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Atom-Photon Thermal Entanglement in Nonlinear Jaynes-Cummings Models

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Abstract

In this work we investigate the thermal entanglement between two-level atoms and photons in a nonlinear cavity. We consider intensity-dependent couplings and calculate the negativity, as a measure of atom-photon entanglement. The cavity is assumed to be at a temperature T, so that all number of photons, and at the same time, both atomic states, with definite probabilities, are present. We then demonstrate a condition under which the intensity-dependent coupling leads to entanglement. It is also shown that, as in the case of linear Jaynes-Cummings model, the thermal states of atoms and photons are never separable.

Keywords: Thermal entanglement; atom-photon; nonlinear Jaynes-Cummings

1. Introduction

Quantum entangled states have been well established to form the corner stone of quantum information processing, including quantum communication, quantum teleportation, cryptography, etc. [1-4]. In fact, for such purposes it is required that physical information contained in a composite system be either local or distributed amongst the subsystems, giving rise to entangled states. The research on the entanglement then focuses on two subjects: How the entanglement is implemented and how it may be quantified [5, 6].

One approach to realize entanglement for quantum information processing is cavity quantum electrodynamics [7]. In this approach, the interaction of material qubits (Rydberg quantum states) with a high finesse optical resonator is used for atom-atom, atom-photon or photon-photon entanglements [8]. The main challenge in this approach is to avoid de-coherence induced by the cavity modes that leak to the environment. On the other hand, interaction with the environment, as a heat reservoir, is also responsible for the de-coherence of entangled states [9]. It is therefore the main purpose of the present work to investigate entanglement between atomic states (two-level) and photons, inside a cavity filled by a nonlinear dielectric. To be specific, it is assumed that the cavity walls are held at a temperature T, so that both atom and field are in thermal states. Here, the entanglement occurs between the thermal atomic and photonic states [10, 11]. In [11], however, the entanglement of two-level atoms (not being in thermal states) and photons (being in thermal states) is reported. In the present work the occurrence of a particular combination of atomic and photonic states is weighted by the Maxwell factor.

Even though there are several measures to determine the entanglement (inseparability) of mixed states, we use the concept of negativity [12], which proves to be more suitable for the problem in hand. A bipartite quantum system is disentangled (separable) if its density matrix can be written as $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$, where $p_i \geq 0$, $\sum_i p_i = 1$ and $\rho_i^{A,B}$ represents the density matrices for the subsystems.

The elements of ρ is given by $\langle a,b|\rho|a',b'\rangle$ where $|\{a\}\rangle$ and $|\{b\}\rangle$ form the orthonormal basis for each subsystem. It has been shown that for the composite system to be separable, it is necessary that the partially transposed density matrix, defined as, $\left(\rho^{PT}\right)_{a,b:a',b'} = \langle a',b|\rho|a,b'\rangle = \langle a,b'|\rho|a',b\rangle$, has no negative eigenvalues [13]. Conversely, if ρ^{PT} possesses even a single negative eigenvalue, then the quantum system is entangled (inseparable). Quantitatively, this criterion may be expressed in terms of the negativity, defined as $N = \sum_n Max(0, -\lambda_n)$, where λ_n's are the

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eigenvalues of ρ^{PT}. It then follows that the state of the composite system is separable (disentangled) when the negativity is null, otherwise it is entangled [14]. The aforementioned criterion becomes a sufficient condition for 2×2 or 2×3 bipartite quantum systems [15]. We thus calculate the negativity as a measure of entanglement between thermal atoms and photons for different types of nonlinearities.

We proceed to calculate the negativity and analytically show that for intensity-dependent couplings of the form $\left(a^{\dagger}a+1\right)^{c}$, in which $a\left(a^{\dagger}\right)$ is the photonic annihilation (creation) operator, the system of thermal atom-photon is entangled at all temperatures (except at absolute zero and infinity) for $\frac{-1}{2}<c<\frac{1}{2}$. Intensity-dependent couplings have been shown to arise from deformed oscillators commutation relations, inside nonlinear dielectrics [16]. This particular form has also been used in [17-19].

