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A Square representation technique for locating frequencies that have maximum autocorrelations

M. Taghavi

Department of Mathematics, School of Sienses, Shiraz University, Shiraz, Iran
E-mail: taghavi@math.susc.ac.ir

Abstract

From the early 1950s, estimating the autocorrelations of polynomials with coefficients on the unit circle has found applications in Ising spin systems and in surface acoustic wave designs. In this paper, a technique is introduced that not only estimates the autocorrelations, but for some special types of such polynomials, it locates the frequencies at which maximum autocorrelation occurs.

Keywords*:* Square representation; Stable cycles; Golay pair of polynomials

1. Introduction

Let h be a complex polynomial of degree n. The integers less than or equal to n are called the frequencies of h. For any frequency k of h define

$$h^{\wedge}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} h(e^{it})dt.$$

Let T denote the unit circle and C_n denote the class of all degree n complex polynomials. Define

$$\mathsf{U}_n = \{h \in \mathsf{C}_n \,|\, h^{\wedge}(k) \in \mathsf{T}, \quad k = 0,1,\cdots,n\},$$

and

$$\mathsf{V}_n = \{h \in \mathsf{C}_n \,|\, h^{\wedge}(k) \in \{-1,+1\}, \quad k = 0,1,\cdots,n\}.$$

We observe from [1-5] that estimating the coefficients (in modulus) of $|h(e^{it})|^2$, with $h \in \mathsf{V}_n$, have found applications in Ising spin systems of physics, orthogonal designs and Hadamard matrices of combinatorics, and in telecommunications, surface-acoustic wave design, the Loran C precision navigation system, channel-measurement, optical time-domain reectometry, synchronization, spread spectrum communications, and, recently, Orthogonal Frequency Division Multiplexing (OFDM) systems. So in what follows, we introduce techniques for estimating autocorrelations.

Given complex numbers $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_m$ and $\gamma_0, \gamma_1, \cdots, \gamma_m$, the *square representation* of the products $\varepsilon_k \bar{\gamma}_j$ ($k, j = 0,1,\cdots,m$) is formed by $2^{m+1} \times 2^{m+1}$ squares on which $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_m$ and $\bar{\gamma}_0, \bar{\gamma}_1, \cdots, \bar{\gamma}_m$ respectively are the columns and rows of the representation. We label each of the squares by $b_{i,j}$ ($0 \le i, j \le m$). Thus $b_{i,j}$ is the square located in $(i+1)$ th row and $(j+1)$ th column. For example, if $m = 3$, then the square representation and label representation are

	ε_0	ε_1	ε_2	ε_3
$\bar{\gamma}_0$	$\varepsilon_0\bar{\gamma}_0$	$\varepsilon_1\bar{\gamma}_0$	$\varepsilon_2\bar{\gamma}_0$	$\varepsilon_3\bar{\gamma}_0$
$\bar{\gamma}_1$	$\varepsilon_0\bar{\gamma}_1$	$\varepsilon_1\bar{\gamma}_1$	$\varepsilon_2\bar{\gamma}_1$	$\varepsilon_3\bar{\gamma}_1$
$\bar{\gamma}_2$	$\varepsilon_0\bar{\gamma}_2$	$\varepsilon_1\bar{\gamma}_2$	$\varepsilon_2\bar{\gamma}_2$	$\varepsilon_3\bar{\gamma}_2$
$\bar{\gamma}_3$	$\varepsilon_0\bar{\gamma}_3$	$\varepsilon_1\bar{\gamma}_3$	$\varepsilon_2\bar{\gamma}_3$	$\varepsilon_3\bar{\gamma}_3$

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{0,3}$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$
$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$

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Suppose that

$$f(e^{it}) = \sum_{j=1}^{m} \varepsilon_j e^{ijt}, \quad g(e^{it}) = \sum_{j=1}^{m} \gamma_j e^{ijt} \qquad -\infty < t < \infty.$$

If $h = f\overline{g}$, then

$$h(e^{it}) = h^\wedge(m)e^{imt} + \cdots + h^\wedge(1)e^{it} + h^\wedge(0) + h^\wedge(-1)e^{-it} + \cdots + h^\wedge(-m)e^{-imt}.$$

The $2m+1$ complex numbers $h^\wedge(m), \cdots, h^\wedge(-m)$ can be obtained from the square representation of the products $\varepsilon_k \overline{\gamma}_j$ as follows:

