IJST (2011) A4: 301-307
Iranian Journal of Science & Technology

http://www.shirazu.ac.ir/en

Direct and fixed point methods approach to the generalized Hyers–Ulam stability for a functional equation having monomials as solutions

H. Azadi Kenary*[1] and C. Park[2]

[1]*Department of Mathematics, College of Sciences, Yasouj University, Yasouj, Iran*
[2]*Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul, South Korea*
E-mails: azadi@mail.yu.ac.ir, baak@hanyang.ac.kr

Abstract

The main goal of this paper is the study of the generalized Hyers-Ulam stability of the following functional equation $f(2x+y)+f(2x-y)+(n-1)(n-2)(n-3)f(y)=2^{n-2}[f(x+y)+f(x-y)+6f(x)]$ where $n=1,2,3,4$, in non–Archimedean spaces, by using direct and fixed point methods.

Keywords: Hyers- Ulam stability; non -Archimedean normed space; p - adic field

1. Introduction

A classical question in the theory of functional equations is the following: *when is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?*

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940.

In the next year, D. H. Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.

In 1978, Th. M. Rassias proved a generalization of Hyers` theorem for additive mappings. The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations.

Theorem 1. ([3]): Let $f:E\to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon(\|x\|^p+\|y\|^p)$$

for all $x,y\in E$ where ε and p are constants with $\varepsilon>0$ and $0\le p<1$. Then the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x\in E$ and $L:E\to E'$ is the unique additive mapping which satisfies

$$\|f(x)-L(x)\|\le\frac{2\varepsilon}{2-2p}\|x\|^p$$

for all $x\in E$. Also, if for each $x\in E$ the function $f(tx)$ is continuous in $t\in R$, then L is linear.

In 1994, a generalization of Rassias' theorem was obtained by Gavruta [4] by replacing the bound $\varepsilon(\|x\|^p+\|y\|^p)$ with a general control function $\phi(x,y)$.

Let X and Y be vector spaces and let $f:X\to Y$ be a mapping for each $n=1,2,3$, consider the functional equation

$$f(2x+y)+f(2x-y)=2^{n-2}[f(x+y)+f(x-y)+6f(x)] \tag{1}$$

Also, consider the functional equation

$$f(2x+y)+f(2x-y)+6f(y)=4[f(x+y)+f(x-y)+6f(x)] \tag{2}$$

For $X=Y=R$, the monomial $f(x)=cx^n$ is a solution of (1) for each $n=1,2,3$ and the monomial

*Corresponding author

Received: 18 January 2011 / Accepted: 28 Jone 2011

$f(x)=cx^4$ is a solution of (2). It is easy to show that, a mapping $f:X\to Y$ satisfies (1) for $n=1$ if and only if it also satisfies the Cauchy functional equation $f(x+y)=f(x)+f(y)$.

For $n=2$, in [5] it was shown that the equation (1) is equivalent to the quadratic functional equation.

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

In 2002, Jun and Kim [6] solved the functional equation (1) for $n=3$. In 2003, Chung and Sahoo [7] introduced the quartic equation

$$\begin{aligned} &f(x+2y)+f(x-2y)+6f(x)\\ &=4\left[f(x+y)+f(x-y)+6f(y)\right]. \end{aligned} \tag{3}$$

In [8], the equation (2) was shown to be equivalent to the above equation.

In 1897, Hensel [9] introduced a normed space which does not have the Archimedean property.

In this paper, the generalized Hyers-Ulam stability of functional equation

$$\begin{aligned} &f(2x+y)+f(2x-y)+(n-1)(n-2)(n-3)f(y)=\\ &2^{n-2}\left[f(x+y)+f(x-y)+6f(x)\right]. \end{aligned} \tag{4}$$

will be investigated in non- Archimedean normed space.

In [8], Bae and Park obtained the general solution of the functional equation (4) and proved the generalized Hyers-Ulam stability of this functional equation in Banach * -algebra.

