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# Function spaces on tensor product of semigroups

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**Abstract**

In this paper, we characterize the function space and $L^1$-space of the [topological] tensor product of [topological] semigroups. As a consequence, for arbitrary [topological] groups $G_1$and $G_2$, it will be shown that $G_1 \times G_2$ is an extension of $G_1 \otimes_\sigma G_2$ by a proper normal subgroup N i,e.$G_1 \otimes_\sigma G_2 = \frac{G_1 \times G_2}{N}$.

***Keywords**:* Topological semigroup; compactification; tensor product

## 1. Introduction

For many algebraic and analytic structures the tensor product has been defined in many different ways. Following Howie [1], for any two nonempty sets, especially for semigroups, $X$ and $Y$ tensor product $X \otimes Y$ has been defined as the quotient space $\frac{X \times Y}{\tau}$, in which the equivalence relation $\tau$ is generated by the set

$$\left\{\left((xx', y), (x, x'y)\right): x, x' \in X, y \in Y\right\}.$$

Note that this structure does not necessarily inherit the algebraic structure of $X$ and $Y$. In other words $X \otimes Y$, as defined previously, is just a quotient space rather than a semigroup when $X$ and $Y$ are two semigroups with identities. The topological tensor product of topological semigroups was introduced by Medghalchi and the author in 2004 [2, 3]. The special characteristic of this structure is completely different from the Sherier Product [4] and Semiditect Product [5]. The ideal structure of topological tensor product of topological semigroups and their results were characterized in [2]. Since compactification of semigroups and more general function spaces of semigroups play an important role in analysis on semigroups, this tool has been used by many authors (see [5-8], for example). The characterization of almost periodic compactification and weak, almost periodic compactification of topological tensor product of topological semigroups was developed in [3]. An important class of semigroups which has been studied extensively from various points of view, is the class of completely 0-simple and completely simple semigroups [9, 10]. By applying the topological tensor product techniques, the function spaces of 0-simple and completely simple semigroup are characterized by the author [11]. These facts led to the motivation to study function spaces of [topological] tensor product of [topological] semigroups.

This paper is organized as follows. In section two, we introduce our notation and the structure of [topological] tensor product of [topological] semigroups. Section three is devoted to discussing the concepts of $\mathcal{P}$-compactifications where $\mathcal{P}$ is an arbitrary property of compactifications, and function spaces on the [topological] tensor product. In section four we characterize the $l^1$-space on tensor products. Finally, in the last section we apply the results of previous sections to show that $G_1 \otimes_\sigma G_2 = \frac{G_1 \times G_2}{N}$ for an appropriate normal subgroup $N$.

## 2. Preliminaries

In this paper we assume that each semigroup possesses an identity. A semigroup $S$ is called a right [left] topological semigroup if there is a topology on $S$ such that $s \longrightarrow st\,[s \longrightarrow ts]$ is continuous for all $t \in S$. A semigroup $S$ is called semitopological [topological] semigroup if $(s,t) \longrightarrow st$ is separately [jointly] continuous. A topological semigroup $S$ is called a topological group if the inverse mapping $s \longrightarrow s^{-1}$ is continuous.

Let $S$ be a topological semigroup. A right topological semigroup $X$ is called a semigroup compactification of $S$ if $X$ is compact, Hausdorff

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and $\psi: S \longrightarrow X$ is a continuous homomorphism such that $\overline{\psi(S)} = X, \psi(S) \subseteq \Lambda(X)$, where $\Lambda(X) = \{ t \in X: \ s \longrightarrow ts: X \longrightarrow X, is\ continuos \}$. We say that the compactification $(\psi, X)$ of $S$ has left [right] jointly continuity property if the mapping $(s, x) \longrightarrow \psi(s)x$ $[(x, s) \longrightarrow x\psi(s)]$ is continuous.