$h^\wedge(m)$	$\varepsilon_m \overline{\gamma}_0$	the number in $b_{0,m}$ (top right square)
$h^\wedge(m-1)$	$\varepsilon_{m-1}\overline{\gamma}_0 + \varepsilon_m \overline{\gamma}_1$	sum of numbers in $b_{0,m-1}$ and $b_{1,m}$
⋮	⋮	⋮
$h^\wedge(0)$	$\varepsilon_0 \overline{\gamma}_0 + \cdots + \varepsilon_m \overline{\gamma}_m$	sum of numbers in diagonal squares
⋮	⋮	⋮
$h^\wedge(-(m-1))$	$\varepsilon_0 \overline{\gamma}_{m-1} + \varepsilon_1 \overline{\gamma}_m$	sum of numbers in $b_{m-1,0}$ and $b_{m,1}$
$h^\wedge(-m)$	$\varepsilon_0 \overline{\gamma}_m$	the number in $b_{m,0}$ (bottom left quare)

For example, if $f(e^{it}) = 1 + 2e^{it} - 3e^{2it} + e^{3it}$ and $g(e^{it}) = -1 - 4e^{it} + e^{2it} - 2e^{3it}$, then the square representation for $f\overline{g}$ is

	1	2	−3	1
−1	−1	−2	3	−1
−4	−4	−8	12	−4
+1	1	2	−3	1
−2	−2	−4	6	−2

and therefore we have, $h^\wedge(3) = -1$, $h^\wedge(2) = 3-4$, $h^\wedge(1) = -2+12+1$, $h^\wedge(0) = -1-8-3-2$, $h^\wedge(-1) = -4+2+6$, $h^\wedge(-2) = 1-4$, and $h^\wedge(-3) = -2$. That is

$$(f\overline{g})(e^{it}) = -e^{3it} - e^{2it} + 11e^{it} - 14 + 4e^{-it} - 3e^{-2it} - 2e^{-3it}.$$

The square representations here are not just for easy calculations of some polynomial product. In fact, the counter example presented in [6] was found by the use of this technique and, as will be seen, their most important applications are estimating or sometimes evaluating the autocorrelations of Golay type polynomials that we define as follows:

Definition 1.1. A pair of polynomials $(A_n(z), B_n(z))$ ($z \in \mathbb{C}$) of the same degree $d = d_n$ are called a Golay pair of polynomials, if for every $k \in \{0, 1, \cdots, d\}$ and for all real t they satisfy the following two conditions:

$$(A_n)^\wedge(k) = \pm 1 \quad \text{and} \quad (B_n)^\wedge(k) = \pm 1, \tag{1}$$

$$|A_n(e^{it})|^2 + |B_n(e^{it})|^2 = 2d + 2. \tag{2}$$

The most remarkable polynomials which form Golay complementary pairs were discovered by Harold S. Shapiro in his 1951 Master thesis [7]. He accidently made the discovery as he had many stimulating conversations with fellow student D. J. Newman about the Fejer-Riesz Theorem on non negative trigonometric polynomials. Shapiro's Master thesis has never been published, but it was used in an extremal work of Walter Rudin in [8]. The course of those studies has led to what we now call the Rudin-Shapiro polynomials. These polynomials are categorized as flat polynomials. This refers to the fact that the amplitude of the complex polynomials are, on the unit circle, bounded by a constant times the energy of the polynomial. The construction of flat polynomials dates back to the beginning of the 20th century. Of course, at that time the purpose was mainly pure rather than to design signals for use in digital transmission systems. One of the early examples of flat polynomials is a discovery in 1916 by Hardy and Littlewood [9]. Another reason that makes the Rudin-Shapiro polynomials so important is that, in general, polynomials with small autocorrelations are of interest in a number of applications in signal processing and communications (see [10-12]). Since the 1950s, digital communications engineers have sought to identify binary sequences for which the absolute values of the aperiodic autocorrelation function are collectively small, for application in synchronization, pulse compression and, especially, radar [13]. Letting $(p_0, q_0) = (1,1)$, for $n \geq 1$ and $z \in \mathbb{C}$ the Rudin-Shapiro pair of polynomials (p_n, q_n) is defined by

$$p_n(z) = p_{n-1}(z) + z^{2^{n-1}} q_{n-1}(z),$$
$$q_n(z) = p_{n-1}(z) - z^{2^{n-1}} q_{n-1}(z).$$

