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Computing of eigenvalues of sturm-liouville problems with eigenparameter dependent boundary conditions

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Abstract

The purpose of this article is to use the classical sampling theorem, WKS sampling theorem, to derive approximate values of the eigenvalues of the Sturm-Liouville problems with eigenparameter in the boundary conditions. Error analysis is used to give estimates of the associated error. Higher order approximations are also drived, which lead to more complicated computations. We give some examples and make companions with existing results.

Keywords: Eigenvalue problem with eigenparameter in the boundary conditions; sinc methods; computing eigenvalues

1. Introduction

Throughout this paper we consider the differential equation

$$\ell(y) := -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0,1], \tag{1}$$

where $q(\cdot)$ is assumed to be real valued and continuous on $[0,1]$ and $\lambda \in \mathbb{C}$ is an eigenvalue parameter, see [1]. We also consider the following two boundary conditions

$$a_1 y(0) + a_2 y'(0) = \lambda(a_1' y(0) + a_2' y'(0)), \tag{2}$$

$$b_1 y(1) + b_2 y'(1) = \lambda(b_1' y(1) + b_2' y'(1)), \tag{3}$$

where $a_i, a_i', b_i, b_i' \in \mathbb{R}$, $i = 1,2$ and

$$\det\begin{pmatrix} a_1 & a_1' \\ a_2 & a_2' \end{pmatrix} > 0, \quad det\begin{pmatrix} b_1' & b_1 \\ b_2' & b_2 \end{pmatrix} > 0. \tag{4}$$

Let $\phi_\lambda(\cdot)$ be a solution of (1) satisfying the following initial condition

$$\phi_\lambda(0) = a_2 - a_2'\lambda, \quad \phi_\lambda'(0) = a_1'\lambda - a_1. \tag{5}$$

The eigenvalues of the problem (1)--(3) are the zeros of the function

$$\Delta(\lambda) := (b'_1\lambda - b_1)\phi_\lambda(1) + (b'_2\lambda - b_2)\phi_\lambda'(1). \tag{6}$$

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The function $\Delta(\lambda)$ is an entire function of λ of order one and type one. These zeros are real and simple.

The famous classical sampling theorem (WKS) of Whittaker [2], Kotel'nikov [3] and Shannon [4] say that if $f(t) \in PW_\sigma^2$, that is, if $f(t)$ is entire in t of exponential type σ, $\sigma > 0$, which belongs to $L^2(\mathbb{R})$ where restricted to $\mathbb{R}$, then $f(t)$ can be reconstructed via, see also [5],

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\sigma}\right) \operatorname{sinc}(\sigma t - n\pi), \quad t \in \mathbb{C}. \tag{7}$$

Series (7) converges absolutely on $\mathbb{C}$ and uniformly on $\mathbb{R}$ and on compact subsets of $\mathbb{C}$, see [2, 3, 4]. The points $\left\{\frac{n\pi}{\sigma}\right\}_{n\in\mathbb{Z}}$ are called the sampling points and the functions

$$\operatorname{sinc}(\sigma t - n\pi) := \begin{cases} \frac{\sin(\sigma t - n\pi)}{(\sigma t - n\pi)}, & t \neq \frac{n\pi}{\sigma}, \\ 1, & t = \frac{n\pi}{\sigma}, \end{cases} \tag{8}$$

are called the sampling, reconstructing, functions. The series (7) is used extensively in approximating solutions and eigenvalues of boundary value problems, see [6, 7]. Since (7) involves the sine function, then (7)-based methods extensively are called sinc techniques, see [7]. Sinc techniques have been employed in computing eigenvalues of some boundary-value problems, see e.g. [8-18]. The associated error analysis in these articles is based only on the truncation error related to the WKS'theorem. This error is established by Jaggerman, [19], as follows. For $N \in \mathbb{N}$ and $f(t) \in PW_\sigma^2$, let $f_N(t)$ be the truncated cardinal series

$$f_N(t) := \sum_{n=-N}^{N} f\left(\frac{n\pi}{\sigma}\right) \operatorname{sinc}(\sigma t - n\pi). \tag{9}$$