Let $\mathfrak{B}(S)$ be the $C^*$-algebra of all bounded complex valued functions on $S$, $\mathcal{F}$ be a unital $C^*$-subalgebra of $\mathcal{B}(S)$, $S^{\mathcal{F}}$be the set of all multiplicative means on $\mathcal{F}$ and $\varepsilon: S \longrightarrow S^{\mathcal{F}}$ be the evaluation mapping. We say that $\mathcal{F}$ is $m$-admissible if $T_\mu(\mathcal{F}) \subseteq \mathcal{F}$ for all $\mu \in S^{\mathcal{F}}$, where $T_\mu(f)(s) = \mu\big(L_s(f)\big)$, $s \in S, f \in \mathcal{F}$. If we equip $S^{\mathcal{F}}$ with the Gelfand topology then $S^{\mathcal{F}}$ with multiplication $\mu\nu(f) = \mu(T_\nu(f)$, $\mu, \nu \in S^{\mathcal{F}}$ is a compact Hausdorff right topological semigroup. Moreover, the evaluation mapping is a continuous homomorphism into a dense subsemigroup of $S^{\mathcal{F}}$ which is contained in the topological center of $S^{\mathcal{F}}$. Now, if $(\psi, X)$ is a compactification of $S$, then $\psi^*(C(X))$ is an $m$-admissible subalgebra of $C(S)$. Conversely, if $\mathcal{F}$ is an $m$-admissible subalgebra of $C(S)$, then there exists a unique (up to isomorphism) compactification $(\psi, X)$ of $S$ such that $\psi^*\big(C(X)\big) = \mathcal{F}$. In other words, the compactification corresponding to the $m$-admissible subalgebra $\mathcal{F}$ is $(\varepsilon, S^{\mathcal{F}})$. Moreover, $\varepsilon^*\big(C(S^{\mathcal{F}})\big) = \mathcal{F}$ [1].

Let $S$ and $T$ be semitopological semigroups with semigroup compactifications $S'$ and $T'$. A continuous function $\varphi': S' \longrightarrow T'$ is an extension of the continuous function $\varphi: S \to T$ if $\varphi' o \varepsilon_S = \varepsilon_T o \varphi$ and $\varphi'$ is uniquely determined by $\varphi$. Such an extension exists if and only if $\varphi^*(B) \subseteq A$, where $A$ and $B$ are the associated function spaces of the compactifications. Let $S'$ and $S''$ be compactifications of $S$. Then $S'$ is a factor of $S''$ if the identity map on $S$ has an extension $\varphi: S'' \longrightarrow S'$. A compactification with a given property $\mathcal{P}$ is called a $\mathcal{P}$-compactification. A universal $\mathcal{P}$-compactification of $S$ is a $\mathcal{P}$-compactification of which, every $\mathcal{P}$-compactification of $S$ is a factor. Universal $\mathcal{P}$-compactifications, if they exist, are unique (up to isomorphism). We denote the universal $\mathcal{P}$-compactification of $S$ by $S^{\mathcal{P}}$. We refer the reader to [12] for more results about compactifications of semigroups.

Following Howie [1], for a relation $l$ on a set $X$, we denote $l^\infty$ by $l^\infty = \{l^n: n \geq 1\}$, where $l^n = lolo \ldots ol$. We recall that the equivalence generated by $l$ is the intersection of all equivalence relations containing $l$ [1, sec 1.4]. Following [1, Lemma 1.4.8], if $l$ is a reflexive relation on $X$, then $l^\infty$ is the smallest transitive relation on $X$ containing $l$. We denote $[l \cup l^{-1} \cup 1_X]^\infty$ by $l^e$, where $l^{-1} = \{(y, x) : (x, y) \in l\}$ and $1_X = \{(x, x) : x \in X\}$. By [1, Proposition 1.4.9], $l^e$ is an equivalence generated by $l$. So, if $l^\infty$ is an equivalence generated by $l$, then $(x, y) \in l^e$ if and only if, either $x = y$ or, for some $n \in \mathbb{N}$, there is a sequence of translations $x = z_1 \longrightarrow z_2 \longrightarrow \cdots \longrightarrow z_n = y$ such that, for each $1 \leq i \leq n-1$, either $(z_i, z_{i+1}) \in l$ or, $(z_{i+1}, z_i) \in l$ [1, Proposition 1.4.10].

An equivalence $\tau$ on a semigroup $S$ is called a left [right] $S$-congruence if $(x, y) \in \tau$ and $s \epsilon S$, then $(sx, sy) \in \tau$ $[(xs, ys) \in \tau]$, and is called an $S$-congruence if it is both a right and a left $S$-congruence.

