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Tangent Bishop spherical images of a biharmonic B-slant helix in the Heisenberg group Heis^3

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Abstract

In this paper, biharmonic slant helices are studied according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for slant helices to be biharmonic. The biharmonic slant helices are characterized in terms of Bishop frame in the Heisenberg group Heis^3. We give some characterizations for tangent Bishop spherical images of B-slant helix. Additionally, we illustrate four figures of our main theorem.

Keywords: Biharmonic curve; Bishop frame; Heisenberg group; tangent Bishop spherical images

1. Introduction

Let (M,g) and (N,h) be manifolds and $\phi: M \to N$ a smooth map. Denote by ∇^{ϕ} the connection of the vector bundle $\phi^* TN$ induced from the Levi-Civita connection ∇^h of (N,h). The second fundamental form $\nabla d\phi$ is defined by

$$(\nabla d\phi)(X,Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y),\ X,Y \in \Gamma(TM).$$

Here ∇ is the Levi-Civita connection of (M,g). The tension field $\tau(\phi)$ is a section of $\phi^* TN$ defined by

$$\tau(\phi) = tr\nabla d\phi. \tag{1}$$

A smooth map ϕ is said to be harmonic if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the energy:

$$E(\phi) = \frac{1}{2}\int h(d\phi, d\phi)dv_g$$

over every compact region of M. Now let $\phi: M \to N$ be a harmonic map. Then the Hessian H of E is given by

$$\mathsf{H}_{\phi}(V,W) = \int h(\mathsf{J}_{\phi}(V), W)dv_g,\ V,W \in \Gamma(\phi^* TN).$$

Here the Jacobi operator J_{ϕ} is defined by

$$\mathsf{J}_{\phi}(V) := \bar{\Delta}_{\phi} V - \mathsf{R}_{\phi}(V),\ V \in \Gamma(\phi^* TN), \tag{2}$$

$$\bar{\Delta}_{\phi} := \sum_{i=1}^{m}\left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i}e_i}\right), \mathsf{R}_{\phi}(V) = \sum_{i=1}^{m} R^N(V, d\phi(e_i))d\phi(e_i), \tag{3}$$

where R^N and $\{e_i\}$ are the Riemannian curvature of N, and a local orthonormal frame field of M, respectively [1].

Let $\phi:(M,g) \to (N,h)$ be a smooth map between two Lorentzian manifolds. The bienergy $E_2(\phi)$ of ϕ over compact domain $\Omega \subset M$ is defined by

$$E_2(\phi) = \int_{\Omega} h(\tau(\phi), \tau(\phi))dv_g.$$

A smooth map $\phi:(M,g) \to (N,h)$ is said to be biharmonic if it is a critical point of the $E_2(\phi)$, [2-11].

The section $\tau_2(\phi)$ is called the bitension field of ϕ and the Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) := -\mathsf{J}_{\phi}(\tau(\phi)) = 0. \tag{4}$$

In [12] the authors completely classified the biharmonic submanifolds of codimension greater than one in the n-dimensional sphere. The

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biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. The proper biharmonic curves on Riemannian surfaces were studied in [13]. Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in three-dimensional Sasakian space forms [14]. Then, Sasahara gave, in [15], the explicit representation of the proper biharmonic Legendre surfaces in five-dimensional Sasakian space forms.

The second variation formula for biharmonic maps in spheres was deduced [8] and the stability of certain classes of biharmonic maps in spheres was discussed in [4]. Also, in [16] there were given some sufficient conditions for the instability of Legendre proper biharmonic submanifolds in Sasakian space forms and the author proved the instability of Legendre curves and surfaces in Sasakian space forms.