Remark 1. For convenience, for all x, y, let

$$\begin{aligned} &\Omega_f^n(x,y)=f(2x+y)+f(2x-y)+\\ &(n-1)(n-2)(n-3)f(y)-\\ &2^{n-2}\left[f(x+y)+f(x-y)+6f(x)\right] \end{aligned}.$$

2. Preliminaries

Definition 1. By a non-Archimedean field, we mean a field K equipped with a function (valuation): $K\to[0,\infty)$ such that for all $r,s\in K$, the following conditions hold:

$(i)\,|r|=0$ if and only if $r=0$

$(ii)\,|rs|=|r||s|$

$(iii)\,|r+s|\le\max\{|r|,|s|\}.$

Definition 2. Let X be a vector space over a scalar field K with a non–Archimedean non-trivial valuation. A function $\|.\|:X\to R$ is a non–Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$

(ii) $\|rx\|=|r|\|x\|(r\in K, x\in X)$

(iii) the strong triangle inequality (ultra-metric), namely

$\|x+y\|\le\max\{\|x\|,\|y\|\}. \qquad x,y\in X$

Then $(X,\|.\|)$ is called a non- Archimedean space.

Due to the fact that

$$\|x_n-x_m\|\le\max\{\|x_{j+1}-x_j\|; m\le j<n\} \qquad (n>m)$$

Definition 3. A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non–Archimedean space. By a complete non–Archimedean space, that is, one in which every Cauchy sequence is convergent.

The most important examples of non–Archimedean spaces are $p-$adic numbers. A key property of $p-$adic numbers is that they do not satisfy the Archimedean axiom: for all $x,y>0$, there exists an integer n such that $x<ny$.

Example 1. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x\in z$ such that $x=\frac{a}{b}p^{n_x}$ where a and b are integers not divisible by p. Then $|x|_p:=p^{-n_x}$ defines a non–Archimedean norm on Q. The completion of Q with respect to the metric $d(x,y)=|x-y|_p$ is denoted by Q_p which is called the $p-$adic number field. In fact, Q_p is the set of all formal series $x=\sum_{k\ge n_x}^{\infty}a_kp^k$ where $|a_k|\le p-1$ are integers. The addition and multiplication between any two elements of Q_p are defined naturally. The norm $\left|\sum_{k\ge n_x}^{\infty}a_kp^k\right|_p=p^{-n_x}$ is a non–Archimedean norm on Q_p and it makes Q_p a locally compact filed.

Definition4. Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:

$(i)\,d(x,y)=0$ if and only if $x=y$, for all $x,y\in X$;

$(ii)\,d(x,y)=d(y,x)$ for all $x,y\in X$;

$(iii)\ d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 2. Let (X, d) be a complete generalized metric space and $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$; either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

$(i)\ d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$;

(ii) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(iii) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

$(iv)\ d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

3. Non–Archimedean stability of functional equation (4): direct method

Throughout this section, we assume that G is an additive semi-group and X is a complete non–Archimedean space.

Remark 2. For convenience, for each $n = 1, 2, 3, 4,$ let

$$a_n = \frac{|(n-1)(n-2)(n-3)|}{|2 + (n-1)(n-2)(n-3) - 2^{n+1}|}$$

Theorem 3. For each $n = 1, 2, 3, 4,$ let $\varsigma_n : G^2 \to [0, +\infty)$ be a function such that

$$\lim_{m \to +\infty} \frac{\varsigma_n(2^m x, 2^m y)}{|2|^{mn}} = 0 \qquad (5)$$

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\Omega(x) = \lim_{m\to\infty} \max\left\{ \frac{\varsigma_n(2^k x, 0)}{|2|^{kn}}, \frac{a_n \varsigma_n(0,0)}{|2|^{kn}};\ 0 \le k \le m \right\} \qquad (6)$$

exists. Suppose that $f: G \to X$ be mapping satisfying the inequality

$$\left\| \Omega_f^n(x, y) \right\| \le \zeta_n(x, y) \qquad (7)$$

for all $x, y \in G$. Then the limit

$$\vartheta(x) := \lim_{m\to\infty} \frac{f(2^m x)}{2^{mn}}$$

exists for all $x \in G$ and $\vartheta(x): G \to X$ is a mapping satisfying

$$\| f(x) - \vartheta(x) \| \le \frac{1}{|2|} \Omega(x) \qquad (8)$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{m\to\infty} \max\left\{ \frac{\zeta_n(2^k x, 0)}{|2|^{kn}}, \frac{a_n \zeta_n(0,0)}{|2|^{kn}};\ j \le k < m + j \right\} = 0$$

Then $\vartheta(x)$ is the unique mapping satisfying (8).

