On compact operators on the Riesz $B^m$-difference sequence space

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Abstract

In this paper, we give the characterization of some classes of compact operators given by matrices on the normed sequence space $r_p^a(B^m)$, which is a special case of the paranormed Riesz $B^m$-difference sequence space $r^q(p,B^m)$. For this purpose, we apply the Hausdorff measure of noncompactness and use some results.

Keywords: $B^m$-difference sequence spaces; Hausdorff measure of noncompactness; compact operators

1. Introduction

Let $\omega$ be the space of all real valued sequences. Any vector subspace of $\omega$ is called a sequence space. We write $c_{00}$, $c$, $c_0$ and $\ell^p$ the sets of all bounded, convergent, null and finite sequences, respectively. Also, by $\ell^p_1$ and $\ell^p_p$ ($1 < p < \infty$) we denote the sequence spaces of all absolutely and $p$-absolutely convergent series, respectively. Throughout this paper, if $x \in \omega$, then we write $x = (x_k)$ instead of $x = (x_k)_k$. Further, we use the conventions that $e = (1,1,\ldots)$ and $e^{(k)}$ is the sequence whose only non-zero term is 1 in the $k$th place for each $k \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,\ldots\}$. Moreover, by $F_r(r \in \mathbb{N})$, we denote the subcollection of $F$ consisting of all nonempty and finite subsets of $\mathbb{N}$ with elements that are greater than $r$, that is

$$F_r = \{N \in F: n > r \text{ for all } n \in N\}; \ (r \in \mathbb{N}).$$

It is quite natural to find conditions for a matrix map between $BK$-spaces to define a compact operator since a matrix transformation between $BK$-spaces is continuous.

This can be achieved by applying the Hausdorff measure of noncompactness. In the past, several authors characterized classes of compact operators given by infinite matrices on some sequence spaces by using this method [1-12].

Recently, Malkowsky and Rakočević [13], Djolović and Malkowsky [14] and Mursaleen and Noman [15] have established some identities or estimates for the operator norms and Hausdorff measures of noncompactness of linear operators given by infinite matrices that map an arbitrary BK-space or the matrix domains of triangles in arbitrary BK-spaces. Also, Mursaleen [16] has determined the Hausdorff measure of noncompactness on the sequence space $n(\phi)$ of W. L. C. Sargent and applied the technique of measure of noncompactness to the theory of infinite systems of differential equations.

In this paper, by taking a special case of the paranormed Riesz $B^m$-difference sequence space $r^q(p,B^m)$, we obtain a $BK$-space and investigate the classes of some compact operators given by matrices on this space by applying the Hausdorff measure of noncompactness and using some results in [15] and [13].

2. Preliminaries and notations

The $\beta$-dual of a subset $X$ of $\omega$ is defined by

$$X^{\beta} = \left\{ a \in \omega: \sum_k a_k x_k \text{ converges for all } x \in X \right\}.$$

If $A$ is an infinite matrix with real entries $a_{nk}$ ($n,k \in \mathbb{N}$), then we write $A_n$ for the sequence in the $n$th row of $A$, that is, $A_n = (a_{nk})_k$. Also, if $x = (x_k) \in \omega$, then we define the $A$-transform of $x$ as the sequence $Ax = (A_{nk}(x))_k$, where

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provided the series on the right side converges for each \( n \in \mathbb{N} \).

Let \( X \) and \( Y \) be subsets of \( \omega \). We say that \( A \) defines a matrix mapping from \( X \) into \( Y \), and we denote it by writing \( A: X \to Y \), if for every sequence \( x = (x_n) \in X \) the sequence \( Ax = (A_n(x)) \) is in \( Y \). Furthermore, the set

\[ X_A = \{ x \in \omega : Ax \in X \} \]  

(1)
is called the matrix domain of \( A \) in \( X \) and \( (X, Y) \) denotes the class of matrices that maps \( X \) into \( Y \), that is \( A \in (X, Y) \) if and only if \( X_A \subset Y \), or equivalently \( A_n \in X_A \) for all \( n \in \mathbb{N} \) and \( Ax \in Y \) for all \( x \in X \).