Biharmonic functions are utilized in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them. In linear elasticity, if the equations are formulated in terms of displacements for two-dimensional problems, then the introduction of a stress function leads to a fourth-order equation of biharmonic type. For instance, the stress function is proved to be biharmonic for an elastically isotropic crystal undergoing phase transition, which follows spontaneous dilatation. Biharmonic functions arise when dealing with transverse displacements of plates and shells. They can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions also arise in fluid dynamics, particularly in Stokes flow problems (i.e., low-Reynolds-number flows). There are many applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [17-19]). Fluid flow through a narrow pipe or channel, such as that used in micro-fluidics, involves low Reynolds number. Seepage flow through cracks and pulmonary alveolar blood flow can also be approximated by Stokes flow. Stokes flow also arises in flow through porous media, which have been long applied by civil engineers to groundwater movement. The industrial applications include the fabrication of microelectronic components, the effect of surface roughness on lubrication, the design of polymer dies and the development of peristaltic pumps for sensitive viscous materials. In natural systems, creeping flows are important in biomedical applications and studies of animal locomotion.

In this paper, biharmonic slant helices are studied according to Bishop frame in the Heisenberg group Heis^3. Necessary and sufficient conditions are given for slant helices to be biharmonic. We characterize the biharmonic slant helices in terms of Bishop frame in the Heisenberg group Heis^3, and give some characterizations for tangent Bishop spherical images of B-slant helix. Additionally, four figures of our main theorem are illustrated (Fig. 1 and Fig. 2).

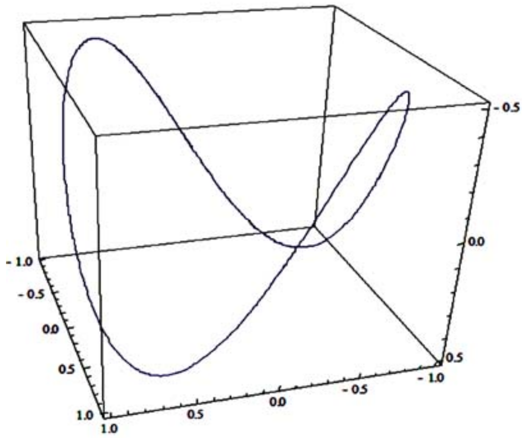

Fig. 1. Biharmonic slant helices according to Bishop frame for different constants with the help of the programme of Mathematica

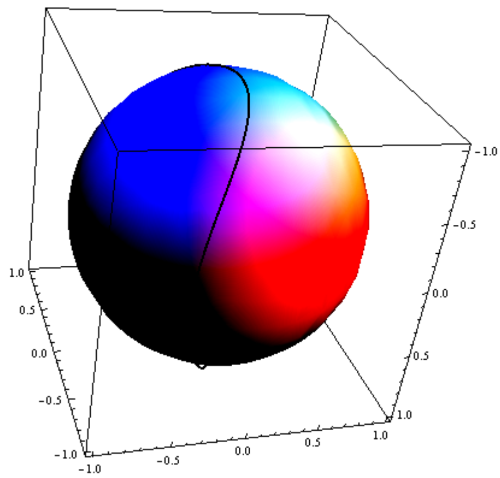

Fig. 2. Tangent spherical indicatrix of γ according to Bishop frame with the help of the programme of Mathematica

2. The Heisenberg group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\bar{x},\bar{y},\bar{z})(x,y,z)=(\bar{x}+x,\bar{y}+y,\bar{z}+z-\frac{1}{2}\bar{x}y+\frac{1}{2}x\bar{y}) \quad (5)$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$e_1=\frac{\partial}{\partial x},\ e_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\ e_3=\frac{\partial}{\partial z}, \qquad (6)$$

for which we have the Lie products [20, 21]

$$[e_1,e_2]=e_3,\ [e_2,e_3]=[e_3,e_1]=0$$

with

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.$$

We obtain

$$\nabla_{e_1}e_1=\nabla_{e_2}e_2=\nabla_{e_3}e_3=0,$$
$$\nabla_{e_1}e_2=-\nabla_{e_2}e_1=\frac{1}{2}e_3,$$
$$\nabla_{e_1}e_3=\nabla_{e_3}e_1=-\frac{1}{2}e_2,$$
$$\nabla_{e_2}e_3=\nabla_{e_3}e_2=\frac{1}{2}e_1.$$

