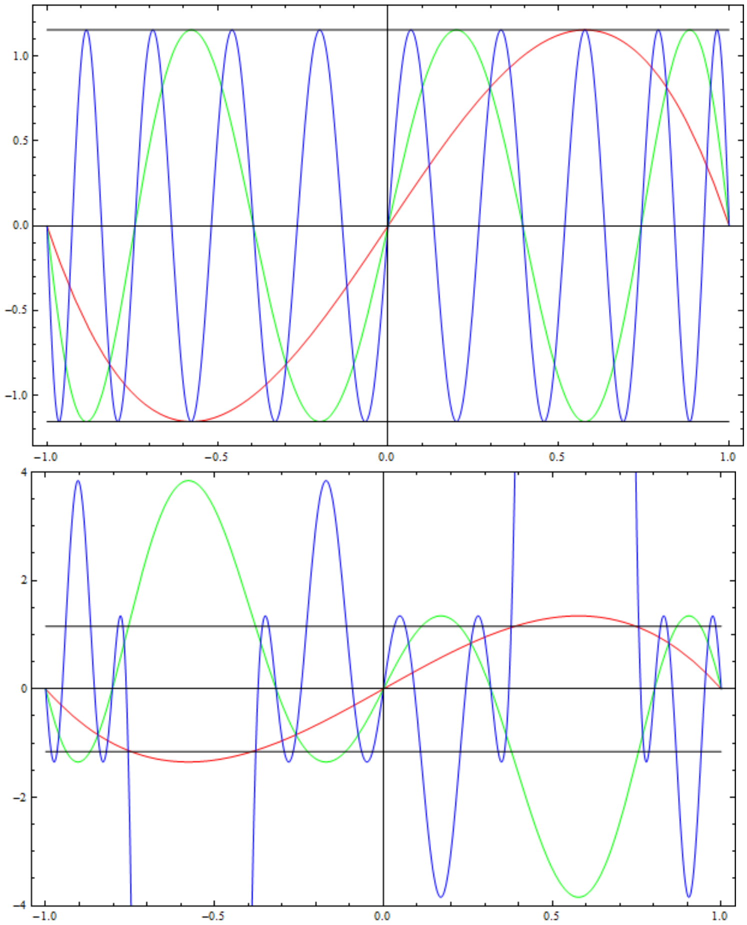

Fig. 6. Shows two diagrams; three iterations of f_r with $r = 2$ in the upper and three iterations of f_r with $r = 2.2$ in the lower

4. BIFURCATION AND HYPERBOLICITY

While considering bifurcation, we see that there is a strong relation between hyperbolicity and bifurcation leading to chaos. As we have restricted our discussion to discrete models, in any discrete model the invariant set always exists for any parameter value. But this invariant set contains only a single point till that value of parameter, after which bifurcation starts. Later on as the value of the parameter is increasing or decreasing (according to the discrete map), the cardinality of the invariant set keeps on increasing. As we have defined the hyperbolic set above and given a detailed idea to get the hyperbolic set in one dimensional discrete map, we know that hyperbolic set is an invariant set with a certain condition. That condition we have shown clearly while leading towards the computation of the hyperbolic set in the above three one dimensional discrete models.

Now we are moving towards the answer of the question which we have left in the last section of this paper.

To answer that question, we have considered three one dimensional models in this paper and studied many more that are not included in this paper. One of which is the most general, Logistic map, from which we have started the study of this hyperbolic set, but in that particular map our interval for *x* and *f(x)* was, fortunately, the same. Jonassen, [9], has extended his further calculations without any discussion of the considered interval. But after a detailed observation we found that these intervals are not, in general, the same for *x* and *f(x)* while calculating the hyperbolic set. In the computations which have been done till now, we have taken three discrete models in which for two of these, the intervals were same, but for the third it was different. But still a question remains, how can this interval be found out? The answer to this question can be obtained by studying the bifurcation diagrams of the following discrete maps:

Consider the discrete map $x_{n+1} = r + x_n - x_n^2$. In this map the interval for *x* was taken as $[-1.5, 2.5]$ earlier while computing hyperbolicity through iterations. Now we have to find the interval for *f(x)*. Drawing of the bifurcation diagram of this map is given below in Fig. 7, for *r = 0* to *r = 2.5*.