Note that $p_{n+1}(e^{it}) = p_n(e^{it}) + e^{i2^n t} q_n(e^{it})$ and if $|j| < 2^n$, then $(e^{i2^n t} q_n(e^{it}))^\wedge(j) = 0$ (because $deg(e^{i2^n t} q_n(e^{it})) \geq 2^n$). Therefore, $(p_{n+1})^\wedge(j) = (p_n)^\wedge(j)$ and so the first 2^n coefficients of p_{n+1} are identical with those of p_n and these coefficients do not depend on n. Hence for each n, there are numbers $\zeta_n, \eta_n \in \{-1, +1\}$ such that

$$p_n(z) = \sum_{n=0}^{2^n - 1} \zeta_n z^n \quad and \quad q_n(z) = \sum_{n=0}^{2^n - 1} \eta_n z^n.$$

Thus the relation (1) holds and for (2) see [10] or [11].

Definition 1.2. Each of the $2^{n+1} - 1$ numbers which form the coefficients of $|p_n(e^{it})|^2$ and each of the $2^{n+1} - 1$ numbers which form the coefficients of $p_n^2(e^{it})$ are, respectively, called an autocorrelation and a correlations of p_n.

Definition 1.3. Given positive integers n and m, a sequence (of frequencies) $\{\alpha_n\}$ of p_n^2 is said to have an m-stable cycle (for correlation) if there are absolute constants A_1, A_2 and A_3 such that

$$(p_n^2)^\wedge(\alpha_n) = A_1 (p_{n-m}^2)^\wedge(\alpha_{n-m})$$
$$+ A_2 (q_{n-m}^2)^\wedge(\alpha_{n-m}) + A_3 (p_{n-m} q_{n-m})^\wedge(\alpha'_{n-m}),$$

where $\alpha'_{n-m} = \alpha_{n-m} + r 2^{n-m}$ for some unspecified number r.

Definition 1.4. A sequence (of frequencies) $\{\alpha_n\}$ of $|p_n|^2$ is said to have an m-stable cycle (for autocorrelation), if there are absolute constants B_1, B_2 and B_3 such that

$$(|p_n|^2)^\wedge(\alpha_n) = B_1 (|p_{n-m}|^2)^\wedge(\alpha'_{n-m})$$
$$+ B_2 (\bar{p}_{n-m} q_{n-m})^\wedge(\beta'_{n-m})$$
$$+ B_3 (p_{n-m} \bar{q}_{n-m})^\wedge(\beta'_{n-m}),$$

where $\alpha'_{n-m} = |\alpha_n - r_1 2^{n-m}|$ and $\beta'_{n-m} = |\alpha_n - r_2 2^{n-m}|$ for unspecified numbers r_1 and r_2.

We observe from the references cited in [14] (resp. [15]) that for a positive integer n, there is a nonzero frequency α_n, in a 2-stable cycle mode, so that the correlation (resp. autocorrelation) of p_n at α_n dominates a universal constant multiple of $2^{0.73n}$.

It should not be surprising that there are similarities between the bounds for correlations and the bounds for autocorrelations of p_n (both in absolute value). The lower bound result for correlation in [14] was achieved by the solution of the following (recurrence) system of three equations in three unknowns $(p_j^2)^\wedge(\alpha_j)$, $(q_j^2)^\wedge(\alpha_j)$, and $(p_j q_j)^\wedge(\alpha'_j)$ (for appropriate frequencies α'_j),

$$\begin{cases} (p_n^2)^\wedge(\alpha_n) = & 2(p_{n-2}^2)^\wedge(\alpha_{n-2}) & +(q_{n-2}^2)^\wedge(\alpha_{n-2}) + 2(p_{n-2}q_{n-2})^\wedge(\alpha'_{n-2}) \\ (q_n^2)^\wedge(\alpha_n) = & -2(p_{n-2}^2)^\wedge(\alpha_{n-2}) & +(q_{n-2}^2)^\wedge(\alpha_{n-2}) + 2(p_{n-2}q_{n-2})^\wedge(\alpha'_{n-2}) \\ (p_n q_n)^\wedge(\alpha'_j) = & & -(q_{n-2}^2)^\wedge(\alpha_{n-2}) + 2(p_{n-2}q_{n-2})^\wedge(\alpha'_{n-2}). \end{cases}$$