A sequence space \( X \) is called \( FK \)-space if it is a complete linear metric space with continuous coordinates \( p_n: X \to \mathbb{R} \) \((n \in \mathbb{N})\), where \( \mathbb{R} \) denotes the real field and \( p_n(x) = x_n \) for all \( x = (x_n) \in X \) and every \( n \in \mathbb{N} \). A BK-space is a normed \( FK \)-space, that is, a \( BK \)-space is a Banach space with continuous coordinates. An \( FK \) space \( X \to \phi \) is said to have \( AK \) if every sequence \( x = (x_n) \in X \) has a unique representation \( x = \sum_{k=0}^{\infty} x_k e^k \), that is \( x = \lim_{n \to \infty} x[n] \). Here, \( x[n] \) is called the \( n \)-section of \( x \) \((n \in \mathbb{N})\).

The sequence spaces \( c_0, c \) and \( \ell_\infty \) are \( BK \)-spaces with the usual sup-norm given by \( \|x\|_\ell_\infty = \sup_{k \in \mathbb{N}} |x_k| \) and the space \( \ell_p \) is a \( BK \)-space with the usual \( \ell_p \)-norm defined by \( \|x\|_{\ell_p} = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p} \), where \( 1 \leq p < \infty \). Also, all of the \( c_0 \) and \( \ell_p \) \((1 \leq p < \infty)\) have \( AK \) [17, Example 1.13 and 1.20].

Let \( X \) be a normed space. Then, we write \( S_X \) for the unit sphere in \( X \), that is \( S_X = \{ x \in X : \|x\| = 1 \} \). If \( X \) and \( Y \) be Banach spaces and \( B(X, Y) \) is the set of all continuous linear operators \( L: X \to Y \); \( B(X, Y) \) is a Banach space with the operator norm defined by \( \|L\| = \sup \{ \|L(x)\| : \|x\| \leq 1 \} \) for all \( L \in B(X, Y) \).

If \( (X, \|\cdot\|) \) is a normed sequence space, then we write \( \|a\|_A = \sup_{x \in X} \sum_{k=0}^{\infty} a_k x_k \) for \( a \in \omega \), provided the expression on the right hand side exists and is finite, which is the case whenever \( X \) is a BK space and \( a \in X \) \([18, \text{Theorem 7.2.9, p.107}]\). Throughout, let \( 1 \leq p < \infty \) and \( q \) denote the conjugate of \( p \), that is, \( q = p/(p - 1) \) for \( 1 < p < \infty \) or \( q = \infty \) for \( p = 1 \).

The following well-known results are fundamental for our investigation.

Lemma 2.1. \([18, \text{Theorem 4.2.8}]\). Let \( X \) and \( Y \) be \( BK \)-spaces. Then we have \( (X, Y) \subset B(X, Y) \), that is,

\[
A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \quad (n \in \mathbb{N})
\]
every \( A \in (X, Y) \) defines a linear operator \( L_A \in B(X, Y) \), where \( L_A(x) = Ax \) for all \( x \in X \).

Lemma 2.2. \([17, \text{Theorem 1.29(b)}]\). Let \( 1 \leq p < \infty \). Then, we have \( \|e_p^q \| = \ell_q \) and \( \|a\|_{\ell_p} = \|a\|_{\ell_q} \) for all \( a \in (a_k) \in \ell_q \).

3. The Riesz \( B^m \)-difference sequence space \( r^q_p(B^m) \) \((1 \leq p < \infty)\)

Let us give the definition of some triangle limitation matrices which are used in the text. Let \( (q_k) \) be a sequence of positive numbers and

\[
Q_n = \sum_{k=0}^{n} q_k; \quad (n \in \mathbb{N}).
\]

Then the matrix \( R^q_p = (r^q_{nk}) \) of the Riesz mean is given by

\[
r^q_{nk} = \begin{cases} 
q_k & (0 \leq k \leq n) \\
0 & (k > n).
\end{cases}
\]

The difference and generalized difference matrices \( \Delta = (\Delta_{nk}) \) and \( B = (b_{nk}) \) are defined by

\[
\Delta_{nk} = \begin{cases} 
(-1)^n & (k = n) \\
0 & (0 \leq k < n - 1) \text{ or } (k > n)
\end{cases}
\]

and

\[
b_{nk} = \begin{cases} 
r & (k = n) \\
0 & (0 \leq k < n - 1) \text{ or } (k > n)
\end{cases}
\]

for all \( k, n \in \mathbb{N} \) and \( r, s \in \mathbb{R} - \{0\} \) (for the matrix \( B \) see \([19, 20])\). If we take \( r = 1 \) and \( s = -1 \) in the matrix \( B \), then we have \( B = \Delta \). Thus, for any sequence space \( X \), the space \( X_B \) is more general and more comprehensive than the corresponding consequences of the space \( X_\Delta \).