Let $S, T$ be two [topological] semigroups with identities and $X$ be a non-empty [topological] space. Then $X$ is called a [topological] left $S$-system if there is an action $(s, x) \longrightarrow sx$ of $S \times X$ into $X$ which [is jointly continuous and] $s_1(s_2x) = (s_1s_2)x$, $1_Sx = x$ $(s_1, s_2 \in S, x \in X)$. A [topological] right $S$-system is defined similarly. A [topological] left $S$-system which is also a [topological] right $T$-system is called a [topological] $(S, T)$-bisystem if $(sx)t = s(xt)$ $(s \in S, t \in T, x \in X)$.

Let $X, Y$ be two [topological] left $S$-systems and $\varphi: X \longrightarrow Y$ be a [continuous] map. We say that $\varphi$ is a [topological] left $S$-map if $\varphi(sx) = s\varphi(x)(x \in X, s \in S)$. Similarly, we can define a [topological] right $T$-map.

Now, let $X$ be a [topological] $(S, U)$-bisystem, $Y$ be a [topological] $(U, T)$-bisystem and $Z$ be a [topological] $(S, T)$-bisystem. Then $X \times Y$ has the structure of a [topological]$(S, T)$-bisystem $(i.e., s_1s_2(x, y) = s_1(s_2x, y)$, $1_S(x, y) = (x, y)$, $(x, y)t_1t_2 = (x, yt_1)t_2$, $(x, y)1_T = (x, y)$, for all $s_1, s_2 \in S$ and $t_1, t_2 \in T)$.

Let [$X \times Y$ be equipped with the product topology and] $\beta: X \times Y \longrightarrow Z$ be a [topological] $(S, T)$-map (i.e., $\beta$ is a [topological] left $S$-map and a [topological] right $T$-map). We say that $\beta$ is a [topological] bimap if further $\beta(xu, y) = \beta(x, uy)$ $(u \in U)$. Let $S$ and $T$ be two [topological] semigroups with identities $1_S, 1_T$, respectively. Let $\sigma: S \longrightarrow T$ be a continuous homomorphism. Then $T$ can obviously be regarded as a [topological] $(S, T)$-bisystem by $s * t = st$ $(s \in S, t \in T)$, and $S$ can be regarded as a [topological] $(S, S)$-bisystem where the action of $S$ on $S$ is just its multiplication. Let $C$ be a [topological] $(S, T)$-bisystem and $\beta: S \times T \longrightarrow C$be a [topological] $(S, T)$-map. We say that $\beta$ is a [topological] $\sigma$-bimap if $\beta(ss', t) = \beta\big(s, \sigma(s')t\big)$ $(s, s \in S, t \in T)$.

By a [topological] tensor product we mean a pair $(P, \varphi)$ where $P$ is a [topological] $(S, T)$-bisystem and $\varphi: S \times T \longrightarrow P$ is a [topological] $\sigma$-bimap such that for every [topological] $(S, T)$-bisystem $C$ and every [topological] $\sigma$-bimap $\beta: S \times T \longrightarrow P$ there exists a unique [topological] $(S, T)$-map $\bar{\beta}: P \longrightarrow C$ such that the diagram

$$\begin{array}{c} S \times T \xrightarrow{\varphi} P \\ \beta \downarrow \ \swarrow \overline{\beta} \\ C \end{array}$$

commutes [2, 3].

In the following theorem the existence of the [topological] tensor product of $S$ and $T$ with respect to $\sigma$, which is denoted by $S \otimes_\sigma T$, was proved.

**Theorem 2.1.** [3, Theorem 3.3] Let S and T be two [topological] semigroups with identities, and $\sigma: S \longrightarrow T$ be a [continuous] homomorphism. Then there is a unique [topological] tensor product of S and T.

**proof:** (sketch) We regard $S \times T$[with the product topology] as a [topological] $(S,T)$-bisystem. Let $\tau$ be the equivalence relation on $S \times T$ generated by $\{((ss', t), (s, \sigma(s')t)) : s, s' \in S, t \in T\}$. Let

$$\rho = \{(a,b) \in (S \times T) \times (S \times T) : u, v \in S \times T, (uav, ubv) \in \tau\}.$$