We adopt the following notation and sign convention for Riemannian curvature operator on Heis^3 defined by

$$R(X,Y)Z=-\nabla_X\nabla_Y Z+\nabla_Y\nabla_Y Z+\nabla_{[X,Y]}Z,$$

while the Riemannian curvature tensor is given by

$$R(X,Y,Z,W)=g(R(X,Y)Z,W),$$

where X,Y,Z,W are smooth vector fields on Heis^3.

The components $\{R_{ijkl}\}$ of R relative to $\{e_1,e_2,e_3\}$ are defined by

$$g\left(R(e_i,e_j)e_k,e_l\right)=R_{ijkl}.$$

The non vanishing components of the above tensor fields are

$$R_{121}=-\frac{3}{4}e_2,\quad R_{131}=\frac{1}{4}e_3,\quad R_{122}=\frac{3}{4}e_1,$$
$$R_{232}=\frac{1}{4}e_3,\quad R_{133}=-\frac{1}{4}e_1,\quad R_{233}=-\frac{1}{4}e_2,$$

and

$$R_{1212}=-\frac{3}{4},\quad R_{1313}=R_{2323}=\frac{1}{4}. \quad (7)$$

3. Biharmonic B-Slant Helices with Bishop Frame In The Heisenberg Group Heis^3

Let $\gamma:I\to Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}&=\kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N}&=-\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B}&=-\tau\mathbf{N},\end{aligned} \qquad (8)$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}&g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{N},\mathbf{N})=1,\ g(\mathbf{B},\mathbf{B})=1,\\ &g(\mathbf{T},\mathbf{N})=g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.\end{aligned} \qquad (9)$$

In the rest of the paper, we suppose everywhere $\kappa\neq 0$ and $\tau\neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined, even when the curve has vanishing second derivative [22]. The Bishop frame is expressed as

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}&=k_1\mathbf{M}_1+k_2\mathbf{M}_2,\\ \nabla_{\mathbf{T}}\mathbf{M}_1&=-k_1\mathbf{T},\\ \nabla_{\mathbf{T}}\mathbf{M}_2&=-k_2\mathbf{T},\end{aligned} \qquad (10)$$

where

$$\begin{aligned}&g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{M}_1,\mathbf{M}_1)=1,\ g(\mathbf{M}_2,\mathbf{M}_2)=1,\\ &g(\mathbf{T},\mathbf{M}_1)=g(\mathbf{T},\mathbf{M}_2)=g(\mathbf{M}_1,\mathbf{M}_2)=0.\end{aligned} \qquad (11)$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures [23, 24], where $\zeta(s)=\arctan\frac{k_2}{k_1}$, $\tau(s)=\zeta'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$. Thus, Bishop curvatures are defined by

$$k_1=\kappa(s)\cos\zeta(s),\qquad (12)$$
$$k_2=\kappa(s)\sin\zeta(s).$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\mathbf{T}=T^1\mathbf{e}_1+T^2\mathbf{e}_2+T^3\mathbf{e}_3,$$
$$\mathbf{M}_1=M_1^1\mathbf{e}_1+M_1^2\mathbf{e}_2+M_1^3\mathbf{e}_3,\qquad (13)$$
$$\mathbf{M}_2=M_2^1\mathbf{e}_1+M_2^2\mathbf{e}_2+M_2^3\mathbf{e}_3.$$

Theorem 3.1. $\gamma:I\to Heis^3$ is a biharmonic curve with Bishop frame if and only if

$$k_1^2+k_2^2=\text{constant}=C\neq 0,$$
$$k_1''-Ck_1=k_1\left[\frac{1}{4}-\left(M_2^3\right)^2\right]-k_2M_1^3M_2^3,\qquad (14)$$
$$k_2''-Ck_2=k_1M_1^3M_2^3+k_2\left[\frac{1}{4}-\left(M_1^3\right)^2\right].$$