Jagerman proved that if $\lambda \in \mathbb{R}$ and, in addition $\lambda^k f(\lambda) \in L^2(\mathbb{R})$, for some integer $k > 0$, then for $N \in \mathbb{N}$, $|\lambda| < N\pi/\sigma$, we have

$$|f(\lambda) - f_N(\lambda)| \le \frac{E_k(f)|\sin\sigma\lambda|}{\pi(\pi/\sigma)^k\sqrt{1-4^{-k}}}\left(\frac{1}{\sqrt{N\pi/\sigma-\lambda}} + \frac{1}{\sqrt{N\pi/\sigma+\lambda}}\right)\frac{1}{(N+1)^k}, \quad \lambda \in \mathbb{R}, \tag{10}$$

where

$$E_k(f) := \left\{\int_{-\infty}^{\infty} \lambda^{2k}|f(\lambda)|^2\, dt\right\}^{\frac{1}{2}} \tag{11}$$

2. Preliminaries

Consider the eigenvalue problem studied in the above section, with $\lambda = \mu^2$

$$-y''(x,\mu) + q(x)y(x,\mu) = \mu^2 y(x,\mu), \quad 0 \le x \le 1, \tag{12}$$

$$a_1 y(0,\mu) + a_2 y'(0,\mu) = \mu^2[a'_1 y(0,\mu) + a'_2 y'(0,\mu)], \tag{13}$$

$$b_1 y(1,\mu) + b_2 y'(1,\mu) = \mu^2[b'_1 y(1,\mu) + b'_2 y'(1,\mu)]. \tag{14}$$

Let $y(\cdot,\mu)$ denote the solution of (12) satisfying the following initial conditions

$$y(0,\mu) = a_2 - a'_2\mu^2, \quad y'(0,\mu) = a'_1\mu^2 - a_1. \tag{15}$$

Thus $y(\cdot,\mu)$ satisfies the boundary condition (13). The eigenvalues of the problem (12)--(14) are the zeros of the function

$$\Delta(\mu) := (b'_1\mu^2 - b_1)y(1,\mu) + (b'_2\mu^2 - b_2)y'(1,\mu). \tag{16}$$

The function $\Delta(\mu)$ is an entire function of μ of order one and type one. These zeros are real and simple. We aim to approximate $\Delta(\mu)$ and hence its zeros, i.e. the eigenvalues by use of the sampling theorem. The idea is to split $\Delta(\mu)$ into two parts, one is known and the other is unknown, but lies in a Paley-Wiener space. We approximate the unknown part to get the approximate $\Delta(\mu)$ and then compute the approximate zeros. Using the method of variation of constants, the solution $y(x,\mu)$ satisfies Volterra integral equation

$$y(x,\mu) = (a_2 - a'_2\mu^2)\cos\mu x - (a_1 - a'_1\mu^2)\frac{\sin\mu x}{\mu} + T[y](x,\mu), \tag{17}$$

where T is the Volterra operator defined by

$$T[y](x,\mu) = \int_0^x \frac{\sin\mu(x-t)}{\mu}\, q(t)y(t,\mu)dt. \tag{18}$$

Differentiating (17), we get

$$y'(x,\mu) = (a'_2\mu^2 - a_2)\mu\sin\mu x + (a'_1\mu^2 - a_1)\cos\mu x + \tilde{T}[y](x,\mu), \tag{19}$$

where $\tilde{T}$ is the Volterra operator

$$\tilde{T}[y](x,\mu) = \int_0^x \cos\mu(x-t)\, q(t)y(t,\mu)dt. \tag{20}$$

Define $f(\cdot,\mu)$ and $g(\cdot,\mu)$ to be

$$f(x,\mu) := T[y](x,\mu), \quad g(x,\mu) := \tilde{T}[y](x,\mu). \tag{21}$$

In the following, we shall make use of the estimates [20],

$$|\cos z| \le e^{|Imz|}, \qquad \left|\frac{\sin z}{z}\right| \le \frac{c_0}{1+|z|}e^{|Imz|}, \tag{22}$$

where c_0 is some constant (we may take $c_0 \cong 1.72$ cf. [20]). For convenience, we define the constants

$$\tau := \int_0^1 |q(t)|\, dt, \quad c_1 := |a_2| + c_0|a_1|, \quad c_2 := |a'_2| + c_0|a'_1|, \quad c_3 := c_0\tau, \quad c_4 := \exp c_3, \quad c_5 := \max\{c_1, c_2, |b_1| + |b_2|\tau, |b'_1| + |b'_2|\tau\}. \tag{23}$$