By [1, Proposition 1.5.10], $\rho$ is the largest congruence on $S \times T$ contained in $\tau$. Now, we denote $\frac{S\times T}{\rho}$ by $S \otimes_\sigma T$ and the elements of $\frac{S\times T}{\rho}$ by $s \otimes_\sigma t$. We use the techniques of [1, Proposition 8.1.8] to show that if $s_1 \otimes_\sigma t_1 = s_2 \otimes_\sigma t_2$ then $s_1 = s_2$ and $t_1 = t_2$, or there exist $a_1, a_2, \dots a_{n-1} \in S, b_1, \dots, b_{n-1} \in T, u_1, \dots, u_n, v_1, \dots, v_n \in S$ (see the introduction) such that

$$s_1 = a_1 u_1, \qquad \sigma(u_1)t_1 = \sigma(v_1)b_1,$$

$$a_1 v_1 = a_2 u_2, \qquad \sigma(u_2)b_1 = \sigma(v_2)b_2,$$

$$\vdots$$

$$a_i v_i = a_{i+1}u_{i+1}, \quad \sigma(u_{i+1})b_i = \sigma(v_{i+1})b_{i+1} \quad (i=2,\dots, n-2), (*)$$

$$\vdots$$

$$a_{n-1}v_{n-1} = s_2 u_n, \quad \sigma(u_n)b_{n-1} = t_2.$$

Let $\varphi: S \times T \longrightarrow S \otimes_\sigma T$ be defined by $\varphi(s,t) = s \otimes_\sigma t$. $\varphi$ is a [topological] $\sigma$-bimap and $(S \otimes_\sigma T, \varphi)$ is a unique (up to isomorphism) [topological] tensor product of $S$ and $T$.

## 3. Function spaces on topological tensor product of topological semigroups

Let $S$ and $T$ be two topological semigroups and $S \otimes_\sigma T$ be their topological tensor product. Let $\mathcal{P}$ be the property of compactifications. In this setting it is natural to ask whether universal $\mathcal{P}$-compactification of $(S \otimes_\sigma T)^{\mathcal{P}}$ of $S \otimes_\sigma T$ is canonically isomorphic to $S^{\mathcal{P}} \otimes_\sigma T^{\mathcal{P}}$. Results of this type are known for $ap$-compactification and $sap$-compactification in [5]. In this chapter we generalize these results, obtaining compactification theorem of the form $(S \otimes_\sigma T)^{\mathcal{P}} \simeq S^{\mathcal{P}} \otimes_\sigma T^{\mathcal{P}}$. Remember that the following results were proved in [3].

**Theorem3.1.** [3, Theorem 3.6] Let $(\psi_1, X_1)$ and $(\psi_2, X_2)$ be two topological semigroup compactifications of topological semigroups S and T, respectively. Let $\sigma: S \longrightarrow T$, $\eta: X_1 \longrightarrow X_2$ be two continuous homomorphisms such that $\eta o \psi_1 = \psi_2 o \sigma$. Then $X_1 \otimes_\sigma X_2$ is a topological semigroup compactification of $S \otimes_\sigma T$.

**Theorem3.2.** [3, Corollary 3.7] Let $(\varepsilon_i, S_i^{\mathcal{F}_i})(i = 1,2)$ be two canonical compacitifications of topological semigroups $S_i$ such that $S_i^{\mathcal{F}_i}$ is a topological semigroup. Let $\sigma: S \longrightarrow T$ be a continuous homomorphism such that $\sigma^*(\mathcal{F}_2) \subseteq \mathcal{F}_1$. Then $S_1^{\mathcal{F}_1} \otimes_\sigma S_2^{\mathcal{F}_2}$ exists and is a compactification of $S \otimes_\sigma T$.

**Theorem 3.3.** Let $S$ and $T$ be two topological semigroups with identities, and $\boldsymbol{\sigma}$ be a continuous homomorphism of $S$ into $T$. Let $S^{\mathcal{P}}.T^{\mathcal{P}}$ and $(S \otimes_\sigma T)^{\mathcal{P}}$ be the universal topological semigroup $\mathcal{P}$-compactifications of $S$, $T$ and $S \otimes_\sigma T$, respectively where $\mathcal{P}$ has joint continuity property and is invariant under multiplication. Then $(S \otimes_\sigma T)^{\mathcal{P}} \simeq S^{\mathcal{P}} \otimes_\sigma T^{\mathcal{P}}$.