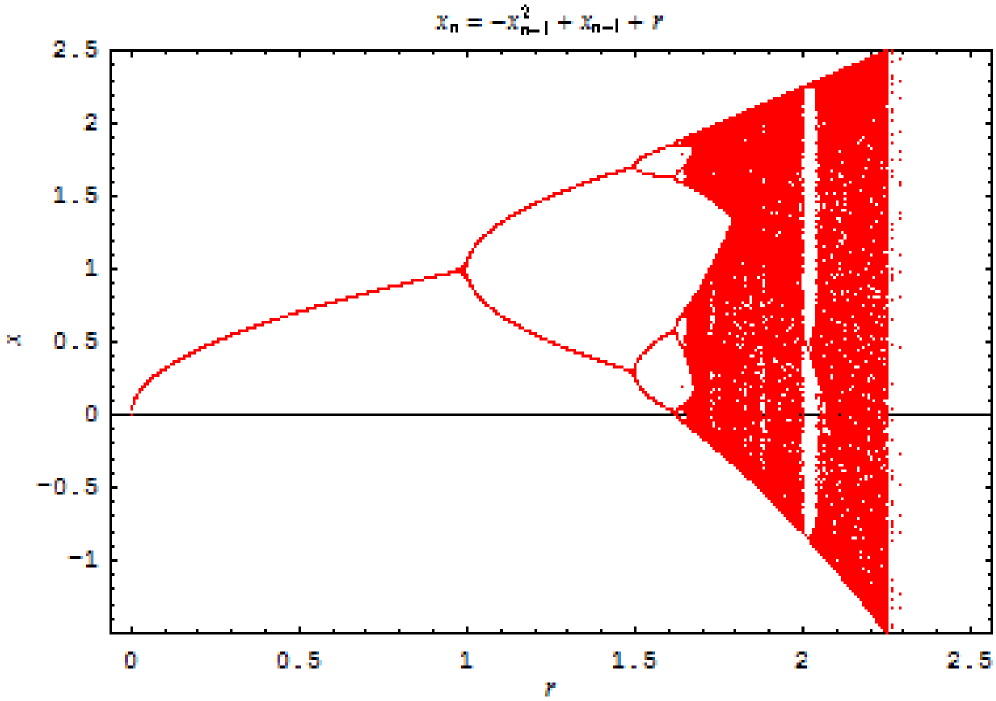

Fig. 7. Shows the Bifurcation diagram of the discrete map $x_{n+1} = r + x_n - x_n^2$ from parameter *r = 0* to *r = 2.5*

In this diagram it can be easily observed that the bifurcation stops at parameter *r = 2.25* (approx.). And for *r > 2.25*, we can get the hyperbolic set, as we have seen in earlier computations through iteration diagrams. So for this parametric value let us draw the first iteration diagram in the given interval of *x* and then find the limiting interval of *f(x)* from this diagram. This interval will be the required interval of *f(x)* for finding the hyperbolic set. In this discrete map this interval is the same as that of *x* as shown in Fig. 8 below. Further, the interval of *x* is taken as $[-1.5, 2.5]$ because all iterations are meeting on the end points of this interval as shown for the three iterations in the lower diagram of Fig. 8.