Similarly the lower bound result for autocorrelation in [13] was achieved by the solution of the following (recurrence) system of three equations in three unknowns $(|p_j|^2)^\wedge(\alpha_j)$, $(\bar{p}_j q_j)^\wedge(\beta_j)$ and $(p_j \bar{q}_j)^\wedge(\beta_j)$ (for appropriate frequencies β_j),

$$\begin{cases} (|p_n|^2)^\wedge(\alpha_n) = & 2(|p_{n-2}|^2)^\wedge(\alpha_{n-2}) - (\bar{p}_{n-2}q_{n-2})^\wedge(\beta_{n-2}) & \\ (\bar{p}_n q_n)^\wedge(\beta_n) = & 2(|p_{n-2}|^2)^\wedge(\alpha_{n-2}) + 2(\bar{p}_{n-2}q_{n-2})^\wedge(\beta_{n-2}) & -(p_{n-2}\bar{q}_{n-2})^\wedge(\beta_{n-2}) \\ (p_n \bar{q}_n)^\wedge(\beta_n) = & 2(|p_{n-2}|^2)^\wedge(\alpha_{n-2}) + 2(\bar{p}_{n-2}q_{n-2})^\wedge(\beta_{n-2}) & +(p_{n-2}\bar{q}_{n-2})^\wedge(\beta_{n-2}). \end{cases}$$

In the following, we refer to the above two systems as the co-system and the auto-system respectively. The matrix equation of the co-system is of the form $w_n = Aw_{n-2}$, where A is a 3×3 nonsingular matrix. The eigenvalues λ_1, λ_2 and λ_3 of A satisfy $\lambda_2 = \bar{\lambda}_3$, $|\lambda_1|>|\lambda_2|=|\lambda_3|$ and $|\lambda_1| = 2^{1.46}$. So, if Λ is the diagonal matrix having the eigenvalues of A on its diagonal entries, then for odd n, there is an invertible S such that $w_n = S\Lambda^{\frac{n}{2}}S^{-1}w_0$ with $w_0 = [1\ 1\ 0]^T$, while for even n, $w_n = S\Lambda^{\frac{n-1}{2}}S^{-1}w_1$ with $w_1 = [1\ 1\ 1]^T$.

2. Locating the appropriate frequencies by the representation

In what follows, we present two different uses of the square representations.

- Given any frequency of $|p_n|^2$ as input, the output is a system in 2-stable cycle mode such that its equations are similar to those in the auto-system.
- Locate the frequency (or frequencies) that the matrix of its system, in terms of eigenvalues, is similar to the matrix of the co-system.

The square representations of $|p_n|^2$ and $|q_n|^2$ in one step backwards are

$\bar{p}_n$ \ p_n	p_{n-1}	q_{n-1}
$\bar{p}_{n-1}$	$\lvert p_{n-1}\rvert^2$	$\bar{p}_{n-1}q_{n-1}$
$\bar{q}_{n-1}$	$p_{n-1}\bar{q}_{n-1}$	$\lvert q_{n-1}\rvert^2$

$\bar{q}_n$ \ q_n	p_{n-1}	$-q_{n-1}$
$\bar{p}_{n-1}$	$\lvert p_{n-1}\rvert^2$	$-\bar{p}_{n-1}q_{n-1}$
$-\bar{q}_{n-1}$	$-p_{n-1}\bar{q}_{n-1}$	$\lvert q_{n-1}\rvert^2$

and the square representation of $|p_n|^2$ in two steps backwards is

		p_n			
		p_{n-1}		q_{n-1}	
$\bar{p}_n$		p_{n-2}	q_{n-2}	p_{n-2}	$-q_{n-2}$
$\bar{p}_{n-1}$	$\bar{p}_{n-2}$	$\lvert p_{n-2}\rvert^2$	$\bar{p}_{n-2}q_{n-2}$	$\lvert p_{n-2}\rvert^2$	$-\bar{p}_{n-2}q_{n-2}$
	$\bar{q}_{n-2}$	$p_{n-2}\bar{q}_{n-2}$	$\lvert q_{n-2}\rvert^2$	$p_{n-2}\bar{q}_{n-2}$	$-\lvert q_{n-2}\rvert^2$
$\bar{q}_{n-1}$	$\bar{p}_{n-2}$	$\lvert p_{n-2}\rvert^2$	$\bar{p}_{n-2}q_{n-2}$	$\lvert p_{n-2}\rvert^2$	$-\bar{p}_{n-2}q_{n-2}$
	$-\bar{q}_{n-2}$	$-p_{n-2}\bar{q}_{n-2}$	$-\lvert q_{n-2}\rvert^2$	$-p_{n-2}\bar{q}_{n-2}$	$\lvert q_{n-2}\rvert^2$