Recently, the generalized \( B^m \)-Riesz difference sequence space \( r^q(p, B^m) \) has been introduced by Başarır and Kayıkçı \([21]\) as follows:

\[
r^q(p, B^m) = \{ x = (x_k) \in \omega : T^m(x) \in \ell(p) \}; \quad (1 \leq p < H),
\]

where \( \ell(p) \) is the paranormed sequence space defined by Maddox \([22]\) and the matrices \( B^m = (b_{nk}^m) \) and \( R^qB^m = T^m = (t_{nk}^m) \) are defined by

\[
b_{nk}^m = \begin{cases} 
m_n & (0 \leq k \leq m) \\
0 & (m < k \leq n) \text{ or } (k > n)
\end{cases}
\]

and
for all \( k,n \in \mathbb{N} \). It is obvious that the matrix \( B^m \) reduced the difference matrix \( \Delta^m \) in case \( r = 1 \) and \( s = -1 \), where \( \Delta^m = (\Delta^m)^{-1} \).

If we take \( p_n = p \) for all \( n \in \mathbb{N} \), then we have

\[
t^m_{nk} = \begin{cases} 
\frac{1}{Q_n} \sum_{i=k}^{n} \binom{m}{i-k} r^{m-i+k} s^{i-k} q_i & (k < n) \\
r^m & (k = n) \\
0 & (k > n)
\end{cases}
\]

(2)

Also, if we take \( m = 1 \) in (2) then we have

\[
r^q_p(B) = \{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} |T_n(x)|^p < \infty \} ; \quad (1 \leq p < \infty).
\]

With the notation of (1), we can redefine the sequence spaces \( r^q_p(B^m) \) and \( r^q_p(B) \) as follows:

\[
r^q_p(B^m) = (\ell p)_1 \text{ and } r^q_p(B) = (\ell p) ; \quad (1 \leq p < \infty).
\]

It is easy to see that the spaces \( r^q_p(B^m) \) and \( r^q_p(B) \) are BK-spaces with the norm, respectively, as follows:

\[
\| x \|_{r^q_p(B^m)} = \| T^m(x) \|_{\ell p} = (\sum_{n=0}^{\infty} |T_n(x)|^p)^{1/p} ; \quad (1 \leq p < \infty).
\]

(3)

and

\[
\| x \|_{r^q_p(B)} = \| T(x) \|_{\ell p} = (\sum_{n=0}^{\infty} |T_n(x)|^p)^{1/p} ; \quad (1 \leq p < \infty).
\]

(4)

Throughout, for any sequence \( x = (x_k) \), we define the associated sequence \( y = (y_k) \), which will be frequently used, as the \( T^m \)-transform of \( x \), i.e., \( y = T^m(x) \) and so

\[
y_k = \frac{1}{q_k} \sum_{j=0}^{m} \binom{m}{j} r^{m-j} s^j q_j x_j + \frac{r^m}{q_k} \sum_{j=0}^{m} \binom{m}{j} r^{m-j} s^j q_j x_j \quad (k \in \mathbb{N}).
\]

(5)

Obviously, if the sequences \( x \) and \( y \) are connected by the relation (5), then \( x \in r^q_p(B^m) \) if and only if \( y \in \ell p \), further if \( x \in r^q_p(B^m) \), then

\[
\| x \|_{r^q_p(B^m)} = \| y \|_{\ell p}.
\]

In this paper, we characterize classes of compact operators given by infinite matrices from \( r^q_p(B^m) \) to \( c_0 \), \( c \), \( \ell_\infty \) and \( \ell_1 \). Also, we give the necessary and sufficient conditions for \( A \in (r^q_p(B^m), \ell p) \) to be compact, where \( 1 \leq p < \infty \).