Proof: Letting $x = y = 0$ in (7), we get

$$\| f(0) \| \le \frac{\zeta_n(0,0)}{|2 + (n-1)(n-2)(n-3) - 2^{n+1}|} \qquad (9)$$

Putting $y = 0$ in (7), we get

$$\left\| 2f(2x) + (n-1)((n-2)(n-3) f(0) - 2^{n+1} f(x) \right\| \le \zeta_n(x, 0) \qquad (10)$$

for all $x \in G$. By the above two inequalities, we have

$$\begin{aligned} \| 2f(2x) - 2^{n+1} f(x) \| &= \| 2f(2x) \pm (n-1)(n-2)(n-3) f(0) - 2^{n+1} f(x) \| \\ &\le \max\{ \| 2f(2x) + (n-1)(n-2)(n-3) f(0) - 2^{n+1} f(x) \| \\ &\quad , \| (n-1)(n-2)(n-3) f(0) \| \} \\ &\le \max\{ \zeta_n(x,0), a_n \zeta_n(0,0) \}. \end{aligned}$$

(11)

for all $x \in G$. So

$$\left\| \frac{f(2x)}{2^n} - f(x) \right\| \le \frac{1}{|2|^{n+1}} \max\{ \zeta_n(x, 0), a_n \zeta_n(0,0) \} \qquad (12)$$

for all $x \in G$. Replacing x by $2^m x$ and dividing both sides by $|2|^{mn}$ in (12), we get

$$\left\| \frac{f(2^{m+1} x)}{2^{(m+1)n}} - \frac{f(2^m x)}{2^{mn}} \right\| \le \frac{1}{|2|^{(m+1)n+1}} \max\{ \zeta_n(2^m x, 0), a_n \zeta_n(0,0) \} \qquad (13)$$

for all $x \in G$. It follows from (5) and (13) that sequence $\left\{ \frac{f(2^m x)}{2^{mn}} \right\}_{m \ge 1}$ is a Cauchy sequence in complete non-Archimedean space X, and so is convergent. Set

$$\vartheta(x) := \lim_{m\to\infty} \frac{f(2^m x)}{2^{mn}}$$

Using induction on m, one can easily see that

$$\left\| \frac{f(2^m x)}{2^{mn}} - f(x) \right\| \le \max\left\{ \frac{1}{|2|^{(k+1)n+1}} \zeta_n(2^k x, 0), \frac{1}{|2|^{(k+1)n+1}} a_n \zeta_n(0,0);\ 0 \le k \le m \right\}. \tag{14}$$

By taking m to approach infinity in (14) and using (6) one obtains (8). To show $\vartheta(x)$ satisfies (4), replace x and y by $2^m x$ and $2^m y$, respectively, in (7) and divide by 2^{mn}, we obtain

$$\begin{aligned}\frac{1}{|2|^{mn}} \| f(2^{m+1}x + 2^m y) + f(2^{m+1}x - 2^m y) + (n-1)(n-2)(n-3) f(2^m y) \\ - 2^{n-2}\left[f(2^m x + 2^m y) + f(2^m x - 2^m y) + 6 f(2^m x) \right] \| \\ \le \frac{1}{|2|^{mn}} \zeta_n(2^m x, 2^m y)\end{aligned}$$

for all $x, y \in G$ and all $m \in N$. Taking the limit as $m \to \infty$, we find that $\vartheta(x)$ satisfies (4) for all $x, y \in G$.

To prove the uniqueness of the mapping $\vartheta(x)$. Let η be another mapping satisfying (8), then for $x \in G$, we get

$$\begin{aligned}\|\vartheta(x) - \eta(x)\|_x &= \lim_{j\to\infty} |2|^{-jn} \|\vartheta(2^j x) - \eta(2^j x)\|_X \\ &\le \lim_{j\to\infty} |2|^{-jn} \max\left\{ \|\vartheta(2^j x) - f(2^j x)\|, \|\eta(2^j x) - f(2^j x)\| \right\} \\ &\le \frac{1}{|2|} \lim_{j\to\infty} \lim_{m\to\infty} \left\{ \frac{\zeta_n(2^k x, 0)}{|2|^{kn}}, \frac{a_n \zeta_n(0,0)}{|2|^{kn}};\ j \le k < m + j \right\} \\ &= 0.\end{aligned}$$

Therefore, $\vartheta = \eta$. This completes the proof.