The remainder of this article is organized as follows: In Section 2 we present the model along with the corresponding Hamiltonian. The dependence of negativity on temperature is investigated in Section 3. The atom-photon entanglement with nonlinear intensity-dependent coupling of the form $\left(a^{\dagger}a+1\right)^{c}$ is investigated in Section 4. Finally, concluding remarks are made in Section 5.

2. Nonlinear Jaynes-Cummings model

The system of two-level atoms, interacting with a single-mode quantized radiation field, is described by the celebrated Jaynes-Cummings Hamiltonian which, in a nonlinear medium and in the rotating wave approximation, reads [18],

$$H=\frac{1}{2}\hbar\Omega\sigma_{z}+\hbar\omega a^{\dagger}a+\hbar g(f(a^{\dagger}a)\sigma_{+}a+a^{\dagger}\sigma_{-}f(a^{\dagger}a)), \tag{1}$$

where $a(a^{\dagger})$ is the field annihilation (creation) operator, σ_{z} and $\sigma_{\pm}$ are the pseudo spin matrices. In equation (1) ω and Ω are, respectively, the field and atomic transition frequencies and $g\, f(a^{\dagger}a)$ is due to intensity-dependent coupling.

In terms of atom-field bare states, $\left|\{n\},+\right\rangle$ and $\left|\{n\},-\right\rangle$, where $\left|\{n\}\right\rangle$ is the field number states and $\left|+\right\rangle\left(\left|-\right\rangle\right)$ describes the atomic upper (lower) state, the Hamiltonian of equation (1) becomes,

$$H=-\frac{1}{2}\hbar\Omega\left|0,-\right\rangle\left\langle 0,-\right|+\sum_{n=0}^{\infty}H_{n}, \tag{2}$$

where,

$$\begin{aligned}H_{n}&=\left(n\hbar\omega+\frac{1}{2}\hbar\Omega\right)\left|n,+\right\rangle\left\langle n,+\right|\\&+\left((n+1)\hbar\omega-\frac{1}{2}\hbar\Omega\right)\left|n+1,-\right\rangle\left\langle n+1,-\right|\\&+\hbar g f(n)\sqrt{n+1}\left(\left|n,+\right\rangle\left\langle n+1,-\right|+\left|n+1,-\right\rangle\left\langle n,+\right|\right).\end{aligned} \tag{3}$$

From equations (2) and (3) it is noted that $\left|0,-\right\rangle$, which is separable, forms the ground state. Furthermore, it is also noted that the presence of $f(n)$, depending on its form, may decrease or increase the degree of entanglement. This point shall be addressed again in sections 4 and 5. In the bases formed by the bare states, each H_{n} is a 2×2 matrix, which may be diagonalized separately. Diagonalizing each block gives the orthonormal dressed states as,

$$\begin{aligned}\left|\psi_{1n}\right\rangle&=u_{1n}\left|n,+\right\rangle+u_{2n}\left|n+1,-\right\rangle\\\left|\psi_{2n}\right\rangle&=u_{2n}\left|n,+\right\rangle-u_{1n}\left|n+1,-\right\rangle,\end{aligned} \tag{4}$$

where

$$\begin{aligned}u_{1n}&=\frac{\hbar^{2}g^{2}f^{2}(n)(n+1)}{\sqrt{\hbar^{2}g^{2}f^{2}(n)(n+1)+\left(E_{1n}-(n+\tfrac{1}{2})\hbar\omega+\tfrac{\hbar\delta}{2}\right)^{2}}},\\u_{2n}&=\sqrt{1-u_{1n}^{2}},\end{aligned} \tag{5}$$

along with the dressed energies,

$$E_{1n,2n}=\hbar\omega\left(n+\frac{1}{2}\right)\pm\frac{\hbar}{2}\sqrt{4(n+1)g^{2}f^{2}(n)+\delta^{2}}, \tag{6}$$

where $\delta=\left(\Omega-\omega\right)$ is the detuning.