The following result is immediate by [13, Theorem 3.2].

Lemma 3.1. Let

\[
\forall(i,j,k) = (-1)^{i-k} s^{i-j} q^{j-i} \frac{1}{(m+j-i-1)!} \quad (i,j,k,m \in \mathbb{N}).
\]

If \( a = (a_k) \in (r^q_p(B^m))^p \), then \( \bar{a} = (\bar{a}_k) \in \ell_q \) and the equality

\[
\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \bar{a}_k y_k
\]

holds for every \( x = (x_k) \in r^q_p(B^m) \), where \( y = T^m(x) \) is given by (5) and

\[
\bar{a}_k = \frac{1}{q_k} \left( \frac{a_k}{r_m q_k} + \sum_{j=k+1}^{\infty} \sum_{i=k}^{m} \frac{1}{r_m} \forall(i,j,k) \right) \quad k \in \mathbb{N}.
\]

(6)

On the other hand, let \( 1 \leq p < \infty \). Then, it can easily be shown that the inclusion \( r^q_p(B^m) \supseteq \phi \) holds if and only if \( 1/q \in \ell_p \), where \( \frac{1}{q} = \left( \frac{1}{q_k} \right) \). So, we shall assume that \( 1/q \in \ell_p \) whenever we study the space \( r^q_p(B^m) \).

Lemma 3.2. Let \( 1 \leq p < \infty \) and \( \bar{a} = (\bar{a}_k) \) be defined as in Lemma 3.1. Then, we have

\[
\| a \|_{r^q_p(B^m)}^* = \bar{a}_k = \left( \sum_{k=0}^{\infty} |\bar{a}_k|^q \right)^{1/q} \quad (1 < p < \infty)
\]

(7)

for all \( a = (a_k) \in (r^q_p(B^m))^p \).

Proof: Let \( a = (a_k) \in (r^q_p(B^m))^p \). Then we have from Lemma 3.1 that \( \bar{a} = (\bar{a}_k) \in \ell_q \) and the equality (6) holds for all \( x = (x_k) \in r^q_p(B^m) \) and \( y = (y_k) \in \ell_p \), which are connected by the relation (5). Also, we can write by (3) that \( x \in S^q_p(B^m) \) if and only if \( y \in \ell_p \). Thus, we have from (6) that

\[
\| a \|_{r^q_p(B^m)}^* = \sup_{x \in S^q_p(B^m)} \left| \sum_{k=0}^{\infty} a_k x_k \right| = \sup_{y \in \ell_p} \left| \sum_{k=0}^{\infty} \bar{a}_k y_k \right|.
\]

Further, since \( \bar{a} \in \ell_q \), we get by Lemma 2.2 and (7) that

\[
\| a \|_{r^q_p(B^m)}^* = \| \bar{a} \|_{\ell_q}^* = \| \bar{a} \|_{\ell_q} < \infty
\]

which concludes the proof.

Lemma 3.3. Let \( X \) be a sequence space, \( A = (a_{nk}) \) an infinite matrix and \( 1 \leq p < \infty \). If \( A \in \ell_p \)
(r^q_p (B^m), X), then \( \tilde{A} \in (\ell_p, X) \) such that \( Ax = \tilde{A} y \)
for all \( x \in r^q_p (B^m), X \) and \( y \in \ell_p \), where the sequences \( x \) and \( y \) are connected
by the relation (5) and \( \tilde{A} = (a_{nk}) \) is the associated matrix with \( A = (a_{nk}) \) defined by

\[
\tilde{a}_{nk} = Q_k \left( \frac{a_{nk}}{r_k} + \sum_{i=k+1}^{\infty} \|v(i, j, k)\| a_{nj} \right); \quad (n, k \in \mathbb{N}).
\]

Proof: Let \( x \in r^q_p (B^m) \) and \( A \in (r^q_p (B^m), X) \). Then \( A_n \in \left( r^q_p (B^m) \right)^\beta \) for all \( n \in \mathbb{N} \). Thus, it follows
by Lemma 3.1 that \( \tilde{A}_n \in (\ell_p)^\beta = \ell_q \) for all \( n \in \mathbb{N} \) and the equality \( Ax = \tilde{A} y \) holds.
Hence, \( \tilde{A} y \in X \). Since every \( y \in \ell_p \) is the associated sequence of \( x \in r^q_p (B^m) \), we obtain that \( \tilde{A} \in (\ell_p, X) \).
This completes the proof.