Corollary 1. For each $n = 1,2,3,4,$ let $\eta : [0,\infty) \to [0,\infty)$ be a function satisfying

$$\eta(|2|t) \le \eta(2)\, \eta(t)\, (t \ge 0), \quad \eta(|2|) < |2|^n.$$

Let $\delta > 0$ and $f : G \to X$ be a mapping satisfying

$$\|\Omega_f^n(x, y)\|_X \le \delta\left(\eta(|x|) + \eta(|y|)\right)$$

for all $x, y \in G$. Then there exists a unique mapping $\vartheta : G \to X$ such that

$$\|f(x) - \vartheta(x)\|_X \le \frac{\delta \eta(|x|)}{|2|}$$

Proof: Defining $\zeta_n : G^2 \to [0,\infty)$ by $\zeta_n(x, y) := \delta\left(\eta(|x|) + \eta(|y|)\right)$, since $|2|^{-n} \eta(|2|) < 1$, then we obtain that for all $x, y \in G$

$$\lim_{m\to\infty} \frac{\zeta_n(2^m x, 2^m y)}{|2|^{mn}} \le \lim_{m\to\infty} \left(\frac{\eta(|2|)}{|2|^n} \right)^m \zeta_n(x, y) = 0$$

Also,

$$\begin{aligned}\Omega(x) &= \lim_{m\to\infty} \max\left\{ \frac{\zeta_n(2^k x, 0)}{|2|^{kn}}, \frac{a_n \zeta_n(0,0)}{|2|^{kn}};\ 0 \le k \le m \right\} \\ &= \max\left\{ \zeta_n(x, 0), a_n \zeta_n(0,0) \right\}\end{aligned}$$

and,

$$\lim_{j\to\infty} \lim_{m\to\infty} \left\{ \frac{\zeta_n(2^k x, 0)}{|2|^{kn}}, \frac{a_n \zeta_n(0,0)}{|2|^{kn}};\ j \le k < m + j \right\} = 0.$$

Applying Theorem 3, the desired result is obtained.

Theorem 4. For each $n = 1,2,3,4$, let $\varsigma_n : G^2 \to [0,+\infty)$ be a function such that

$$\lim_{m\to\infty} 2^{mn} \varsigma_n\left(\frac{x}{2^m}, \frac{y}{2^m} \right) = 0 \tag{15}$$

for all $x, y \in G$. Let for each $x \in G$, the limit

$$\Omega(x) = \lim_{m\to\infty} \max\left\{ |2|^{kn} \zeta_n\left(\frac{x}{2^{k+1}}, 0 \right), |2|^{kn} a_n \zeta_n(0,0);\ 0 \le k < m \right\} \tag{16}$$

exists. Suppose that $f : G \to X$ be a mapping satisfying the inequality

$$\|\Omega_f^n(x, y)\| \le \zeta_n(x, y) \tag{17}$$

for all $x, y \in G$. Then the limit

$$\vartheta(x) := \lim_{m\to\infty} 2^{mn} f\left(\frac{x}{2^m} \right)$$

exists for all $x \in G$ and $\vartheta(x) : G \to X$ is a mapping satisfying

$$\|f(x) - \vartheta(x)\| \le \frac{1}{|2|} \Omega(x) \tag{18}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{m\to\infty} \max\left\{ |2|^{kn} \zeta_n\left(\frac{x}{2^{k+1}}, 0 \right), |2|^{kn} a_n \zeta_n(0,0);\ j \le k < m + j \right\} = 0$$

Then $\vartheta(x)$ is the unique mapping satisfying (18).

Proof: By (12), we have

$$\left\|f(2x)-2^{n}f(x)\right\|\le\frac{1}{|2|}\max\left\{\zeta_{n}(x,0),a_{n}\zeta_{n}(0,0)\right\} \qquad (19)$$