Proof: Using (10), we have

$$\tau_2(\gamma)=\nabla^3_{\mathbf{T}}\mathbf{T}-R(\mathbf{T},\nabla_{\mathbf{T}}\mathbf{T})\mathbf{T}$$
$$=(-3k_1'k_1-3k_2'k_2)\mathrm{T}+(k_1''-k_1^3-k_1k_2^2)\mathbf{M}_1$$
$$+(k_2''-k_2^3-k_2k_1^2)\mathbf{M}_2$$
$$-k_1R(\mathbf{T},\mathbf{M}_1)\mathbf{T}-k_2R(\mathbf{T},\mathbf{M}_2)\mathbf{T}.$$

By (4), we see that γ is a biharmonic curve if and only if

$$k_1'k_1+k_2'k_2=0,$$
$$k_1''-k_1^3-k_1k_2^2=k_1R(\mathbf{T},\mathbf{M}_1,\mathbf{T},\mathbf{M}_1)$$
$$+k_2R(\mathbf{T},\mathbf{M}_2,\mathbf{T},\mathbf{M}_1),\qquad (15)$$
$$k_2''-k_2^3-k_2k_1^2=k_1R(\mathbf{T},\mathbf{M}_1,\mathbf{T},\mathbf{M}_2)$$
$$+k_2R(\mathbf{T},\mathbf{M}_2,\mathbf{T},\mathbf{M}_2).$$

Making necessary calculations from (15), we have

$$k_1^2+k_2^2=\text{constant}=C\neq 0,$$
$$k_1''-Ck_1=k_1R(\mathbf{T},\mathbf{M}_1,\mathbf{T},\mathbf{M}_1)$$
$$+k_2R(\mathbf{T},\mathbf{M}_2,\mathbf{T},\mathbf{M}_1),\qquad (16)$$
$$k_2''-Ck_2=k_1R(\mathbf{T},\mathbf{M}_1,\mathbf{T},\mathbf{M}_2)+k_2R(\mathbf{T},\mathbf{M}_2,\mathbf{T},\mathbf{M}_2).$$

A direct computation using (7) yields

$$R(\mathbf{T},\mathbf{M}_1,\mathbf{T},\mathbf{M}_1)=\frac{1}{4}-\left(M_2^3\right)^2,$$
$$R(\mathbf{T},\mathbf{M}_2,\mathbf{T},\mathbf{M}_1)=-M_1^3M_2^3,\qquad (17)$$
$$R(\mathbf{T},\mathbf{M}_1,\mathbf{T},\mathbf{M}_2)=M_1^3M_2^3,$$
$$R(\mathbf{T},\mathbf{M}_2,\mathbf{T},\mathbf{M}_2)=\frac{1}{4}-\left(M_1^3\right)^2.$$

These, together with (11), complete the proof of the theorem.

Definition 3.2. A regular curve $\gamma:I\to Heis^3$ is called a slant helix, provided the unit vector $\mathbf{M}_1$ of the curve γ has constant angle θ with some fixed spacelike unit vector u, that is

$$g(\mathbf{M}_1(s),u)=\cos\theta \text{ for all } s\in I.\qquad (18)$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as B-slant helix.