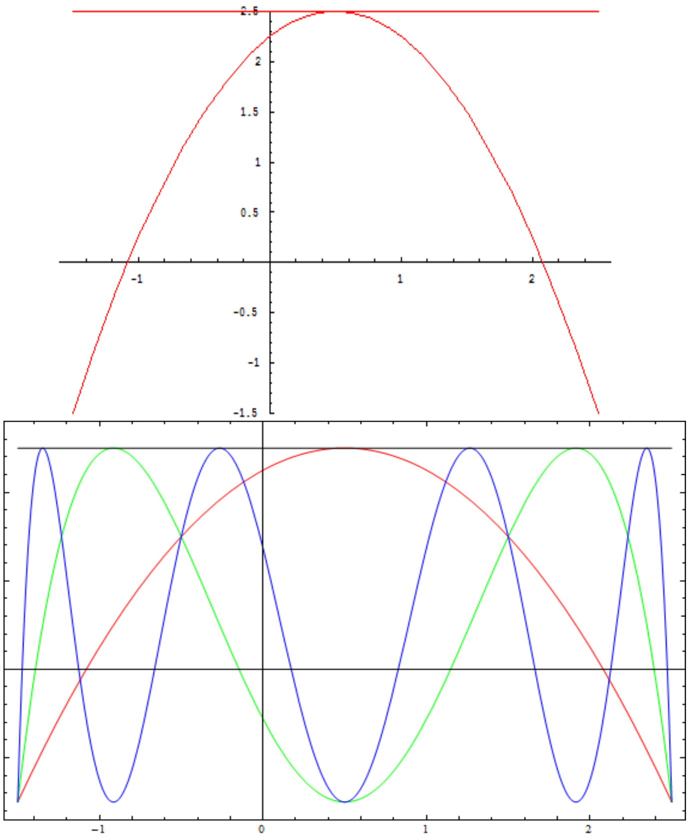

Fig. 8. Shows the first iteration for the discrete map $x_{n+1} = r + x_n - x_n^2$ with parameter *r = 2.25* in the interval $[-1.5, 2.5]$ in the upper diagram, and three iterations with the same parameter *r = 2.25* meeting at the end points of the interval in the lower one

Further, for the second map $x_{n+1} = -(1+r)x_n + x_n^3$, we can see the same observation. We have drawn the bifurcation diagram of this map as shown in Fig. 9.

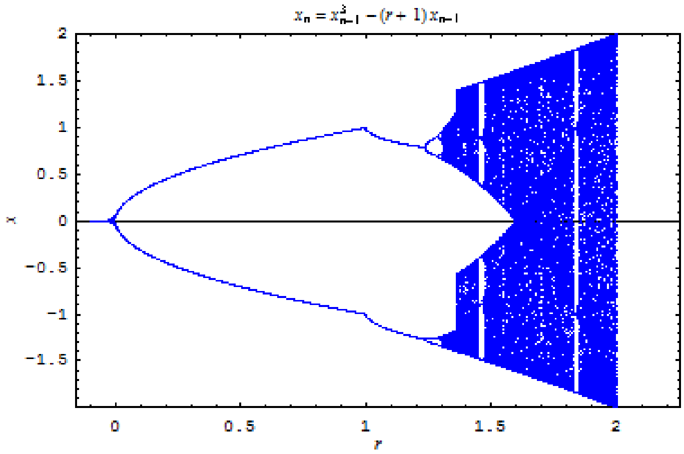

Fig. 9. Shows the Bifurcation diagram of the discrete map $x_{n+1} = -(1+r)x_n + x_n^3$ from parameter *r = -0.1* to *r = 2.2*

In this bifurcation diagram it can be observed that bifurcation stops at parameter *r = 2*. And for this parameter the first iteration diagram lies in the limiting interval [-2, 2]. So, the interval for *f(x)* is [-2, 2] for finding the hyperbolic set. For this map, as computed earlier, the interval for *x* was also taken as [-2, 2]. In this map, the interval for *x* and *f(x)* is also the same as in the earlier case.

Now, let us consider the third discrete map $x_{n+1} = r\,x_n(1 - x_n^2)$. Take the interval for x as [-1, 1] where all the iterations meet at the end points of this interval. Now consider the bifurcation diagram for this map as shown in Fig. 10. In this diagram bifurcation stops at parameter $r = 3$, and for this parameter we found the interval for $f(x)$ is [-1.155, 1.155], which is not same as interval of x.

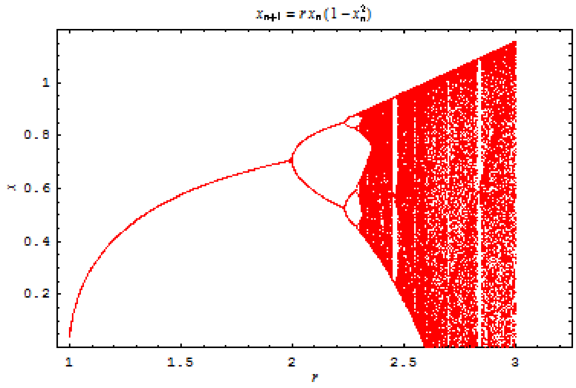

Fig. 10. Shows the Bifurcation diagram of the discrete map $x_{n+1} = r\,x_n(1 - x_n^2)$ from parameter $r = 1$ to $r = 3.2$

So we came to the conclusion that while calculating the hyperbolic set, in general, intervals for x and $f(x)$ are not necessarily the same. So, bifurcation plays a very important role in hyperbolicity.