**Proof:** Let $(\varepsilon_{S\otimes_\sigma T}, (S \otimes_\sigma T)^{\mathcal{P}})$, $(\varepsilon_S, S^{\mathcal{P}})$, $(\varepsilon_T, T^{\mathcal{P}})$ be universal topological semigroup $\mathcal{P}$-compactifications of $S \otimes_\sigma T$, $S$ and $T$ respectively. By Theorem 3.2, $(\delta_{S\otimes_\sigma T}, S^{\mathcal{P}} \otimes_\eta T^{\mathcal{P}})$ is a topological semigroup compactification of $S \otimes_\sigma T$. The universal property of $\mathcal{P}$-compactification $(\varepsilon_{S\otimes_\sigma T}, (S \otimes_\sigma T)^{\mathcal{P}})$ gives a continuous homomorphism $\phi: (S \otimes_\sigma T)^{\mathcal{P}} \longrightarrow S^{\mathcal{P}} \otimes_\eta T^{\mathcal{P}}$ such that the following diagram commutes.

$$\begin{array}{c} S \otimes_\sigma T \xrightarrow{\varepsilon_{S\otimes_\sigma T}} (S \otimes_\sigma T)^{\mathcal{P}} \\ \delta_{S\otimes_\sigma T} \downarrow \qquad \swarrow \phi \\ S^{\mathcal{P}} \otimes_\eta T^{\mathcal{P}}. \end{array}$$

Also, since $(\varepsilon_S \times \varepsilon_T, (S \times T)^{\mathcal{P}}$ is a topological semigroup compactification of $S \times T$, via the homomorphism $\theta: S \times T \xrightarrow{\pi} S \otimes_\sigma T \xrightarrow{\varepsilon_{S\otimes_\sigma T}} (S \otimes_\sigma T)^{\mathcal{P}}$, there is a continuous homomorphism $\phi_1: (S \times T)^{\mathcal{P}} \longrightarrow (S \otimes_\sigma T)^{\mathcal{P}}$ such that the following diagram commutes.

$$\begin{array}{c} S \times T \xrightarrow{\theta} (S \otimes_\sigma T)^{\mathcal{P}} \\ \varepsilon_S \times \varepsilon_T \downarrow \qquad \swarrow \phi_1 \\ (S \times T)^{\mathcal{P}}. \end{array}$$

On the other hand, $(S\times T)^{\mathcal{P}}=S^{\mathcal{P}}\times T^{\mathcal{P}}$, thus we can assume that $\phi_1: S^{\mathcal{P}}\times T^{\mathcal{P}}\longrightarrow (S\otimes_\sigma T)^{\mathcal{P}}$. Observe that $\phi_1$ preserves congruence, because, if $vv'\otimes_\eta \mu = v\otimes_\eta \eta(v')\mu$, where $v, v'\in S^{\mathcal{P}}, \mu\in T^{\mathcal{P}}$, we can get the nets $\{s_\alpha\}$, $\{s'_\beta\}$ in $S$ and $\{t_\gamma\}$ in $T$ such that $\lim_\alpha \varepsilon_S(s_\alpha)=v$, $\lim_\beta \varepsilon_S(s'_\beta)=v'$ and $\lim_\gamma \varepsilon_T(t_\gamma)=\mu$. Therefore,

$$\begin{aligned}\phi_1(vv'\otimes_\eta \mu) &= \phi_1(\lim_{\alpha,\beta,\gamma} \varepsilon_S\times\varepsilon_T(s_\alpha s'_\beta, t_\gamma))\\ &= \lim_{\alpha,\beta,\gamma}\phi_1(\varepsilon_S\times\varepsilon_T(s_\alpha s'_\beta, t_\gamma))\\ &= \lim_{\alpha,\beta,\gamma}\varepsilon_{S\otimes_\sigma T}(\pi_1(s_\alpha s'_\beta, t_\gamma))\\ &= \lim_{\alpha,\beta,\gamma}\varepsilon_{S\otimes_\sigma T}(\pi_1(s_\alpha, \sigma(s'_\beta)t_\gamma)).\end{aligned}$$

For the reverse calculations we have

$$\begin{aligned}\phi_1(v\otimes_\eta \eta(v')\mu) &= \phi_1(\lim_{\alpha,\beta,\gamma}\varepsilon_S\times\varepsilon_T(s_\alpha, \sigma(s'_\beta)t_\gamma))\\ &= \lim_{\alpha,\beta,\gamma}\varepsilon_{S\otimes_\sigma T}(\pi_1(s_\alpha, \sigma(s'_\beta), t_\gamma)).\end{aligned}$$