Replacing x by $\frac{x}{2^{m}}$ in (19), we obtain

$$\left\|2^{(m-1)n}f\left(\frac{x}{2^{m-1}}\right)-2^{mn}f\left(\frac{x}{2^{m}}\right)\right\|_{x}\le|2|^{n(m-1)-1}\max\left\{\zeta_{n}\left(\frac{x}{2^{m}},0\right),a_{n}\zeta_{n}(0,0)\right\} \qquad (20)$$

for all $x\in G$ and all non-negative integer m. It follows from (15) and (20) that the sequence $\left\{2^{mn}f\left(\frac{x}{2^{m}}\right)\right\}_{m=1}^{\infty}$ is a Cauchy in X for all $x\in G$. Since X is complete, the sequence $\left\{2^{mn}f\left(\frac{x}{2^{m}}\right)\right\}_{m=1}^{\infty}$ converges for all $x\in G$. On the other hand, it follows from (20) that

$$\begin{aligned}\left\|2^{pn}f\left(\frac{x}{2^{p}}\right)-2^{qn}f\left(\frac{x}{2^{q}}\right)\right\|&=\left\|\sum_{k=p}^{q-1}2^{(k+1)n}f\left(\frac{x}{2^{k+1}}\right)-2^{kn}f\left(\frac{x}{2^{k}}\right)\right\|\\&\le\max\left\{\left\|2^{(k+1)n}f\left(\frac{x}{2^{k+1}}\right)-2^{kn}f\left(\frac{x}{2^{k}}\right)\right\|;p\le k<q-1\right\}\\&\le\frac{1}{|2|}\max\left\{|2|^{kn}\zeta_{n}\left(\frac{x}{2^{k+1}},0\right),|2|^{kn}a_{n}\zeta_{n}(0,0);p\le k<q\right\},\end{aligned}$$

for all $x\in G$ and all non–negative integers p,q with $q>p\ge 0$. Letting $p=0$ and passing the limit $q\to\infty$ in the last inequality and using (16), we obtain (18).

The rest of the proof is similar to the proof of Theorem 3.

Corollary 2. For each $n=1,2,3,4,$ let $\eta:[0,\infty)\to[0,\infty)$ be a function satisfying

$$\eta\left(|2|^{-1}t\right)\le\eta\left(|2|^{-1}\right)\eta(t)\ (t\ge 0),\quad \eta\left(|2|^{-1}\right)<|2|^{-n}$$

Let $\delta>0$ and $f:G\to X$ is a mapping satisfying

$$\left\|\Omega_{f}^{n}(x,y)\right\|_{X}\le\delta\left(\mu(|x|)+\mu(|y|)\right)$$

for all $x,y\in G$. Then there is a unique mapping $\vartheta:G\to X$ such that

$$\left\|f(x)-\vartheta(x)\right\|_{x}\le\frac{\delta\eta(|2|)}{|2|^{n+1}}$$

Proof. Defining $\zeta_{n}:G^{2}\to[0,\infty)$ by $\zeta_{n}(x,y):=\delta\left(\mu(|x|)+\mu(|y|)\right)$, then we obtain

$$\lim_{m\to\infty}2^{mn}\zeta_{n}\left(\frac{x}{2^{m}},\frac{y}{2^{m}}\right)=0.$$

Also,

$$\begin{aligned}\Omega(x)&=\lim_{m\to\infty}\max\left\{|2|^{kn}\zeta_{n}\left(\frac{x}{2^{k+1}},0\right),|2|^{kn}a_{n}\zeta_{n}(0,0);0\le k<m\right\}\\&=\varsigma_{n}\left(\frac{x}{2},0\right)\\&\le|2|^{-n}\delta\mu(|x|)\end{aligned}$$

And

$$\lim_{j\to\infty}\lim_{m\to\infty}\max\left\{|2|^{kn}\zeta_{n}\left(\frac{x}{2^{k+1}},0\right),|2^{kn}|a_{n}\zeta_{n}(0,0);j\le k<m+j\right\}=0.$$

4. Non- Archimedea stability of functional equation (4): fixed point method

Throughout this section, assume that X is a non-Archimedean normed vector space and that Y is a non- Archimedean Banach space. In the rest of the present paper, let $|2|\ne 1$.

Theorem 5. For $n=1,2,3,4,$ $\zeta_{n}:X\to[0,\infty)$ be a function such that there exists an $L<1$ with

$$\zeta_{n}(2x,2y)\le|2|^{n}L\zeta_{n}(x,y) \qquad (21)$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying

$$\left\|\Omega_{f}^{n}(x,y)\right\|\le\zeta_{n}(x,y) \qquad (22)$$

for all $x,y\in X$. Then there is a unique mapping $C:X\to Y$ such that

$$\left\|f(x)-C(x)\right\|\le\frac{\max\{\zeta_{n}(x,0),a_{n}\zeta_{n}(0,0)\}}{|2|^{n+1}(1-L)} \qquad (23)$$