In the following, z is restricted to satisfy $|z|=1$ and we put $l_n = 2^n$ for all n. The Rudin-Shapiro polynomials p_n and q_n are of degree $(l_n - 1)$, and so the frequencies of $|p_n|^2$, $p_n\bar{q}_n$ and $\bar{p}_n q_n$ must be integers in the closed (frequency) interval $[1-l_n, l_n-1]$. One can easily verify that $(|p_n(z)|^2)^\wedge(2j) = 0$ for all j and $(|p_n(z)|^2)^\wedge(0) = 2^n$. Moreover, $(|p_n|^2)^\wedge$ at the two endpoints of the frequency interval is either $+1$ or -1. We "divide" the frequency interval into eight open subintervals as follows: For each $i \in \{0,1,\cdots,7\}$ define

$$F_{n_i} = \{\alpha_n : (3-i)l_{n-2} < \alpha_n < (4-i)l_{n-2}\}.$$

Because of what was said about the even frequencies, the odd frequencies in $F_{n_0},\cdots,F_{n_7}$ are of the interest. Now, in more detail, we are able to present the square representation of $|p_n(z)|^2$ with two steps backwards as follows:

$\lvert p_{n-2}\rvert^2$	$\overline{p}_{n-2}q_{n-2}$	$\lvert p_{n-2}\rvert^2$	$-\overline{p}_{n-2}q_{n-2}$	F_{n_0}	$4l_{n-2}$
$p_{n-2}\overline{q}_{n-2}$	$\lvert q_{n-2}\rvert^2$	$p_{n-2}\overline{q}_{n-2}$	$-\lvert q_{n-2}\rvert^2$	F_{n_1}	$3l_{n-2}$
$\lvert p_{n-2}\rvert^2$	$\overline{p}_{n-2}q_{n-2}$	$\lvert p_{n-2}\rvert^2$	$-\overline{p}_{n-2}q_{n-2}$	F_{n_2}	$2l_{n-2}$
$-p_{n-2}\overline{q}_{n-2}$	$-\lvert q_{n-2}\rvert^2$	$-p_{n-2}\overline{q}_{n-2}$	$\lvert q_{n-2}\rvert^2$	F_{n_3}	l_{n-2}
F_{n_7}	F_{n_6}	F_{n_5}	F_{n_4}		0
$-4l_{n-2}$	$-3l_{n-2}$	$-2l_{n-2}$	$-l_{n-2}$	0	

Suppose, for example, that $\alpha_n \in F_{n_0}$ (top right square). Then $(\lvert p_n\rvert^2)^\wedge$ at α_n is the same as $(-\overline{p}_{n-2}q_{n-2})^\wedge$ at $\alpha_n - 3l_{n-2}$. As another example, suppose that $\alpha_n \in F_{n_3}$. Then the $(\lvert p_n\rvert^2)^\wedge$ at α_n is $[2\times\lvert p_{n-2}\rvert^2 + 2\times\lvert q_{n-2}\rvert^2]^\wedge$ at α_n (four diagonal squares) plus $(p_{n-2}\overline{q}_{n-2})^\wedge$ at $-\alpha_n$ (the $b_{1,2}$ box). The plus and minus elements in the $b_{0,1}$ and $b_{2,3}$ boxes cancel out. We identify the frequency related to both $\lvert p_{n-2}\rvert^2$ and $\lvert q_{n-2}\rvert^2$ by α'_{n-2}. We also identify the frequency related to both $\overline{p}_{n-2}q_{n-2}$ and $p_{n-2}\overline{q}_{n-2}$ by β'_{n-2}. Given a frequency α_n of $\lvert p_n\rvert^2$, there is an $i \in \{0,\cdots,7\}$ so that $\alpha_n \in F_{n_i}$ and there are constants A, B, C, and D such that

$$
\begin{aligned}
(\lvert p_n\rvert^2)^\wedge(\alpha_n) &= A(\lvert p_{n-2}\rvert^2)^\wedge(\alpha'_{n-2}) \\
&+ B(\lvert q_{n-2}\rvert^2)^\wedge(\alpha'_{n-2}) \\
&+ C(\overline{p}_{n-2}q_{n-2})^\wedge(\beta'_{n-2}) \\
&+ D(p_{n-2}\overline{q}_{n-2})^\wedge(\beta'_{n-2}),
\end{aligned} \tag{3}
$$