In the next section the thermal density operator (matrix) and partially transposed one, formed by the dressed states and dressed energies (equations (4) to (6)) are calculated.

3. Atom-photon thermal states

The atom-field thermal state, in equilibrium with the environment at temperature T, is given by,

$$\rho(T) = \frac{1}{Z}\sum_{n=0}^{\infty}\sum_{i=1,2} e^{-\beta E_{in}} \left|\psi_{in}\right\rangle\left\langle\psi_{in}\right|, \quad (7)$$

where $Z = \sum_{n=0}^{\infty}\sum_{i=1,2} e^{-\beta E_{in}}$ is the partition function, $\beta = \frac{1}{kT}$ and the dressed states and energies are given in equations (4) to (6), respectively. The corresponding partially transposed (with respect to the atomic states) density operator, defined as $\left\langle i,n\right|\rho^{PT}\left|j,m\right\rangle = \left\langle j,n\right|\rho\left|i,m\right\rangle$ with $i,j=\pm$, upon using equations (4) to (7), becomes,

$$\begin{aligned}\rho^{PT}(T) &\equiv \frac{1}{Z}\left(A_0\left|0,-\right\rangle\left\langle 0,-\right| + \sum_{n=0}^{\infty}\rho_n^{PT}(T)\right) \\ &= \frac{1}{Z}[A_0\left|0,-\right\rangle\left\langle 0,-\right| + \sum_{n=0}^{\infty}\{A_n\left|n,+\right\rangle\left\langle n,+\right| + B_{n+1}\left|n+1,-\right\rangle\left\langle n+1,-\right| \\ &+ C_n(\left|n,-\right\rangle\left\langle n+1,+\right| + \left|n+1,+\right\rangle\left\langle n,-\right|)\}],\end{aligned} \quad (8)$$

where,

$$\begin{aligned}B_0 &= e^{\frac{1}{2}\beta\hbar\Omega} \\ B_{n+1} &= u_{2n}^2 e^{-\beta E_{1n}} + u_{1n}^2 e^{-\beta E_{2n}} \\ A_n &= u_{1n}^2 e^{-\beta E_{1n}} + u_{2n}^2 e^{-\beta E_{2n}} \\ C_n &= u_{1n}u_{2n}(e^{-\beta E_{1n}} - e^{-\beta E_{2n}}).\end{aligned} \quad (9)$$

Again, each $\rho_n^{PT}(T)$ is a 2 × 2 matrix, whose eigenvalues are easily obtained as,

$$\lambda_n^{\pm} = \frac{(B_n + A_{n+1}) \pm \sqrt{(B_n + A_{n+1})^2 - 4(B_n A_{n+1} - C_n^{\ 2})}}{2Z}. \quad (10)$$

Clearly, only λ_n^-, which may assume negative values, is relevant to the negativity. As we shall see in the next section, one can analytically determine the sign of λ_n^-, indicating the conditions under which atom-photon entanglement forms. Moreover, we also present numerical analysis of λ_n^- and thereby the negativity, confirming our analytical results in this section.

4. Atom-photon entanglement

The aim of this section is to check the sign of λ_n^- and thus determine the conditions under which the system of atoms and photons is entangled.