4. The Hausdorff measure of noncompactness and compact operators on the space \( r^q_p (B^m) \) (1 ≤ p < ∞)

The Hausdorff measure of noncompactness was defined by Goldstein, Golberg and Markus in
1957, and later studied by Goldstein and Markus in 1968.

In this section, we give some classes of compact operators on the space \( r^q_p (B^m) \) for 1 ≤ p < ∞.

We recall that if \( X \) and \( Y \) are Banach spaces and \( L \) is a linear operator from \( X \) to \( Y \), then \( L \) is said to be
compact if its domain is all of \( X \) and for every bounded sequence \( (x_n) \) in \( X \), the sequence \( (L(x_n)) \)
has a convergent subsequence in \( Y \). We denote the class of such operators by \( C(X, Y) \).

If \( (X, d) \) is a metric space, we write \( \mathcal{M}_X \) for the class of all bounded subsets of \( X \). By \( B(x, r) = \{ y \in X : d(x, y) < r \} \) we denote the open ball of
radius \( r > 0 \) with the centre in \( x \). Then the Hausdorff measure of noncompactness of the set
\( Q \in \mathcal{M}_X \), denoted by \( \chi(Q) \), is given by

\[
\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{i=0}^{n} B(x_i, r_i), \quad x_i \in X, \quad r_i < \varepsilon \right\}.
\]

The function \( \chi : \mathcal{M}_X \to [0, \infty) \) is called the Hausdorff measure of noncompactness.

The basic properties of the Hausdorff measure of noncompactness can be found in [17], for example
if \( Q, Q_1 \) and \( Q_2 \) are bounded subsets of a metric space \( (X, d) \), then

\[
\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded,}
\]

\[
Q_1 \subset Q_2 \implies \chi(Q_1) \leq \chi(Q_2).
\]

Further, if \( X \) is a normed space, then the function \( \chi \) has some additional properties connected with the
linear structure, e.g.

\[
\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),
\]

\[
\chi(\alpha Q) = |\alpha| \chi(Q) \text{ for all } \alpha \in \mathbb{C}.
\]

The following lemma is related to the Hausdorff
measure of noncompactness of a bounded linear operator.

Lemma 4.1. [17, Theorem 2.25, Corollary 2.26]. Let \( X \) and \( Y \) be Banach spaces and \( L \in B(X, Y) \).
Then we have

\[
\|L\|_X = \chi(L(S_X))
\]

and

\[
L \in C(X, Y) \text{ if and only if } \|L\|_X = 0.
\]

Lemma 4.2. [6, Lemma 5.5]. Let \( Q \) be a bounded subsets of the normed space \( X \), where \( X \) is \( \ell_p \) for
1 ≤ p < ∞ or \( c_0 \). If \( P_n : X \to X \) is the operator defined by

\[
P_n(x) = x^{[n]} = (x_0, x_1, x_2, ..., x_n, 0, 0, ...)
\]

for all \( x = (x_k) \in X \), then we have

\[
\chi(Q) = \lim_{n \to \infty} \left( \sup_{x \in Q} \| (I - P_n)(x) \| \right).
\]

Lemma 4.3. [15, Theorems 3.7 and 3.11]. Let \( X \) be a BK-space. Then, we have
(a) If \( A \in (X, c_0) \), then

\[
\|L_A\|_X = \limsup_{n \to \infty} \|A_n\|_X
\]

and

\[
L_A \text{ is compact if and only if } \lim_{n \to \infty} \|A_n\|_X = 0.
\]

(b) If \( X \) has \( AK \) and \( A \in (X, c) \), then

\[
\frac{1}{2} \limsup_{n \to \infty} \|A_n - \alpha\|_X \leq \|L_A\|_X
\]

\[
\leq \limsup_{n \to \infty} \|A_n - \alpha\|_X
\]

and

\[
L_A \text{ is compact if and only if } \lim_{n \to \infty} \|A_n - \alpha\|_X = 0,
\]

where \( \alpha = (\alpha_k) \) with \( \alpha_k = \lim_{n \to \infty} a_{nk} \) for all \( k \in \mathbb{N} \).