for appropriate α'_{n-2} and β'_{n-2}. Depending on the location of α_n in any of the sets F_{n_0}, F_{n_1}, F_{n_2} or F_{n_3}, Table 1 provides two frequencies α'_{n-2} and β'_{n-2} together with the four constants in (3). We omit the other four intervals F_{n_4}, F_{n_5}, F_{n_6} and F_{n_7}, because $(\lvert p_n\rvert^2)^\wedge(k) = (\lvert p_n\rvert^2)^\wedge(-k)$ for all k.

Table 1. The value of each constant in (3) at different locations for the frequency α_n

Location of α_n	α'_{n-2}	A	B	β'_{n-2}	C	D
F_{n_0}	Does not exist	0	0	$\alpha_n - 3l_{n-2}$	-1	0
F_{n_1}	$\alpha_n - 2l_{n-2}$	1	-1	$3l_{n-2} - \alpha_n$	-1	0
F_{n_2}	$l_{n-2} - \alpha_n$	1	-1	$\alpha_n - l_{n-2}$	0	1
F_{n_3}	α_n	2	2	$-\alpha_n$	0	1

For example, if $\alpha_n \in F_{n_2}$, then

$$(|p_n|^2)^\wedge(\alpha_n) = (|p_{n-2}|^2)^\wedge(\alpha'_{n-2}) - (|q_{n-2}|^2)^\wedge(\alpha'_{n-2}) + (p_{n-2}\overline{q}_{n-2})^\wedge(\beta'_{n-2})$$

$$= (|p_{n-2}|^2 - |q_{n-2}|^2)^\wedge(l_{n-2} - \alpha_n) + (p_{n-2}\overline{q}_{n-2})^\wedge(\alpha_n - l_{n-2}).$$

Locating noncentral frequencies at which the maximum (in modulus Fourier coefficients) occurs: At first, consider the representation of $|p_4|^2$:

D_1 D_2	1	1	-1	1	1	-1	1	D_2	1	1	•	D_3	-1	1	-1
1	D_1	1	-1	1	1	-1	1	1	D_2	1	-1	•	D_3	1	-1
1	1	D_1	-1	1	1	-1	1	1	1	D_2	-1	D_4	•	D_3	-1
-1	-1	-1	D_1	-1	-1	1	-1	-1	-1	-1	D_2	1	D_4	•	D_3
1	1	1	-1	D_1	1	-1	1	1	1	1	-1	D_2	-1	1	•
1	1	1	-1	1	D_1	-1	1	1	1	1	-1	-1	D_2	1	-1
-1	-1	-1	1	-1	-1	D_1	-1	-1	-1	-1	1	1	1	D_2	1
1	1	1	-1	1	1	-1	D_1	1	1	1	-1	-1	-1	1	D_2
1	1	1	-1	1	1	-1	1	D_1	1	1	-1	-1	-1	1	-1
1	1	1	-1	1	1	-1	1	1	D_1	1	-1	-1	-1	1	-1
1	1	1	-1	1	1	-1	1	1	1	D_1	-1	-1	-1	1	-1
•	-1	-1	1	-1	-1	1	-1	-1	-1	-1	D_1	1	1	-1	1
-1	•	-1	1	-1	-1	1	-1	-1	-1	-1	1	D_1	1	-1	1
-1	-1	•	1	-1	-1	1	-1	-1	-1	-1	1	1	D_1	-1	1
1	1	1	•	1	1	-1	1	1	1	1	-1	-1	-1	D_1	-1
-1	-1	-1	1	•	-1	1	-1	-1	-1	-1	1	1	1	-1	D_1

It is formed by four $2^3 \times 2^3$ squares, each of which is formed by four $2^2 \times 2^2$ squares and so on. The maximum modulus Fourier coefficient is 5, and occurs at the frequencies 11 and -11. These two frequency lines are shown by •s (each • has value -1).