From (17) and (21), we get

$$f(x,\mu) = \int_0^x \frac{\sin\mu(x-t)}{\mu} q(t)\left[(a_2 - a'_2\mu^2)\cos\mu t - (a_1 - a'_1\mu^2)\frac{\sin\mu t}{\mu}\right] dt + \int_0^x \frac{\sin\mu(x-t)}{\mu} q(t)f(t,\mu)\, dt. \tag{24}$$

Lemma 2.1. *For* $0 \le x \le 1$, $\mu \in \mathbb{C}$, the following estimates hold

$$f(x,\mu) \le \frac{c_3c_4(c_1 + c_2|\mu|^2)}{1+|\mu|}e^{|Im\mu|x} \tag{25}$$

and

$$g(x,\mu) \le \frac{\tau c_3c_4(c_1 + c_2|\mu|^2)}{1+|\mu|}e^{|Im\mu|x}. \tag{26}$$

Proof: We divide $f(\cdot,\mu)$ into two parts, $f_1(\cdot,\mu)$ and $f_2(\cdot,\mu)$, estimate each of them. Indeed, for $x \in [0,1]$ and $\mu \in \mathbb{C}$ we have

$$\begin{aligned} |f_1(x,\mu)| &= \left|\int_0^x \frac{\sin\mu(x-t)}{\mu}\, q(t)\left[(a_2 - a'_2\mu^2)\cos\mu t - (a_1 - a'_1\mu^2)\frac{\sin\mu t}{\mu}\right] dt\right| \\ &\le e^{|Im\mu|x}\int_0^x |q(t)|\frac{c_0(x-t)}{1+|\mu|(x-t)}\left[|a_2| + |a'_2||\mu|^2 + (|a_1| + |a'_1||\mu|^2)\frac{c_0 t}{1+|\mu|t}\right] dt \\ &\le e^{|Im\mu|x}\frac{c_0 x}{1+|\mu|x}\int_0^x |q(t)|[|a_2| + |a'_2||\mu|^2 + (|a_1| + |a'_1||\mu|^2)c_0 t]\, dt \\ &\le e^{|Im\mu|x}\frac{c_0}{1+|\mu|}\int_0^1 |q(t)|[|a_2| + |a'_2||\mu|^2 + (|a_1| + |a'_1||\mu|^2)c_0 t]\, dt. \end{aligned} \tag{27}$$

Moreover, $0 \le x \le 1$, $\mu \in \mathbb{C}$,

$$\begin{aligned} |f_2(x,\mu)| &= \left|\int_0^x \frac{\sin\mu(x-t)}{\mu}\, q(t) f(t,\mu)\, dt\right| \\ &\le \int_0^x \frac{c_0(x-t)}{1+|\mu|(x-t)} e^{|Im\mu|(x-t)}\, |q(t)||f(t,\mu)|\, dt \\ &\le c_0 e^{|Im\mu|x}\int_0^x e^{-|Im\mu|t}\, |q(t)||f(t,\mu)|\, dt. \end{aligned} \tag{28}$$

Combining (27) and (28), we obtain, $0 \le x \le 1$, $\mu \in \mathbb{C}$

$$\begin{aligned} |f(x,\mu)| &\le e^{|Im\mu|x}\frac{c_0}{1+|\mu|}\int_0^1 |q(t)|[|a_2| + |a'_2||\mu|^2 + (|a_1| + |a'_1||\mu|^2)c_0 t]\, dt \\ &\quad + c_0 e^{|Im\mu|x}\int_0^x e^{-|Im\mu|t}\, |q(t)||f(t,\mu)|\, dt. \end{aligned} \tag{29}$$