It is clear from equation (10) that λ_n^- is negative if and only if,

$$B_n A_{n+1} - C_n^2 < 0, \quad (11)$$

which, upon using equations, (5), (6) and (9), becomes,

$$\begin{aligned}&\cosh(\tfrac{x}{2}\sqrt{\Delta^2 + 4nf^2(n-1)})\cosh(\tfrac{x}{2}\sqrt{\Delta^2 + 4(n+2)f^2(n+1)}) \\ &+\frac{\Delta}{\sqrt{\Delta^2 + 4nf^2(n-1)}}\sinh(\tfrac{x}{2}\sqrt{\Delta^2 + 4nf^2(n-1)}) \\ &\cosh(\tfrac{x}{2}\sqrt{\Delta^2 + 4(n+2)f^2(n+1)}) \\ &-\frac{\Delta}{\sqrt{\Delta^2 + 4(n+2)f^2(n+1)}}\sinh(\tfrac{x}{2}\sqrt{\Delta^2 + 4(n+2)f^2(n+1)}) \\ &\cosh(\tfrac{x}{2}\sqrt{\Delta^2 + 4nf^2(n-1)}) \\ &-\frac{\Delta^2}{\sqrt{[\Delta^2 + 4nf^2(n-1)][\Delta^2 + 4(n+2)f^2(n+1)]}} \\ &\sinh(\tfrac{x}{2}\sqrt{\Delta^2 + 4(n+2)f^2(n+1)})\sinh(\tfrac{x}{2}\sqrt{\Delta^2 + 4nf^2(n-1)}) \\ &-\frac{4(n+1)f^2(n)}{\Delta^2 + 4(n+1)f^2(n)}\sinh^2(\tfrac{x}{2}\sqrt{\Delta^2 + 4(n+1)f^2(n)}) < 0,\end{aligned} \quad (12)$$

where $x = \beta\hbar g$ determines the atom-field coupling energy relative to the thermal energy and $\Delta = \frac{\delta}{g}$ is the scaled field mode detuning. If the atom and field are in resonance, $\Delta = 0$, equation (12) simplifies to,

$$\begin{aligned}&e^{2x\tilde{f}(n)}(e^{x\alpha(n)} - 1) + e^{-2x\tilde{f}(n)}(e^{-x\alpha(n)} - 1) \\ &+e^{x\gamma(n)} + e^{-x\gamma(n)} + 2 < 0\end{aligned} \quad (13)$$

where $\tilde{f}(n) = \sqrt{n+1}f(n)$, $\gamma(n) = \tilde{f}(n+1) - \tilde{f}(n-1)$ and,

$$\begin{aligned}&\alpha(n) = \tilde{f}(n+1) - 2\tilde{f}(n) + \\ &\tilde{f}(n-1) \cong \frac{d^2}{dn^2}\tilde{f}(n).\end{aligned} \quad (14)$$

From equation (13) it is evident that when $\alpha(n) \geq 0$, then λ_n^- is never negative and consequently the system of atom-field is *separable (unentangled)*. On the other hand, when $\alpha(n) < 0$ for all n's, then the sign of λ_n^- depends upon $2x\tilde{f}(n)$ (see the first term of equation (13)). That is, if $2x\tilde{f}(n)$ is relatively large, then for some n (and thereafter), λ_n^- is negative and the system of atom-field *is inseparable (entangled).* It is then concluded that for intensity-dependent coupling, $\alpha(n) < 0$, as defined in equation (14), forms a necessary condition for the photon-atom entanglement to occur. As a specific example and to support the above conclusions, we take the widely used expression of the type [17-19], $f(n) = (n+1)^c$, where c is an arbitrary constant.

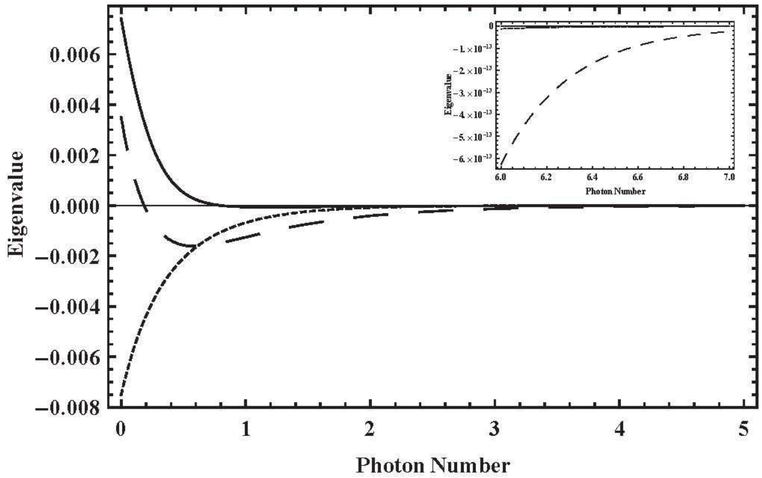

Fig. 1. Plots of λ_n^- versus photon numbers for scaled temperature , $x = 2$. Dotted line; $c = 0$, dashed line; $c = \frac{1}{4}$, and solid line; $c = \frac{1}{2}$. The inset indicates that the negative eigenvalue for $c = \frac{1}{4}$ remains negative for large n