We only concentrate on the positive frequency $\alpha = 11$. The central frequency line of the $|p_4|^2$ representation is formed by D_1s. The D_2s form the central frequency line of the $2^3 \times 2^3$ square $\overline{p}_3 q_3$ (see representation of $|p_n|^2$ in one step backward). The next two squares of size $2^2 \times 2^2$ (for $-\overline{p}_2 q_2$ square) and 2×2 (for $p_1 \overline{q}_1$ square) have central frequency lines D_3s and D_4s respectively. Each of the central frequency lines is located either above the frequency line α or below. We correspond the two digits 1 and 0 to the above and below positions respectively. Four position comparing of α with central frequency lines with the order D_4, D_3, D_2 and D_1 yield 1011.

Similarly, forming the square representation of $|p_6|^2$ and $|p_8|^2$, we see that the maximum modulus Fourier coefficients occur at $\alpha = 43$ and $\alpha = 171$ respectively. The central frequency line comparisons for $|p_6|^2$ give 101011 and for $|p_8|^2$ give 10101011. Interestingly, the result in each case is the binary representation of α. So we suspect that, in general, the "maximum" may occur at the n digit binary representation $1010\cdots1011$. Solving the relation $\alpha_n = 2^{n-1} + 2^{n-3} + \cdots + 2^3 + 2^1 + 1$ yields $\alpha_n = \frac{1}{3}(2^{n+1} + 1)$, which is an integer whenever

n is even. The same technique for odd n s yields $\alpha_n = \frac{1}{3}(2^n + 1)$.

Suppose that n is even. Define $\alpha_n = \frac{1}{3}(2^{n+1} + 1)$, which is clearly in F_{n_1} and define $\beta_n = \frac{1}{3}(2^n - 1)$. Then $\alpha_{n-2} = \frac{1}{3}(2^{n-1} + 1) = \alpha_n - 2l_{n-2}$ and $\beta_{n-2} = \frac{1}{3}(2^{n-2} - 1) = 3l_{n-2} - \alpha_n$.

Therefore, $\alpha_{n-2} = \alpha'_{n-2}$ and $\beta_{n-2} = \beta'_{n-2}$. Thus by (3) and Table 1, we have

$$(|p_n|^2)^\wedge(\alpha_n) = (|p_{n-2}|^2 - |q_{n-2}|^2)^\wedge(\alpha_{n-2}) - (\overline{p}_{n-2}q_{n-2})^\wedge(\beta_{n-2})$$

$$= 2(|p_{n-2}|^2 - l_{n-1})^\wedge(\alpha_{n-2}) - (\overline{p}_{n-2}q_{n-2})^\wedge(\beta_{n-2}).$$

Now, the first equation of the auto-system follows, since $(l_{n-2})^\wedge(\alpha_{n-2}) = 0$. Because of the second term in that equation, we consider the square representation of $\overline{p}_n q_n$ with two steps backward. Our interest is the frequency $\beta_n = \frac{1}{3}(2^n - 1)$.

q_n: p_{n-1} = (p_{n-2}, q_{n-2}); $-q_{n-1}$ = ($-p_{n-2}$, q_{n-2})

	p_{n-2}	q_{n-2}	$-p_{n-2}$	q_{n-2}	
	$\|p_{n-2}\|^2$	$\overline{p}_{n-2}q_{n-2}$	$-\|p_{n-2}\|^2$	$\overline{p}_{n-2}q_{n-2}$	F_{n_0}
	$p_{n-2}\overline{q}_{n-2}$	$\|q_{n-2}\|^2$	$-p_{n-2}\overline{q}_{n-2}$	$\|q_{n-2}\|^2$	F_{n_1}
$\overline{p}_n$	$\|p_{n-2}\|^2$	$\overline{p}_{n-2}q_{n-2}$	$-\|p_{n-2}\|^2$	$\overline{p}_{n-2}q_{n-2}$	F_{n_2}
	$-p_{n-2}\overline{q}_{n-2}$	$-\|q_{n-2}\|^2$	$p_{n-2}\overline{q}_{n-2}$	$-\|q_{n-2}\|^2$	F_{n_3}

Since β_n lies in F_{n_2}, we have

$$(\overline{p}_n q_n)^\wedge(\beta_n) = (-|p_{n-2}|^2 + |q_{n-2}|^2)^\wedge(\alpha_{n-2}) + (2\overline{p}_{n-2}q_{n-2} - p_{n-2}\overline{q}_{n-2})^\wedge(\beta_{n-2}).$$

Therefore we have the second equation of the auto-system. Similarly, from the square representation of $p_n\overline{q}_n$, we easily get the third equation of auto-system, as desired.

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