Applying Gronwall's inequality, cf. e.g. [21, p. 51], yields, $\mu \in \mathbb{C}$

$$\begin{aligned} e^{-|Im\mu|x}|f(x,\mu)| &\le \left[\frac{c_0}{1+|\mu|}\int_0^1 |q(t)|[|a_2| + |a'_2||\mu|^2 + (|a_1| + |a'_1||\mu|^2)c_0 t]\, dt\right]\exp\left(c_0\int_0^x |q(t)|\, dt\right) \\ &\le \left[\frac{c_0}{1+|\mu|}\int_0^1 |q(t)|[|a_2| + |a'_2||\mu|^2 + (|a_1| + |a'_1||\mu|^2)c_0 t]\, dt\right]\exp\left(c_0\int_0^1 |q(t)|\, dt\right), \end{aligned}$$

from which we get

$$\begin{aligned} |f(x,\mu)| &\le e^{|Im\mu|x}\left[\frac{c_0[|a_2| + |a'_2||\mu|^2 + (|a_1| + |a'_1||\mu|^2)c_0]}{1+|\mu|}\int_0^1 |q(t)|\, dt\right]\exp\left(c_0\int_0^1 |q(t)|\, dt\right) \\ &= \frac{c_3 c_4 (c_1 + c_2|\mu|^2)}{1+|\mu|} e^{|Im\mu|x}. \end{aligned}$$

Then from (21) and (25), we obtain the estimate (26).

3. The method and error estimates

This section contains the method and the associated error analysis. First we decompose $\Delta(\mu)$ into two parts, one is known and the other is unknown. Indeed, let

$$\Delta(\mu) = G(\mu) + S(\mu) \tag{30}$$

where $G(\mu)$ is known part

$$\begin{aligned} G(\mu) = {} & (b'_1\mu^2 - b_1)\left[(a_2 - a'_2\mu^2)\cos\mu x - (a_1 - a'_1\mu^2)\frac{\sin\mu x}{\mu}\right] \\ & + (b'_2\mu^2 - b_2)[(a'_2\mu^2 - a_2)\mu\sin\mu + (a'_1\mu^2 - a_1)\cos\mu], \end{aligned} \tag{31}$$

and $S(\mu)$ is unknown part

$$S(\mu) = (b'_1\mu^2 - b_1)f(1,\mu) + (b'_2\mu^2 - b_2)g(1,\mu). \tag{32}$$

Then, from Lemma 2.1 we have the following lemma.

Lemma 3.1. The function $S(\mu)$ is entire in μ and the following estimate holds

$$|S(\mu)| \le \frac{c_3 c_4 c_5 (1+|\mu|^2)^2}{1+|\mu|} e^{|Im\mu|}. \tag{33}$$

Proof: Since

$$S(\mu) \le (|b'_1||\mu|^2 + |b_1|)|f(1,\mu)| + (|b'_2||\mu|^2 + |b_2|)|g(1,\mu)|,$$

then from (25) and (26) we get (33).

Let $\theta \in (0,1)$ and $m \in \mathbb{Z}^+$, $m \ge 4$ be fixed. Let $\mathcal{F}_{\theta,m}(\lambda)$ be the function

$$\mathcal{F}_{\theta,m}(\mu) := \left(\frac{\sin\theta\mu}{\theta\mu}\right)^m S(\mu), \quad \lambda \in \mathbb{C}. \tag{34}$$

The number θ will be specified later. The number 4 is the smallest positive integer that suites our investigation, as is seen in the next lemma.

Lemma 3.2. $\mathcal{F}_{\theta,m}(\mu)$ is an entire function of μ which satisfies the estimates

$$\left|\mathcal{F}_{\theta,m}(\mu)\right| \le \frac{c_3 c_4 c_5 c_0^m (1+|\mu|^2)^2}{(1+\theta|\mu|)^{m+1}} e^{|Im\mu|(1+m\theta)}. \tag{35}$$

Moreover, $\mu^{m-4}\mathcal{F}_{\theta,m}(\mu) \in L^2(\mathbb{R})$ and

$$E_{m-4}(\mathcal{F}_{\theta,m}) = \sqrt{\int_{-\infty}^{\infty} |\mu^{m-4}\mathcal{F}_{\theta,m}(\mu)|^2\, d\mu} \le \sqrt{2}\, c_3 c_4 c_5 c_0^m \nu_0, \tag{36}$$

where

$$\nu_0 := \sqrt{\frac{(m(2m-1) + 4\theta^2)\Gamma[2m+2] + 144m(4m^2-1)\theta^4(280\theta^4\Gamma[2m-7] + 20\theta^2\Gamma[2m-5] + \Gamma[2m-3])}{m(4m^2-1)\Gamma[2m+2]\theta^{2m+1}}}.$$