This type of nonlinearity is of great interest in applications to two-photon micromasers [19, 20], atom-photon interaction inside photonic crystals [21, 22] and construction of photonic gates [23]. Using equation (14), one finds,

$$\alpha(n) = (c+\frac{1}{2})(c-\frac{1}{2})(n+1)^{(c-\frac{3}{2})}, \qquad (15)$$

which is negative for $\frac{-1}{2} < c < \frac{1}{2}$ only. Furthermore, with this range of values for c, $2x\tilde{f}(n) = 2x\left(n+1\right)^{\left(c+\frac{1}{2}\right)}$ definitely becomes large, so that the system of atom-field is entangled (inseparable). The conclusion that, for $\frac{-1}{2} < c < \frac{1}{2}$, the system of atom-photon is entangled holds for all temperatures (expect at absolute zero and infinity). Verification of this point follows. Using $\tilde{f}(n) = (n+1)^{c+\frac{1}{2}}$, an expansion of $\alpha(n)$, defined in equation (14), for $n >> 1$ leads to

$$\alpha(n) \approx (c^2 - \frac{1}{4})(n+1)^{c-\frac{3}{2}} \qquad (16)$$

which, upon inserting into equation (13), neglecting the second term and using the fact that,

$$e^{\pm\gamma(n)} \cong 1 \pm \left(2c+1\right)n^{\left(c-\frac{3}{2}\right)} \qquad (17)$$

(with $\gamma(n) \approx (2c+1)n^{c-\frac{1}{2}}$, for $n >> 1$), the condition of equation (13)becomes,

$$1 + \frac{1}{4}(c^2 - \frac{1}{4})\,x\,e^{2x(n+1)^{c+\frac{1}{2}}}(n+1)^{c-\frac{3}{2}} < 0. \qquad (18)$$

It is evident from equation (18) that for $\frac{-1}{2} < c < \frac{1}{2}$ the second term is always negative and grows rapidly, regardless of the temperature (x), as the number of photons increases. Thus, for $\frac{-1}{2} < c < \frac{1}{2}$ the system of atom-photon in the presence of intensity-dependent coupling is inseparable (entangled) at any temperature. One may rather easily check the condition of equation (13) for the case $f(n) = (n+1)^{\frac{1}{2}}$ and $f(n) = (n+1)^{\frac{-1}{2}}$, which give: $\cosh(2x) + 1 > 0$ and $\cosh^2(x) - \sinh^2(x) = 1 > 0$, respectively. These conclusions are confirmed from considerations of Figs. (1) to (3). The figures are drawn for a typical value of atom-field linear coupling, $g = 2$. Moreover, these illustrations are based on Eq.(10) for the negative eigenvalues and, when necessary, summations over n are carried out up to $n = 1000$. In Fig. (1) plots of λ_n^- as a function of photon number, at a fixed temperature corresponding to $x = 2$, are presented. From this

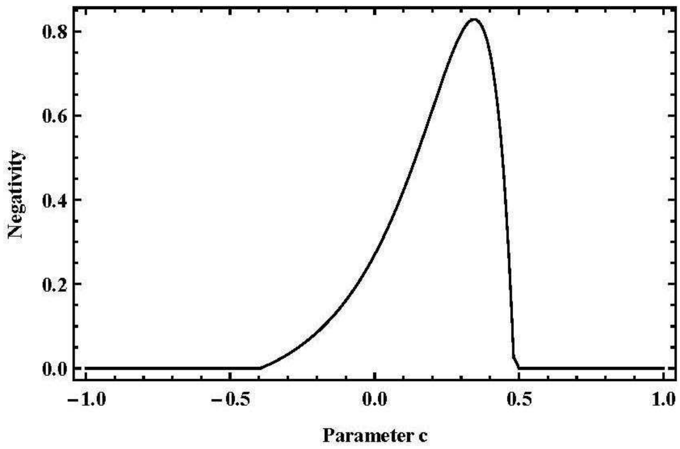

Fig. 2. Plot of negativity versus the parameter c , for scaled temperature, $x = 2$