5. HYPERBOLICITY IN TWO DIMENSIONAL DISCRETE SYSTEMS

Earlier we have computed hyperbolic set for one dimensional maps and suggested a method to get the parameter after which a discrete map will be hyperbolic. Now we want to extend our discussion to higher dimensions. Let us consider two dimensional discrete maps. For such maps, calculations of hyperbolic sets cannot be similar to what has been explained above. However, we can obtain certain set of critical values of pamameter (s), after which the existence of the hyperbolic set can be observed. Graphically, the intersection of iterations of the two or higher dimensional discrete maps cannot be observed. As the bifurcation diagram for two dimensional discrete maps can easily be drawn, we can get the parameter(s) after which hyperbolic set exists.

Let us consider the popular two dimensional discrete Henon map

$$x_{n+1} = 1 - a\,x_n^2 + y_n$$
$$y_{n+1} = b\,x_n$$

The phase plot for the map is shown as in Fig. 11 and the bifurcation diagram is shown in Fig. 12.

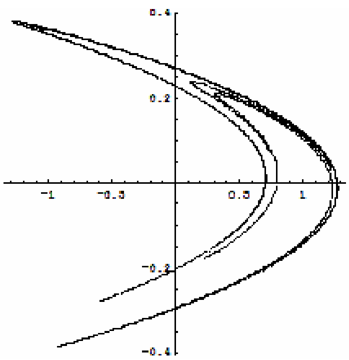

Fig. 11. Phase plot for the Henon map

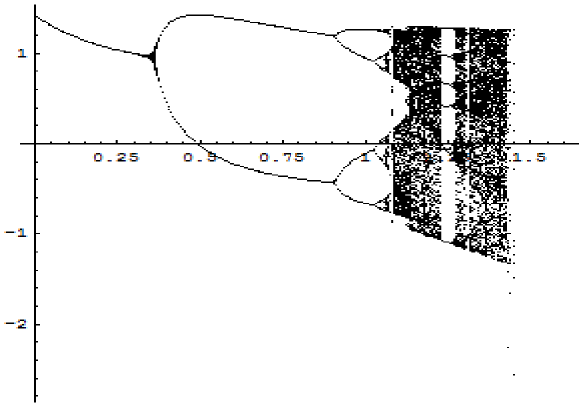

Fig. 12. Bifurcation diagram of the Henon map from parameter $a = 0$ to $a = 1.6$ and fixed $b = 0.3$

In the above bifurcation diagram, bifurcation stops at $a = 1.45$(approx.). So for Henon map, if we fix $b = 0.3$, then hyperbolic set exists for $a > 1.45$.

Similarly, for any two dimensional discrete map this parametric limit can be found out. For example, let us consider a very interesting Burger's map

$$x_{n+1} = a\,x_n - y_n^2$$
$$y_{n+1} = b\,y_n + x_n y_n$$

The phase plot for this map is shown in Fig. 13 and the bifurcation diagram is shown in Fig. 14:

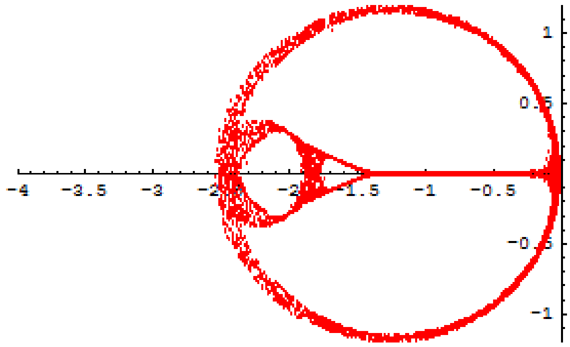

Fig. 13. Phase plot for the Burger's map

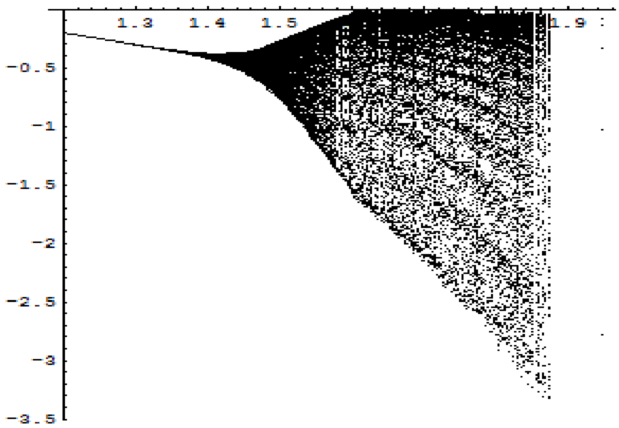

Fig. 14. Bifurcation diagram of the Burger's map from parameter $b = 0.2$ to $b = 2$ and fixed $a = 0.75$

In this bifurcation diagram one can observe that the bifurcation stops at parameter $b = 1.88$. So if we fixed the parameter $a = 0.75$, then hyperbolic set exists in Burger's map for $b > 1.88$.

6. PERIOD DOUBLING PHENOMENA

We observe through the bifurcation diagram of one dimensional discrete maps, period doubling phenomena introduced by Feigenbaum [15] holds in the form of two cycles, four cycles, eight cycles etc. from one cycle (e.g. in Fig.7, Fig. 9, Fig. 10) and then leading to chaos. But during this period, doubling phenomena may not hold for all two or higher dimensional systems. For example, in Henon map it holds as shown in Fig. 12, but for Burger's map it does not hold as shown in Fig. 14. To get a clearer picture, a very small magnified region of the bifurcation diagram of Burger's map is shown in Fig. 15.

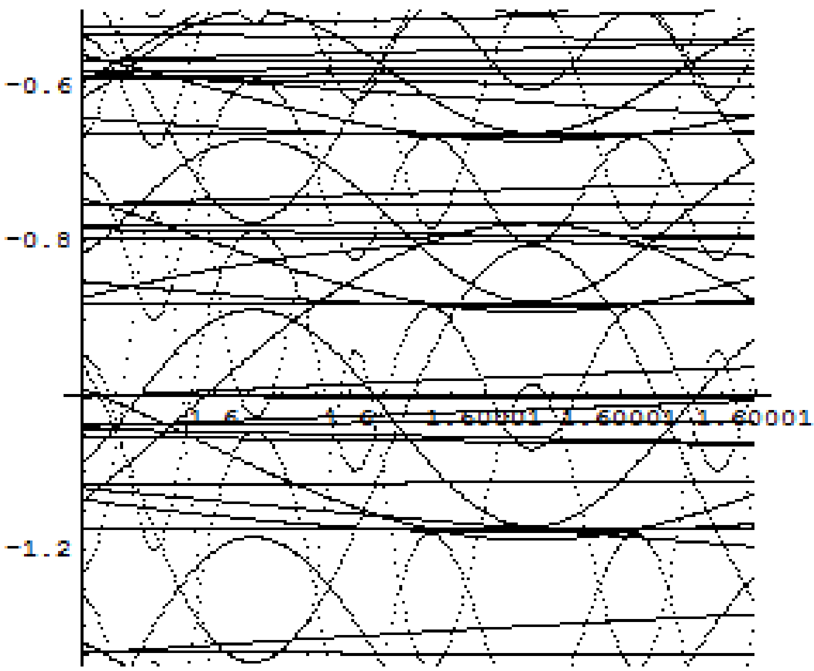

Fig. 15. Shows the magnified region of Bifurcation diagram shown in Fig. 12

In this diagram one can easily observe that no period doubling phenomena appears in it. It has been observed that the steady state solution of one cycle suddenly changes into chaotic. The orbits are random in nature.

7. CONCLUDING REMARKS

Through the above investigation we find that the property *hyperbolicity* has a great role while discussing regular as well as chaotic motion in the dynamical system. Parameters of the systems have a specific role in deciding hyperbolic and invariant sets. As systems are nonlinear, the parameters variations may not be exactly similar. Also, the bifurcation diagrams play an important role in deciding hyperbolic and invariant sets. The results shown above are mostly graphical rather than numerical. For two or higher dimensional systems period doubling phenomena may not necessarily be observed. However, limiting values of the parameters can be obtained which play an important role in obtaining hyperbolic and invariant sets. One can extend the problem to other two or higher dimensional maps as well as for continuous maps. This part of the program may be considered in our future studies.

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