Theorem 3.3. Let $\gamma:I\to Heis^3$ be a unit speed curve with non-zero natural curvatures. Then γ is a B-slant helix if and only if

$$\frac{k_1}{k_2}=\text{constant}.\qquad (19)$$

Proof: Differentiating (18) and by using the Bishop frame (3.3), we find

$$g(\nabla_{\mathbf{T}}\mathbf{M}_1,u)=g(k_1\mathbf{T},u)=k_1g(\mathbf{T},u)=0.$$

From the above equation, we get

$$g(\mathbf{T},u)=0.$$

Again differentiating from the last equality, we obtain

$$\begin{aligned} g(\nabla_{\mathbf{T}}\mathbf{T},u) &= g(k_1\mathbf{M}_1 + k_2\mathbf{M}_2, u) \\ &= k_1 g(\mathbf{M}_1, u) + k_2 g(\mathbf{M}_2, u) \\ &= k_1\cos\theta + k_2\sin\theta = 0. \end{aligned}$$

Using the above equation, we get

$$\frac{k_1}{k_2} = -\tan\theta = \text{constant}. \quad (20)$$

The converse statement is trivial. This completes the proof.

Theorem 3.4. Let $\gamma : I \to Heis^3$ be a unit speed biharmonic B-slant helix with non-zero natural curvatures. Then

$$k_1 = \text{constant and } k_2 = \text{constant}. \quad (21)$$

Proof: Suppose that γ be a unit speed biharmonic slant helix. From (19), we have

$$k_1 = -\tan\theta k_2. \quad (22)$$

On the other hand, using the first equation of (14), we obtain that k_2 is a constant. Similarly, k_1 is a constant.

Hence, the proof is completed.

Corollary 3.5. $\gamma : I \to Heis^3$ is biharmonic B-slant helix if and only if

$$\begin{aligned} &k_1 = \text{constant} \neq 0, \\ &k_2 = \text{constant} \neq 0, \\ &\frac{M_1^3 M_2^3}{C + \frac{1}{4} - (M_2^3)^2} = -\tan\theta, \\ &\frac{C + \frac{1}{4} - (M_1^3)^2}{M_1^3 M_2^3} = \tan\theta, \end{aligned} \quad (23)$$

where $-\tan\theta = \dfrac{k_1}{k_2}$ and $C = k_1^2 + k_2^2$.

Corollary 3.6. If $\gamma : I \to Heis^3$ is biharmonic B-slant helix, then

$$(M_1^3 M_2^3)^2 + \left(C + \frac{1}{4} - (M_2^3)^2\right)\left(C + \frac{1}{4} - (M_1^3)^2\right) = 0, \quad (24)$$

where $C = k_1^2 + k_2^2$.

Theorem 3.7. Let $\gamma : I \to Heis^3$ be a unit speed biharmonic B-slant helix with non-zero natural curvatures. Then, the parametric equation of γ are

$$\begin{aligned} x(s) &= \frac{1}{k}\cos\theta\sin\Omega(s) + c_1, \\ y(s) &= -\frac{1}{k}\cos\theta\cos\Omega(s) + c_2, \\ z(s) &= (\sin\theta + \cos^2\theta)s - \frac{1}{4k}\cos 2\Omega(s) - \frac{c_1}{k}\cos\Omega(s) + c_3, \end{aligned} \quad (25)$$

where c_1, c_2, c_3 are constants of integration, $\Omega(s) = \left(\sqrt{k_1^2 + k_2^2 - \cos^2\theta} - \sin\theta\right)s + \sigma$ and $k = \sqrt{k_1^2 + k_2^2 - \cos^2\theta} - \sin\theta.$

Proof: Assume that $\gamma : I \to Heis^3$ be a unit speed biharmonic B-slant helix. So, without loss of generality, we take the axis of γ parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{M}_1, \mathbf{e}_3) = M_1^3 = \cos\theta, \quad (26)$$

where θ is constant angle.

On the other hand, the vector $\mathbf{M}_1$ is a unit vector, so the following condition is satisfied

$$(M_1^1)^2 + (M_1^2)^2 = 1 - \cos^2\theta. \quad (27)$$

Noting that $\cos^2\theta + \sin^2\theta = 1$, we have

$$(M_1^1)^2 + (M_1^2)^2 = \sin^2\theta. \quad (28)$$

The general solution of (28) can be written in the following form

$$\begin{aligned} M_1^1 &= \sin\theta\cos\Omega(s), \\ M_1^2 &= \sin\theta\sin\Omega(s), \end{aligned} \quad (29)$$

where Ω is an arbitrary function of s.