Now, by an argument similar to equations (*) of Theorem 2.1, $\phi_1$ preserves congruence. Thus there exists a continuous homomorphism $\phi_2$: $S^{\mathcal{P}}\otimes_\eta T^{\mathcal{P}}\longrightarrow (S\otimes_\sigma T)^{\mathcal{P}}$ such that the following diagram commutes.

$$\begin{array}{ccc} S^{\mathcal{P}}\times T^{\mathcal{P}} & \xrightarrow{\phi_1} & (S\otimes_\sigma T)^{\mathcal{P}}\\ \pi_2 \downarrow & \nearrow \phi_2 & \\ S^{\mathcal{P}}\otimes_\eta T^{\mathcal{P}}. & & \end{array}$$

Now, $\phi o\phi_2$ is an identity map on $S^{\mathcal{P}}\otimes_\eta T^{\mathcal{P}}$, because if $v\otimes_\eta \mu\in S^{\mathcal{P}}\otimes_\eta T^{\mathcal{P}}$, then we can find a net $\{s_\alpha\}$ in $S$ and $\{t_\beta\}$ in $T$ such that $\lim_\alpha \varepsilon_S(s_\alpha)=v$, and $\lim_\beta \varepsilon_T(t_\beta)=\mu$. Thus

$$\begin{aligned} o\phi_2(v\otimes_\eta \mu) &= \phi o\phi_2(\pi_2(v,\mu))\\ &= \lim_{\alpha,\beta}\phi(\phi_1(\varepsilon_S\times\varepsilon_T(s_\alpha, t_\beta))\\ &= \lim_{\alpha,\beta}\phi(\theta(s_\alpha, t_\beta))\\ &= \lim_{\alpha,\beta}\phi(\pi_1(\varepsilon_{S\otimes_\sigma T}(s_\alpha\otimes_\sigma t_\beta))\\ &= \lim_{\alpha,\beta}\delta_{S\otimes_\sigma T}(s_\alpha\otimes_\sigma t_\beta) = v\otimes_\eta \mu.\end{aligned}$$

Therefore $(S\otimes_\sigma T)^{\mathcal{P}}\simeq S^{\mathcal{P}}\otimes_\sigma T^{\mathcal{P}}$.

**Corollary 3.4.** Let $S$ and $T$ be two topological semigroups with identities, and $\sigma: S\longrightarrow T$ be a continuous homomorphism. Then $(S\otimes_\sigma T)^{ap}\simeq S^{ap}\otimes_\sigma T^{ap}$.

**Corollary 3.5.** Let $S$ and $T$ be two topological semigroups with identities, and $\sigma: S\longrightarrow T$ be a continuous homomorphism. Then $(S\otimes_\sigma T)^{sap}\simeq S^{sap}\otimes_\sigma T^{sap}$.

## 4. $L^1$-Spaces on tensor products of semigroups

We recall that for semigroup $S$,

$$l^1(S)=\{f: f: S\longrightarrow \mathbb{C}, \sum_{s\in S}|f(s)|<\infty\}.$$

With pointwise addition and scalar multiplication, with convolution

$$(f*g)(s)=\sum_{s=uv} f(u)g(v)$$

as product ($(f*g)(s)$ =0 if $s=uv$ has no solutions) and with the norm

$$\|f\|_1=\sum_{s\in S}|f(s)|$$

is a Banach algebra that we call it Discrete semigroup algebra.

**Theorem 4.1.** Let S and T be two semigroups with identities, and $\sigma: S\longrightarrow T$ be a continuous homomorphism. Then $l^1(S\otimes_\sigma T)\simeq \frac{l^1(S\times T)}{k}$, where k is a closed subspace of $l^1(S\times T)$.

**Proof:** For every $f\in l^1(S\times T)$, consider the function

$$(s,t)\longrightarrow \sum_{u\otimes_\sigma v=s\otimes_\sigma t} f(u,v).$$

Since this function is constant on each congruence class, it is of the form $\bar{f}o\pi_{S\otimes_\sigma T}$, where $\bar{f}$ is a function on the quotient space $l^1(S\otimes_\sigma T)$. Now put

$$\psi: l^1(S\times T)\longrightarrow l^1(S\otimes_\sigma T)$$
$$\psi(f)=\bar{f}$$

In fact,

$$\psi(f)(s\otimes_\sigma t)=\bar{f}(s\otimes_\sigma t)=\sum_{u\otimes_\sigma v=s\otimes_\sigma t} f(u,v).$$