Proof: Since $S(\mu)$ is entire, then $\mathcal{F}_{\theta,m}(\mu)$ is also entire in λ. Combining the estimates $\left|\frac{\sin z}{z}\right| \le \frac{c_0}{1+|z|} e^{|Imz|}$ and (33), we obtain

$$\left|\mathcal{F}_{\theta,m}(\mu)\right| \le \left(\frac{c_0}{1+\theta|\mu|}\right)^m e^{|Im\mu|m\theta} \cdot \frac{c_3 c_4 c_5 (1+|\mu|^2)^2}{1+|\mu|} e^{|Im\mu|}, \quad \mu \in \mathbb{C}, \tag{37}$$

leading to (35). Therefore

$$\left|\mu^{m-4}\mathcal{F}_{\theta,m}(\mu)\right| \le \frac{c_3 c_4 c_5 c_0^m |\mu|^{m-4}(1+|\mu|^2)^2}{(1+\theta|\mu|)^{m+1}}, \mu \in \mathbb{R}, \tag{38}$$

i.e. $\mu^{m-4}\mathcal{F}_{\theta,m}(\mu) \in L^2(\mathbb{R})$. Moreover

$$\int_{-\infty}^{\infty} |\mu^{m-1}\mathcal{F}_{\theta,m}(\mu)|^2\, d\mu \le c_3^2 c_4^2 c_5^2 c_0^{2m}\int_{-\infty}^{\infty} \frac{|\mu|^{2m-8}(1+|\mu|^2)^4}{(1+\theta|\mu|)^{2m+2}}\, d\mu = 2c_3^2 c_4^2 c_5^2 c_0^{2m}\nu_0^2. \tag{39}$$

What we have just proved is that $\mathcal{F}_{\theta,m}(\mu)$ belongs to the Paley-Wiener space PW_σ^2 with $\sigma = 1 + m\theta$. Hence, $\mathcal{F}_{\theta,m}(\mu)$ can be recovered from its values at the points $\mu_n = \frac{n\pi}{\sigma}$, $n \in \mathbb{Z}$ via the sampling expansion

$$\mathcal{F}_{\theta,m}(\mu) := \sum_{n=-\infty}^{\infty} \mathcal{F}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \text{sinc}\,(\sigma\mu - n\pi). \quad (40)$$

Let $N \in \mathbb{Z}^+$, $N > m$ and approximate $\mathcal{F}_{\theta,m}(\mu)$ by its truncated series $\mathcal{F}_{\theta,m,N}(\mu)$, where

$$\mathcal{F}_{\theta,m,N}(\mu) := \sum_{n=-N}^{N} \mathcal{F}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \text{sinc}\,(\sigma\mu - n\pi). \quad (41)$$

Since $\mu^{m-4}\mathcal{F}_{\theta,m}(\mu) \in L^2(\mathbb{R})$, the truncation error is given for $|\mu| < \frac{N\pi}{\sigma}$ by

$$\left|\mathcal{F}_{\theta,m}(\mu) - \mathcal{F}_{\theta,m,N}(\mu)\right| \le T_N(\mu), \quad (42)$$

where

$$T_N(\mu) := \frac{E_{m-4}(\mathcal{F}_{\theta,m})}{\sqrt{1-4^{-m+4}}\,\pi(\pi/\sigma)^{m-4}} \frac{|\sin\sigma\mu|}{(N+1)^{m-4}} \left[\frac{1}{\sqrt{\frac{N\pi}{\sigma}-\mu}} + \frac{1}{\sqrt{\frac{N\pi}{\sigma}+\mu}}\right]. \quad (43)$$

Let $\Delta_N(\mu) := G(\mu) + \left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu)$. Then (42) implies

$$|\Delta(\mu) - \Delta_N(\mu)| \le \left|\frac{\sin\theta\mu}{\theta\mu}\right|^{-m} T_N(\mu), \quad |\mu| < \frac{N\pi}{\sigma} \quad (44)$$

and θ is chosen sufficiently small for which $|\theta\mu| < \pi$.
Let μ^{*2} be an eigenvalue, that is $\Delta(\mu^*) = G(\mu^*) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m}(\mu^*) = 0$. Then it follows that

$$G(\mu^*) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu^*) = \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu^*) - \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m}(\mu^*)$$

and so

$$\left|G(\mu^*) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu^*)\right| \le \left|\frac{\sin\theta\mu^*}{\theta\mu^*}\right|^{-m} T_N(\mu^*).$$