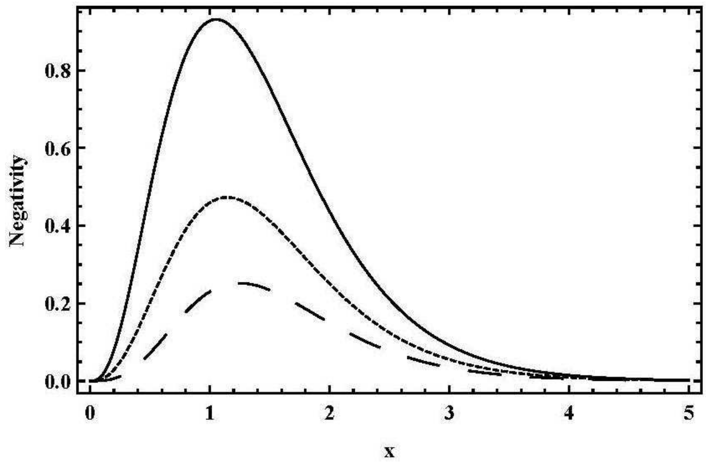

Fig. 3. Plot of negativity versus the scaled temperature, x . Solid line; $c = 0.1$, dotted line; $c = 0$ and dashed line; $c = -0.1$

figure it is clear (at least for $x = 2$) that λ_n^- becomes negative for $\frac{-1}{2} < c < \frac{1}{2}$ only. In Fig. (2) the negativity, $\sum_n Max(0, -\lambda_n^-)$, as a function of c , is plotted. It is again observed from this figure that the system of atom-photon is entangled for $\frac{-1}{2} < c < \frac{1}{2}$. We present the behavior of negativity, as a function of x (inverse temperature) for $c = \pm 0.1, 0$ in Fig. (3). This figure clearly indicates that the system of atom photon, even at very high temperature (low x) is entangled. Furthermore, from this figure it is observed that the entanglement is enhanced for larger c 's (within the admissible range) due to stronger couplings. The figure is also consistent with the commonly accepted fact that the maximal entanglement occurs at higher temperature for higher couplings. It is also pointed out that for $c = 0$, as the dotted line in Fig. (3), the case of linear JCM is reproduced [10].

5. Conclusions

In this paper we have reported the characteristics of entanglement between thermal states of two-level atoms and photons in a nonlinear medium, through the calculation of negativity, as a measure of it. The nonlinearity we consider is an intensity-dependent coupling. From the material and graphs presented in sections 4, the following points are observed.

One may define a parameter, $\alpha(n)$, with n the number of thermal photons, which asymptotically approaches the second derivative of $\sqrt{n+1}f(n)$ ($f(n)$ describes the intensity-dependent coupling). It then follows that a necessary condition for the formation of entanglement between atoms and thermally induced photons is $\alpha(n) < 0$. Taking the intensity-dependent coupling of the form $(n+1)^c$, it is shown that the atom-photon entanglement definitely occurs for $\frac{-1}{2} < c < \frac{1}{2}$. This conclusion is verified from Figs. 1 and 2 in which the possibly negative eigenvalues of $\rho^{PT}(n)$ versus n for different c's and negativity, as a function of the parameter c, all for a fixed value of scaled temperature are illustrated. Moreover, as the parameter c increases (within the admissible range) the system of atom-photon becomes more entangled (Figs. (2) and (3)). From Fig. (3) it is also noted that the maximal entanglement occurs at larger temperatures (lower x) for larger values of c.

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