We have $\psi(f*g)=\psi(f)*\psi(g)$, for

$$\begin{aligned}&\psi(f*g)(s\otimes_\sigma t)=\overline{f*g}(s\otimes_\sigma t)\\ &=\sum_{u\otimes_\sigma v=s\otimes_\sigma t} f*g(u,v)\\ &=\sum_{u\otimes_\sigma v=s\otimes_\sigma t}[\sum_{(u,v)=(p,q)(n,m)} f(p,q)g(n,m)]\\ &=\sum_{u\otimes_\sigma v=s\otimes_\sigma t}[\sum_{u=pn,\ v=qm} f(p,q)g(n,m)]\\ &=\sum_{s\otimes_\sigma t=pn\otimes_\sigma qm}[\sum_{p\otimes_\sigma q=p'\otimes_\sigma q',\ n\otimes_\sigma m=m'\otimes_\sigma n'} f(p',q')g(n',m')]\\ &=\sum_{s\otimes_\sigma t=pn\otimes_\sigma qm}[\sum_{p\otimes_\sigma q=p'\otimes_\sigma q'} f(p',q')]\sum_{n\otimes_\sigma m=n'\otimes_\sigma m'} g(n',m')]\\ &=\sum_{s\otimes_\sigma t=(p\otimes_\sigma q)(n\otimes_\sigma m)}\bar{f}(p,q)\bar{g}(n,m)\\ &=\psi(f)*\psi(g)(s\otimes_\sigma t).\end{aligned}$$

Also, we assert that $\psi$ maps $l^1(S\times T)$ onto $l^1(S\otimes_\sigma T)$. Indeed, let any $\bar{f}\in l^1(S\otimes_\sigma T)$ be given; then we can obtain an $f\in l^1(S\times T)$ such that $\psi(f)=\bar{f}$ as follows. Put

$$N = \{\, s \otimes_\sigma t : \bar{f}(s \otimes_\sigma t) \neq 0 \,\}$$

and

$$M = \pi^{-1}_{S\otimes_\sigma t}(\mathrm{N}).$$

Now define for $(s,t) \in S \times T$,

$$f(s,t) = \begin{cases} \bar{f} o \pi_{S\otimes_\sigma T}(s,t), & \pi_{S\otimes_\sigma T}(s,t) \in N \\ 0 & , \ othewise \end{cases}$$

Then $f \in l^1(S \times T)$, for

$$\sum_{(s,t)\in S\times T} |f(s,t)| = \sum_{\pi_{S\otimes_\sigma T}(s,t)\in N} \left|\bar{f} o \pi_{S\otimes_\sigma T}(s,t)\right|$$
$$= \sum_{\pi_{S\otimes_\sigma T}(s,t)\in N} \left|\bar{f}(s \otimes_\sigma t)\right| < \infty$$

and

$$\psi(f) = \bar{f}.$$

Let
$k = \ker(\psi)$={ $f \in l^1(S \times T)$: $\psi(f) = 0$}.

It is clear $\psi$ is a linear operator from $l^1(S \times T)$ onto $l^1(S \otimes_\sigma T)$. Then

$$l^1(S \otimes_\sigma T) \simeq \frac{l^1(S \times T)}{k}.$$

## 5. Topological tensor products and extension group

In this section we study some properties of [topological] tensor products. We will show for arbitrary [topological] groups $G_1$ and $G_2$, $G_1 \times G_2$ is an extension of $G_1 \otimes_\sigma G_2$ by a proper [closed] normal subgroup $N$ i,e. $G_1 \otimes_\sigma G_2 = \frac{G_1\times G_2}{N}$. Also, by extension argument we get a number of interesting results on tensor product.

**Lemma 5.1.** Let $G_1$and $G_2$, be two [topological] groups and $\sigma: G_1 \longrightarrow G_2$ be a [continuous] homomorphism. Let $G_1 \otimes_\sigma G_2 = \frac{G_1\times G_2}{\rho}$, $\pi: G_1 \times G_2 \longrightarrow \frac{G_1\times G_2}{\rho}$ be the quotient map. Then $s \otimes_\sigma t = a \otimes_\sigma b$ if and only if $(s \otimes_\sigma t)(a \otimes_\sigma b)^{-1} \in \pi(1_{G_1}, 1_{G_2})$

**Proof:** Since $G_1 \otimes_\sigma G_2$ is a group, [3, Theorem 2.5], we have $s \otimes_\sigma t = a \otimes_\sigma b$ if and only if $(s \otimes_\sigma t)(a \otimes_\sigma b)^{-1} = \pi(1_{G_1}, 1_{G_2})$ and or $(s \otimes_\sigma t)(a \otimes_\sigma b)^{-1} \in \pi(1_{G_1}, 1_{G_2})$.