(c) If \( A \in (X, \ell_\infty) \), then

\[
0 \leq \|L_A\|_X \leq \limsup_{n \to \infty} \|A_n\|_X.
\]
and

\[ L_A \text{ is compact if } \lim_{n \to \infty} \|A_n\|_X = 0. \]

(d) If \( A \in (X, \ell_1) \), then

\[
\lim_{r \to \infty} \left( \sup_{N \in \mathbb{F}_X} \left\| \sum_{n \in \mathbb{N}} A_n \right\|_X \right) \leq \|L_A\|_X
\]

\[
\leq 4 \lim_{r \to \infty} \left( \sup_{N \in \mathbb{F}_X} \left\| \sum_{n \in \mathbb{N}} A_n \right\|_X \right)
\]

and

\[ L_A \text{ is compact if and only if } \lim_{r \to \infty} \left( \sup_{N \in \mathbb{F}_X} \left\| \sum_{n \in \mathbb{N}} A_n \right\|_X \right) = 0. \]

This lemma gives necessary and sufficient conditions for a matrix transformation from a BK-space \( X \) to \( c_0, c, \ell_1 \) and \( \ell_{\infty} \) to be compact (the only sufficient condition for \( \ell_{\infty} \)). Thus, we have:

Theorem 4.4. Let \( 1 < p < \infty \) and \( q = p/(p - 1) \). Then we have

(a) If \( A \in (r^q_p(B^m), c_0) \), then

\[ \|L_A\|_X = \limsup_{n \to \infty} (\sum_{k=0}^{\infty} |\tilde{a}_{nk}|q)^{1/q} \quad (11) \]

and

\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} (\sum_{k=0}^{\infty} |\tilde{a}_{nk}|q)^{1/q} = 0. \quad (12) \]

(b) If \( A \in (r^q_p(B^m), c) \), then

\[ \frac{1}{2} \limsup_{n \to \infty} (\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k|^q)^{1/q} \leq \|L_A\|_X \leq \limsup_{n \to \infty} (\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k|^q)^{1/q} \quad (13) \]

and

\[ L_A \text{ is compact if and only if } \lim_{n \to \infty} (\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k|^q)^{1/q} = 0, \quad (14) \]

where \( \tilde{a} = (\tilde{a}_k) \) with \( \tilde{a}_k = \lim_{n \to \infty} \tilde{a}_{nk} \) for all \( k \in \mathbb{N} \).

(c) If \( A \in (r^q_p(B^m), \ell_m) \), then

\[ 0 \leq \|L_A\|_X \leq \limsup_{n \to \infty} (\sum_{k=0}^{\infty} |\tilde{a}_{nk}|q)^{1/q} \quad (15) \]

and

\[ L_A \text{ is compact if } \lim_{n \to \infty} (\sum_{k=0}^{\infty} |\tilde{a}_{nk}|q)^{1/q} = 0. \quad (16) \]

Proof: (a) Let \( A \in (r^q_p(B^m), c_0) \). Since \( A_n \in (r^q_p(B^m))^p \) for all \( n \in \mathbb{N} \), we have from Lemma 3.2 that

\[ \|A_n\|_p(B^m) = \|A_n\|_{\ell_q} = (\sum_{k=0}^{\infty} |a_{nk}|^q)^{1/q} \quad (17) \]

for all \( n \in \mathbb{N} \). Hence, we get (11) and (12) from (17) and Lemma 4.3(a).

Parts (b) and (c) can be proved similarly by using Lemma 4.3(b) and (c).

The conclusions of Theorem 4.4 still hold for \( r^q(B^m) \) instead of \( r^q_p(B^m) \) with \( q = 1 \), and on replacing the summations over \( k \) by the supremums over \( k \).