Since $G(\mu^*) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu^*)$ is given and, $\left|\frac{\sin\theta\mu^*}{\theta\mu^*}\right|^{-m} T_N(\mu^*)$ has computable upper bound, we can define an enclosure for μ^*, by solving the following system of inequalities

$$-\left|\frac{\sin\theta\mu^*}{\theta\mu^*}\right|^{-m} T_N(\mu^*) \le G(\mu^*) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu^*) \le \left|\frac{\sin\theta\mu^*}{\theta\mu^*}\right|^{-m} T_N(\mu^*). \quad (45)$$

Its solution is an interval containing μ^*, and over which the graph $G(\mu^*) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu^*)$ is trapped between the graphs $-\left|\frac{\sin\theta\mu^*}{\theta\mu^*}\right|^{-m} T_N(\mu^*)$ and $\left|\frac{\sin\theta\mu^*}{\theta\mu^*}\right|^{-m} T_N(\mu^*)$. Use the fact that

$$\mathcal{F}_{\theta,m,N}(\mu) \to \mathcal{F}_{\theta,m}(\mu)$$

converges uniformly over any compact set, and since μ^* is a simple root, we obtain for large N

$$\frac{\partial}{\partial\mu}\left(G(\mu) + \left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu)\right) \ne 0$$

in a neighborhood of μ^*. Hence the graph of $G(\mu) + \left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu)$ intersects the graphs $-\left|\frac{\sin\theta\mu}{\theta\mu}\right|^{-m} T_N(\mu)$ and $\left|\frac{\sin\theta\mu}{\theta\mu}\right|^{-m} T_N(\mu)$ at two points with abscissae $a_-(\mu^*, N) \le a_+(\mu^*, N)$ and the solution of the system of inequalities (45) is the interval

$$I_N(\mu^*) := [a_-(\mu^*, N), a_+(\mu^*, N)]$$

and in particular, $\mu^* \in I_N(\mu^*)$. Now, we summarize the above idea in the following lemma, see [22].

Lemma 3.3 For any eigenvalue μ^{*2}
1. there exists N_0 such that $\mu^* \in I_N(\mu^*)$ for $N > N_0$;
2. $[a_-(\mu^*, N), a_+(\mu^*, N)] \to \{\mu^*\}$ as $N \to \infty$.

Proof: Since all eigenvalues are simple, then for N large enough we have $\frac{\partial}{\partial\mu}\left(G(\mu) + \left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu)\right) > 0$ say, in a neighborhood of μ^*. Now we choose N_0 such that

$$G(\mu) + \left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m} \mathcal{F}_{\theta,m,N_0}(\mu) = \pm\left|\frac{\sin\theta\mu}{\theta\mu}\right|^{-m} T_{N_0}(\mu)$$

has two distinct solutions which we denote by $a_-(\mu^*, N_0) \le a_+(\mu^*, N_0)$. The decay of $T_N(\mu) \to 0$ as $N \to \infty$ will ensure the existence of the solutions $a_-(\mu^*, N)$ and $a_+(\mu^*, N)$ as $N \to \infty$. For the second point we recall that $\mathcal{F}_{\theta,m,N}(\mu) \to \mathcal{F}_{\theta,m}(\mu)$ as $N \to \infty$. Hence by taking the limit we obtain

$$G(a_+(\mu^*, \infty)) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m}(a_+(\mu^*, \infty)) = 0,$$

$$G(a_-(\mu^*,\infty)) + \left(\frac{\sin\theta\mu^*}{\theta\mu^*}\right)^{-m} \mathcal{F}_{\theta,m}(a_-(\mu^*,\infty)) = 0,$$

that is, $\Delta(a_+) = \Delta(a_-) = 0$. This leads us to conclude that $a_+ = a_- = \mu^*$, since μ^* is a simple root.

4. Examples

In this section, the above theory is illustrated by looking at two simple examples where eigenvalue enclosures are obtained. We also indicate the effect of the parameters m and θ by several choices. Both numerical results and the associated figures prove the credibility of the method. In the following examples, $\mu_{k,N}$ is considered to be the k th root of $G(\mu) + \left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m} \mathcal{F}_{\theta,m,N}(\mu) = 0$. Also, in the following examples, we observe that $\mu_{k,N}$ and the exact solution μ_k are all inside the interval $[a_-, a_+]$.