**Lemma 5.2.** Let $G_1$and $G_2$, be two [topological] groups and $\sigma: G_1 \longrightarrow G_2$ be a [continuous] homomorphism. Then $N = \{(m,n) \in G_1 \times G_2$: (m,n) $\rho$ $(1_{G_1}, 1_{G_2})\}$ is a [closed] normal subgroup of $G_1 \times G_2$.

**Proof:** Suppose $(m_1,n_1) \in N$ and $(m_2,n_2) \in N$, then $(m_1,n_1)\,\rho\,(1_{G_1},1_{G_2})$, $(m_2,n_2)\,\rho\,(1_{G_1},1_{G_2})$. Since $\rho$ is a congruence, $(m_2,n_2)^{-1}\,\rho\,(1_{G_1},1_{G_2})$ and $(m_1,n_1)(m_2,n_2)^{-1}\,\rho\,(1_{G_1},1_{G_2})(1_{G_1},1_{G_2}) = (1_{G_1},1_{G_2})$. This implies that $N$ is a subgroup of $G_1 \times G_2$. Now, let $(m,n) \in N$ and $(g_1,g_2) \in G_1 \times G_2$. Since $\rho$ is a congruence on $G_1 \times G_2$,
$(g_1,g_2)(m,n)(g_1,g_2)^{-1}\ \rho$
$(g_1,g_2)(1_{G_1},1_{G_2})(g_1,g_2)^{-1}$,
$(g_1,g_2)(m,n)(g_1,g_2)^{-1}\ \rho\ (1_{G_1},1_{G_2})$.
This implies that $(g_1,g_2)(m,n)(g_1,g_2)^{-1}\in$N. Thus $N$ is a normal subgroup of $G_1 \times G_2$. Let $\{(m_\alpha,n_\alpha)\}$ be a net in $N$ such that $(m_\alpha,n_\alpha) \to (m,n)$. By the definition of $N$, $(m_\alpha,n_\alpha)\rho(1_{G_1},1_{G_2})$. Since $\rho$ is a closed congruence on $G_1 \times G_2$, we have $(m,n)\rho(1_{G_1},1_{G_2})$. Thus $(m,n) \in N$.

**Theorem 5.1.** Let $G_1$and $G_2$, be two [topological] groups and $\sigma: G_1 \longrightarrow G_2$ be a [continuous] homomorphism. Then $G_1 \otimes_\sigma G_2 = \frac{G_1\times G_2}{N}$, where $N = \{(m,n) \in G_1 \times G_2$:(m,n)$\rho(1_{G_1},1_{G_2})\}$. In other words, $G_1 \otimes_\sigma G_2$ is an extension of $G_1 \times G_2$ by $N$.

**Proof:** Let $\pi: G_1 \times G_2 \to \frac{G_1\times G_2}{\rho} = G_1 \otimes_\sigma G_2$ be the quotient map and $\pi(x) = g_1 \otimes_\sigma g_2 \in G_1 \otimes_\sigma G_2$. We show that $\pi(x) = Nx$. Let $n \in N$, by Lemma 5.2, $N$ is a subgroup of $G_1 \times G_2$. Now, $n^{-1} = x(nx)^{-1} \in N$. By Lemma 5.1, $nx \in \pi(x)$. This implies that $Nx \subseteq \pi(x)$. Conversely, let $y \in \pi(x)$, so $xy^{-1} \in N$. Since $N$ is a subgroup of $G_1 \times G_2$, so $yx^{-1} = (xy^{-1})^{-1} \in N$. Thus there is an $n \in N$ such that $yx^{-1} = n$ and so $y = nx$. This implies that $\pi(x) \subseteq Nx$. Thus $\pi(x) = Nx$ $(x \in G_1 \times G_2)$. Now, $G_1 \otimes_\sigma G_2 = \cup_{x\in G_1\times G_2}\pi(x) = \cup_{x\in G_1\times G_2} Nx = \frac{G_1\times G_2}{N}$.

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