Proof: By (12), we have

$$\left\|2f(2x)-2^{n+1}f(x)\right\|\le\max\{\zeta_{n}(x,0),\ a_{n}\zeta_{n}(0,0)\}. \qquad (24)$$

for all $x\in X$. Consider the set

$$S:=\{g:X\to Y\}$$

and the generalized metric d in S defined by

$$d(f,g)=\inf\left\{\mu\in R^{+}:\left\|g(x)-h(x)\right\|\le\mu\max\left\{\zeta_{n}(x,0),a_{n}\zeta_{n}(0,0)\right\},\forall x\in X\right\},$$

where $\inf\varphi=+\infty$. It is easy to show that (S,d) is complete. Now, we consider a linear mapping $J:S\to S$ such that

$$Jh(x):=\frac{1}{2^{n}}h(2x)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g,h)=\varepsilon$. Then

$$\|g(x)-h(x)\| \le \varepsilon \max\{\zeta_n(x,0), a_n\zeta_n(0,0\}$$

for all $x \in X$. So

$$\|Jg(x)-Jh(x)\| = \left\|\frac{1}{2^n}g(2x)-\frac{1}{2^n}h(2x)\right\|$$
$$\le \frac{\varepsilon}{|2|^n}\max\{\zeta_n(2x,0), a_n\zeta_n(0,0)\}$$
$$\le \frac{1}{|2|^n}\varepsilon|2|^n L\max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}$$

for $x \in X$. Thus $d(g,h)=\varepsilon$ implies that $d(Jg,Jh)\le L\varepsilon$, this means that $d(Jg,Jh)\le Ld(g,h)$ f or all $g,h \in S$. It follows from (24) that $d(f,Jf)\le \frac{1}{|2|^{n+1}}$.

By Theorem 2, there exists a mapping $C: X \to Y$ satisfying the following :

(i) C is a fixed point of J, that is, for all $x \in X$,

$$C(2x)=2^n C(x) \tag{25}$$

(ii) the mapping C is a unique fixed point of J in the set $\Omega=\{h \in S: d(g,h)<\infty\}$. This implies that C is a unique mapping satisfying (25) such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x)-C(x)\| \le \mu \max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}$, for all $x \in X$.

(iii) $d(J^m f, C)\to 0$ as $m\to\infty$. This implies the equality, $\lim_{m\to\infty}\frac{f(2^m x)}{2^{mn}}=C(x)$, for all $x \in X$.

(iv) $d(f,C)\le \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f,C)\le \frac{1}{|2|^{n+1}(1-L)}$.

This implies that the inequality (23) holds.

Corollary 3. Let $\theta \ge 0$ and p be a real number with $0<p<1$. Let $f: X \to Y$ be a mapping satisfying

$$\|\Omega_f^n(x,y)\| \le \theta\left(\|x\|^p+\|y\|^p\right)$$

for all $x, y \in X$. Then, the limit $C(x)=\lim_{m\to\infty}\frac{f(2^m x)}{2^{mn}}$ exists for all $x \in X$ and $C: X \to Y$ is a unique mapping such that

$$\|f(x)-C(x)\| \le \frac{|2|^{np}\theta\|x\|^p}{|2|^{n+1}\left(|2|^{np}-|2|^n\right)}$$

for all $x \in X$.

Proof: The proof follows from Theorem 5 by taking $\zeta_n(x,y)=\theta\left(\|x\|^p+\|y\|^p\right)$, for all $x, y \in X$. In fact, if we choose $L=\frac{|2|^n}{|2|^{np}}$ we get the desired result.

Theorem 6. For $n=1,2,3,4$, let $\zeta_n: X \to [0,\infty)$ be a function such that there exists an $L<1$ with

$$\zeta_n(x,y)\le \frac{L}{|2|^n}\zeta_n(2x,2y)$$

for all $x, y \in X$. Let $f: X \to Y$ be a mapping satisfying

$$\|\Omega_f^n(x,y)\| \le \zeta_n(x,y)$$

for all $x, y \in X$. Then there is a unique mapping $C: X \to Y$ such that

$$\|f(x)-C(x)\| \le \frac{L\max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}}{|2|^{n+1}(1-L)} \tag{26}$$