Example 4.1. Consider the boundary value problem

$$-y''(x,\mu) - y(x,\mu) = \mu^2 y(x,\mu) \quad 0 \le x \le 1, \quad (46)$$

$$y(0,\mu) = \mu^2 y'(0,\mu), \quad y'(1,\mu) = \mu^2 y(1,\mu). \quad (47)$$

This problem is a special case of the problem when $q = -1$, $a_2 = a_1{}' = b_1 = b_2{}' = 0$ and $a_1 = a_2{}' = b_1{}' = b_2 = 1$. The characteristic determinant of the problem is

$$\Delta(\mu) = (1-\mu^4)\cos\sqrt{\mu^2+1} - (2\mu^2+\mu^4)\frac{\sin\sqrt{\mu^2+1}}{\sqrt{\mu^2+1}}. \quad (48)$$

After some calculations it is found that

$$G(\mu) = (1+\mu^2)\big((1-\mu^2)\cos\mu - \mu\sin\mu\big). \quad (49)$$

Tables [1-2] and Figures [1-2] indicate the application of our technique to this problem.

Example 4.2. Consider the boundary value problem

$$-y''(x,\mu) + xy(x,\mu) = \mu^2 y(x,\mu) \quad 0 \le x \le 1, (50)$$

$$y(0,\mu) + y'(0,\mu) = \mu^2(-y(0,\mu) + y'(0,\mu)), \quad (51)$$

$$-y(1,\mu) + y'(1,\mu) = \mu^2(y(1,\mu) + y'(1,\mu)). \quad (52)$$

In this case $q(x) = x$, $a_1 = a_2 = a_2{}' = b_2 = b_2{}' = b_1{}' = 1$ and $b_1 = a_1{}' = -1$. The characteristic determinant of the problem is

$$\begin{aligned}\Delta(\mu) = \frac{1}{(AiryAiPrime[1-\mu^2]AiryBi[-\mu^2] - AiryAi[-\mu]AiryBiPrime[1-\mu^2])} \\ \times [-(1+\mu^2)AiryAi[1-\mu^2]((1+\mu^2)AiryBi[-\mu^2] - (-1+\mu^2)AiryBiPrime[-\mu^2]) \\ -(-1+\mu^2)AiryAiPrime[1-\mu^2]((1+\mu^2)AiryBi[-\mu^2] - (-1+\mu^2)AiryBiPrime[-\mu^2]) \\ +((1+\mu^2)AiryAi[-\mu^2] - (-1+\mu^2)AiryAiPrime[1-\mu^2])((1+\mu^2)AiryBi[1-\mu^2] \\ +(-1+\mu^2)AiryBiPrime[1-\mu^2]],\end{aligned} \quad (53)$$

where $AiryAi[z]$ and $AiryBi[z]$ are Airy functions $Ai(z)$ and $Bi(z)$, respectively, and $AiryAiPrime[z]$ and $AiryBiPrime[z]$ are derivatives of Airy functions. The function $G(\mu)$ will be

$$G(\mu) = \frac{2\mu(1-\mu^4)\cos\mu + (\mu^6 - 3\mu^4 - \mu^2 - 1)\sin\mu}{\mu}. \quad (54)$$

Tables [3-4] and Figures [2-4] indicate the application of our technique to this problem.

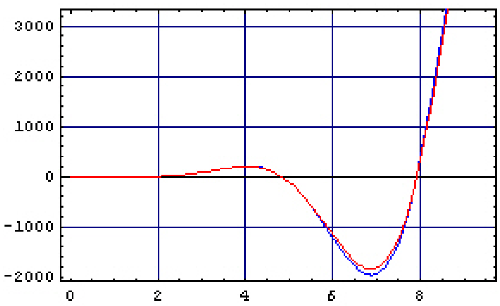

Fig 1. $\Delta(\mu)$, $\Delta_N(\mu)$ with $N = 40$, $m = 8$ and $\theta = 1/32$.

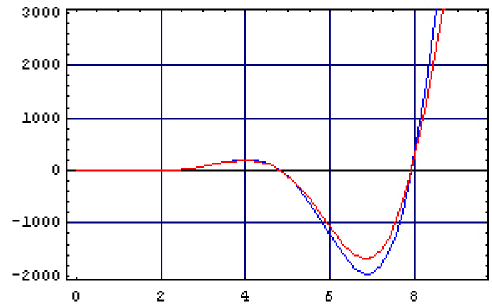

Fig 2. $\Delta(\mu)$, $\Delta_N(\mu)$ with $N = 40$, $m = 14$ and $\theta = 1/26$.