So, substituting the components M_1^1, M_1^2 and M_1^3 in the second equation of (3.6), we have the following equation

$$\begin{aligned} \mathbf{M}_1 &= \sin\theta\cos\Omega(s)\mathbf{e}_1 \\ &\quad + \sin\theta\sin\Omega(s)\mathbf{e}_2 + \cos\theta\mathbf{e}_3. \end{aligned} \quad (30)$$

On the other hand, using Bishop formulas (10) and (6), we have

$$\mathbf{M}_2 = \sin\Omega(s)\mathbf{e}_1 - \cos\Omega(s)\mathbf{e}_2,$$
$$\mathbf{T} = \cos\theta\cos\Omega(s)\mathbf{e}_1 + \cos\theta\sin\Omega(s)\mathbf{e}_2 - \sin\theta\mathbf{e}_3.$$

The covariant derivative of the vector field $\mathbf{T}$ is:

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' + T_2T_3)\mathbf{e}_1 + (T_2' - T_1T_3)\mathbf{e}_2 + T_3'\mathbf{e}_3.$$

Therefore, we use Bishop formulas (10) and the above equation we get

$$\Omega(s) = \left(\sqrt{k_1^2 + k_2^2 - \cos^2\theta} - \sin\theta\right)s + \sigma,$$

where σ is a constant of integration.
Thus proof is complete.

4. Tangent Bishop spherical images of Biharmonic B-Slant Helix in the Heisenberg group Heis^3

Definition 4.1. Let $\gamma : I \to Heis^3$ be a regular curve in Heis^3. If we translate the first (tangent) vector field of Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\xi = \xi(s_\xi)$. This curve is called Tangent Bishop spherical image or indicatrix of the curve $\gamma = \gamma(s)$.

Let $\xi = \xi(s_\xi)$ be tangent Bishop spherical image of a regular curve $\gamma = \gamma(s)$. One can differentiate ξ with respect to s:

$$\xi' = \frac{d\xi}{ds_\xi}\frac{ds_\xi}{ds} = k_1\mathbf{M}_1 + k_2\mathbf{M}_2. \qquad (31)$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_ξ by a dot. In terms of Bishop frame vector fields (10), we have the tangent vector of the spherical image as follows:

$$\mathbf{T}_\xi = \frac{k_1\mathbf{M}_1 + k_2\mathbf{M}_2}{\sqrt{k_1^2 + k_2^2}}, \qquad (32)$$

where

$$\frac{ds_\xi}{ds} = \sqrt{k_1^2 + k_2^2} = \kappa(s).$$

Theorem 4.2. Let $\gamma : I \to Heis^3$ be a unit speed biharmonic slant helix with non-zero natural curvatures. Then the parametric equation of the tangent spherical indicatrix of γ are

$$\begin{aligned} x_\xi(s) &= \cos\theta\cos\Omega(s), \\ y_\xi(s) &= \cos\theta\sin\Omega(s), \\ z_\xi(s) &= \cos^2\theta\cos\Omega(s)\sin\Omega(s) - \sin\theta, \end{aligned} \qquad (33)$$

where

$$\Omega(s) = \left(\sqrt{k_1^2 + k_2^2 - \cos^2\theta} - \sin\theta\right)s + \sigma.$$

Proof: Using (31) we have

$$\xi' = \nabla_{\mathbf{T}}\mathbf{T} = k_1\mathbf{M}_1 + k_2\mathbf{M}_2. \qquad (34)$$

Substituting (20) in (34), we get

$$\xi = \cos\theta\cos\Omega(s)\mathbf{e}_1 + \cos\theta\sin\Omega(s)\mathbf{e}_2 - \sin\theta\mathbf{e}_3. \qquad (35)$$

These, together with (30) give (33).

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