Proof: By (11), we have

$$\left\|f(x)-2^n f\left(\frac{x}{2}\right)\right\| \le \frac{1}{|2|}\max\left\{\zeta_n\left(\frac{x}{2},0\right), a_n\zeta_n(0,0)\right\} \tag{27}$$

for all $x \in X$. Let (S,d) be the generalized metric space defined as in the proof of Theorem 5, we consider a linear mapping $J: S \to S$ such that $Jh(x):=2^n h\left(\frac{x}{2}\right)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h)=\varepsilon$. Then $\|g(x)-h(x)\| \le \varepsilon\max\{\zeta_n(x,0), a_n\zeta_n(0,0\}$ for all $x \in X$. So

$$\|Jg(x) - Jh(x)\| = \left\|2^n g\left(\frac{x}{2}\right) - 2^n h\left(\frac{x}{2}\right)\right\|$$
$$\le |2|^n \varepsilon \max\left\{\zeta_n\left(\frac{x}{2}, 0\right), a_n\zeta_n(0,0)\right\}$$
$$\le |2|^n \varepsilon \frac{L}{|2|^n} \max\left\{\zeta_n(x,0), a_n\zeta_n(0,0)\right\}$$

for all $x \in X$. Thus $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$, this means that $d(Jg, Jh) \le Ld(g,h)$ for all $g, h \in S$. It follows from (27) that $d(f, Jf) \le \frac{L}{|2|^{n+1}}$.

By Theorem 2, there exists a mapping $C: X \to Y$ satisfying the following:

(a) C is a fixed point of J, that is

$$C\left(\frac{x}{2}\right) = \frac{1}{2^n} C(x) \tag{28}$$

for all $x \in X$.

(b) The mapping C is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies C is a unique mapping satisfying (28) such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x) - C(x)\| \le \mu \max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}$, for all $x \in X$.

(c) $d(J^m f, C) \to 0 \text{ as } m \to \infty$, this implies the equality $\lim_{n\to\infty} 2^{mn} f\left(\frac{x}{2^m}\right) = C(x)$ for all $x \in X$.

(d) $d(f,C) \le \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f,C) \le \frac{L}{|2|^{n+1}(1-L)}$.

This implies that the inequality (26) holds.

The rest of the proof is similar to the proof of Theorem 5.

Corollary 4. Let $\theta \ge 0$ and p be a real number with $p > 1$. Let $f: X \to Y$ be a mapping satisfying

$$\|\Omega_f^n(x,y)\| \le \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then, the limit $C(x) = \lim_{m\to\infty} 2^{mn} f\left(\frac{x}{2^m}\right)$ exists for all $x \in X$, and $C: X \to Y$ is a mapping such that

$$\|f(x) - C(x)\| \le \frac{|2|^{np}\,\theta\|x\|^p}{|2|^{n+1}\left(|2|^n - |2|^{np}\right)}$$

for all $x \in X$.

Proof: The proof follows from Theorem 6 by taking $\zeta_n(x,y) = \theta\left(\|x\|^p + \|y\|^p\right)$ for all $x, y \in X$. In fact, if we choose $L = \frac{|2|^{np}}{|2|^n}$, we get the desired result.

Acknowledgement

The second author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2009-0070788).

References

[1] Ulam, S. M. (1964). *Problems in Modern Mathematics, Science Editions*. John Wiley and Sons.

[2] Hyers, D. H. (1941). On the stability of the linear functional equation. *Proc. Nat. Acad. Sci.*, U. S. A., 27, 222-224.

[3] Rassias, Th. M. (1978). On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc. 72*(2), 297-300.

[4] Gavruta, P. (1994). A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl. 184* (3), 431-436.

[5] Bae J. -H. & Park, W. -G. (2007). On a cubic and a Jensen-quadratic equations. *Abstract Appl. Anal.*, 2007, Article ID 45179.

[6] Jun K. & Kim, H. (2007). On the Hyers-Ulam-Rassias stability problem for approximately k-additive mappings and functional inequalities. *Math. Inequal. Appl., 10*(4), 895-908.

[7] Chung J. K. & Sahoo, P. K. (2003). On the general solution of a quartic functional equation. *Bull. Korean Math. Soc., 40*, 565-576.

[8] Bae J. -H. & Park, W. -G. (2010). A functional equation having monomials as solutions. *Appl. Math. Comp., 216*, 87-94.

[9] Hensel, K. (1897). Ubereine news Begrundung der Theorie der algebraischen Zahlen. *Jahresber. Deutsch. Math. Verein 6*, 83-88.

.