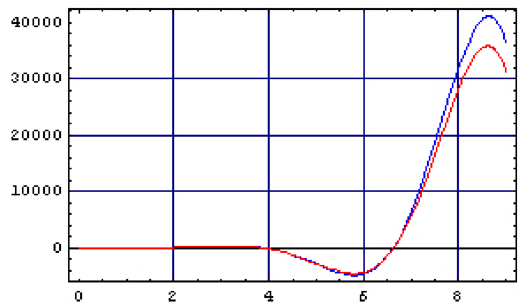

Fig 3. $\Delta(\mu)$, $\Delta_N(\mu)$ with $N = 40$, $m = 10$ and $\theta = 1/30$.

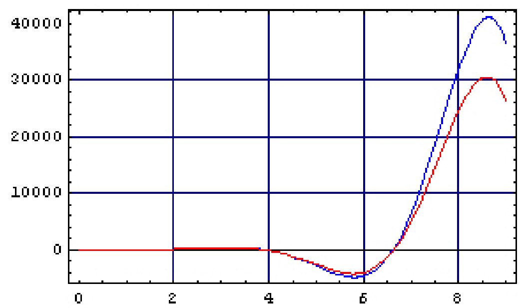

Fig 4. $\Delta(\mu)$, $\Delta_N(\mu)$ with $N = 40$, $m = 15$ and $\theta = 1/25$.

Table 1. $\mu_{k,N}$ and the exact solution μ_k are all inside $[a_-, a_+]$, where $N = 40$, $m = 8$, $\theta = \frac{1}{32}$, $E_4(\mathcal{F}_{\theta,m}) = 2.69987 \times 10^{15}$

Exact μ_k			
0.4828692021748485	0.4793063473441252	0.48646645601871363	0.48287973117971406
1.9663180523504247	1.9658893932751713	1.9667354276372926	1.9663129864196962
4.827089429919572	4.8270824332214	4.827096026486742	4.827089230117779
7.919684444168381	7.919682641882104	7.919686429100161	7.919684535484135

Table 2. $\mu_{k,N}$ and the exact solution μ_k are all inside $[a_-, a_+]$, where $N = 40$, $m = 14$, $\theta = \frac{1}{26}$, $E_{10}(\mathcal{F}_{\theta,m}) = 2.83057 \times 10^{24}$

Exact μ_k			
0.4828692021748485	0.482836191137082	0.4829022140937972	0.48286920219626944
1.9663180523504247	1.9663174453361767	1.9663186593505408	1.9663180523488202
4.827089429919572	4.827089226021884	4.827089633819908	4.827089429920899
7.919684444168381	7.919684430472341	7.919684457864718	7.919684444168529

Table 3. $\mu_{k,N}$ and the exact solution μ_k are all inside $[a_-, a_+]$, where $N = 40$, $m = 10$, $\theta = \frac{1}{30}$, $E_6(\mathcal{F}_{\theta,m}) = 1.25034 \times 10^{18}$

Exact μ_k			
0.5829673818446196	0.5828163858485008	0.5831083879812755	0.5829623977579995
1.8189332937112266	1.818921178705338	1.8189482961554078	1.8189347380634264
3.8046187863352188	3.8046179490207503	3.804620104016852	3.804619026520858
6.63563219383198	6.63563216621949	6.635632256635445	6.635632211427473

Table 4. $\mu_{k,N}$ and the exact solution μ_k are all inside $[a_-, a_+]$, where $N = 40$, $m = 15$, $\theta = \frac{1}{25}$, $E_{11}(\mathcal{F}_{\theta,m}) = 2.23046 \times 10^{25}$

Exact μ_k			
0.5829673818446196	0.5829644938415554	0.5829702688272004	0.5829673813399913
1.8189332937112266	1.8189332114883874	1.818933376024228	1.8189332937563671
3.8046187863352188	3.8046187823511173	3.8046187903286928	3.8046187863399052
6.63563219383198	6.635632192367194	6.635632195290339	6.635632193828767

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