Theorem 4.5. Let \( 1 \leq p < \infty \). If \( A \in (r^q_1(B^m), \ell_p) \), then

\[ \lim_{r \to \infty} \left( \sup_{k \geq r} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|p \right)^{1/p} \right), \quad (18) \]

Proof: Let \( S = S_{r^q_1(B^m)} \). Then, we have by Lemma 2.1 that \( L_A(S) = AS \in \ell_p \). Thus, from (9) and Lemma 4.2 we can write that a 4.2 can we write that

\[ \|L_A\|_X = \chi(AS) = \lim_{r \to \infty} \left( \sup_{k \geq r} \|I - P_r(Ax)\|_{\ell_p} \right). \quad (19) \]

where \( P_r : \ell_p \to \ell_p (r \in \mathbb{N}) \) is the operator defined by \( P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots) \) for all \( x = (x_k) \in \ell_p \).

Now, let \( x = (x_k) \in r^q_1(B^m) \). Since \( A \in (r^q_1(B^m), \ell_p) \), we obtain from Lemma 3.3 that \( \tilde{A} \in (\ell_1, \ell_p) \) and \( Ax = \tilde{A}y \), where \( y = (y_k) \in \ell_1 \) is the associated sequence defined by (5). Therefore, we have that

\[ \|(I - P_r(Ax))\|_{\ell_p} = \|(I - P_r(\tilde{A}y))\|_{\ell_p} \]

\[ = \left( \sum_{r+1}^{\infty} |\tilde{A}_n(y)|^p \right)^{1/p} \]

\[ = \left( \sum_{r+1}^{\infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}y_k|^p \right)^{1/p} \right) \]

\[ \leq \left( \sum_{k=0}^{\infty} \left( \sum_{r+1}^{\infty} |\tilde{a}_{nk}|p \right)^{1/p} \right) \]

\[ \leq \|x\|_{\ell_1} \left( \sup_{k \geq r} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|p \right)^{1/p} \right). \]
Corollary 4.6. Let
\[ L_A \text{ is compact if and only if } \lim_{r \to \infty} \left( \sup_k \left( \sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right) = 0. \]

Proof: This is an immediate consequence of (10) and Theorem 4.5.

Theorem 4.7. Let \( 1 < p < \infty \) and \( q = p/(p-1) \). If \( A \in (r_p^q(B^m), \ell_p^q) \), then
\[
\lim_{r \to \infty} \left( \sup_{N \in F_r} \left( \sum_{k=0}^{\infty} \sum_{n \in N} |\tilde{a}_{nk}|^q \right)^{1/q} \right) \leq \|L_A\|_X
\]
\[
\leq 4 \lim_{r \to \infty} \left( \sup_{N \in F_r} \left( \sum_{k=0}^{\infty} \sum_{n \in N} |\tilde{a}_{nk}|^q \right)^{1/q} \right)
\]
and
\[ L_A \text{ is compact if and only if } \lim_{r \to \infty} \left( \sup_{N \in F_r} \left( \sum_{k=0}^{\infty} \sum_{n \in N} |\tilde{a}_{nk}|^q \right)^{1/q} \right) = 0. \]

Proof: Let \( A \in (r_p^q(B^m), \ell_p^q) \). Since \( A_n \in r_p^q(B^m)^\beta \) for all \( n \in \mathbb{N} \), we derive from Lemma 3.2 that
\[
\| \sum_{n \in N} A_n \|_{r_p^q(B^m)} = \left\| \sum_{n \in N} \bar{A}_n \right\|_{\ell_p^q}. \]

Thus, we get (22) and (23) from Lemma 4.3(d) and (24).

Remark: Let
\[ \nabla(j,k) = (-1)^{j-k} \left( \frac{s^{j-k}}{r^{j-k}q_{k+1}} + \frac{s^{j-k}}{r^{j-k}q_k} \right) : (j,k) \in \mathbb{N}. \]

If we take
\[ \tilde{a}_{nk} = Q_{nk} \left( \frac{a_{nk}}{r q_k} + \sum_{j=k+1}^{\infty} \nabla(j,k)a_{nj} \right) : (n,k) \in \mathbb{N} \]
then, we can obtain the above same results for the sequence space \( r_p^q(B) \) \( (1 \leq p < \infty) \).

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References


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