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On the Goursat problem for a linear partial differential equation

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Abstract

In this paper, the Goursat problem of a general form for a linear partial differential equation is investigated with the help of the Riemann function method. Some results are given concerning the existence and uniqueness for the solution of the suggested problem.

***Keywords**:* General form partial differential equation; Goursat problem; Riemann function

1. Introduction

The studies of several authors [1-22] are devoted to the investigation of various questions about the equation

$$\frac{\partial^{m+n}u}{\partial x^m \partial y^n} + Mu = f(x,y),$$

where M is a linear differential operator with variable coefficients. The above equation is very important for the mathematical modeling of the process of soil-water absorption by plant roots (Aller equation see, [11]). The most general results have been obtained in [19], in particular, where the variant of developing the Riemann method for the solution of the Goursat problem was suggested.

In this paper the symbols ∂_1^k are introduced to indicate the differentiation and integration operators that provide for a more unique and compact writing of formulas that emerge in the process of the formula argument (construction). If $k \ge 1$, then

$$\partial_t^k \vartheta = \frac{\partial^k \vartheta}{\partial t^t},$$

and when $k \le -1$, then

$$\partial_t^k \vartheta = \frac{1}{-(k+1)!}\int_{t_0}^{t} (t-\tau)^{-k-1}\vartheta(\tau)d\tau;$$

where ∂_t^0 is an identity operator.

2. Main results

In the domain $D = \{(x,y); x_0 < x < x_1\,,\, y_0 < y < y_1\}$, we discuss a special case of the equation under consideration having the following form

$$L(u) = \partial_x^m \partial_y u + \sum_{0\le i+j<m+1} a_{ij}(x,y)\partial_x^i \partial_y^j u = f(x,y)$$

for $(x,y) \in D,$ (1)

Where $a_{ij} \in C^{i+j}(D)$ for $i = 0,1,...,m$ and $j = 0, \min\{1, m-i\}$. Here the class C^{k+l} means the existence and continuity for all derivatives

$$\frac{\partial^{r+s}}{\partial^r \partial^s}; \qquad (r = 0,...,k\,; \quad s = 0,...,l).$$

We put $\sum\limits_{0\le i+j<m+1}$ for $\sum\limits_{i=0}^{m} \sum\limits_{j=0}^{\min\{1,m-i\}}$, where $i + j < m+1$.

The Goursat Problem for (1) consists of finding a solution in D from the class

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$$C^{m+1}(D)\cap C^{(m-1)+0}(D\cup P)\cap C^{0+1}(D\cup q)$$

under conditions defined on the part of domain boundary **G** formed by the characteristics of this equation concurrent in the point (x_0, y_o):

$$\partial_x^i(u(x_0,y))=\vartheta_i(y),\quad (0\le i\le m-1),\qquad y\in P,\quad \vartheta_i\in C^1(P); \tag{2}$$

$$\partial_y^i(u(x,y_0))=\psi_j(x),\quad (j=0),\qquad x\in q,\quad \psi_j\in C^m(q); \tag{3}$$

where

$$y\in P=[y_0,y_1],\qquad y\in q=[x_0,x_1];\qquad (x_0,y_0)\in D$$

Here, we consider the conditions coincidence from (2) and (3) on the boundary of their definitions (co-ordination conditions) as satisfied:

$$\partial_y^j(\vartheta_i(y_0))=\partial_x^i(\psi_j(x_0)). \tag{4}$$

The solution of the following integral equation is called the Riemann function, as in the previous case [22]

$$v(x,y)+\sum_{0\le i+j<m+1}(-1)^{m+1-(i+j)}\partial_x^{i-m}\partial_y^{j-1}(a_{ij}v)=1, \tag{5}$$

which exists and unique in [10]. It is clear that the properties $R(x,y;\xi,\eta)$ described for the third order equation are also saved in this case. Besides, $v(x,y)$ remains the solution of the following equation adjoint to (1):

$$L^*(u)=\partial_x^m\partial_y v+\sum_{0\le i+j<m+1}(-1)^{m+1-(i+j)}\partial_x^i\partial_y^j(a_{ij}v)=0.$$

We discuss several auxiliary results for the solution of the above-stated problem. Let us have the following statement:

Lemma: If R is the Riemann function defined on

$$C^{m+1}(D)\cap C^{(m-1)+0}(D\cup P)\cap C^{0+0}(D\cup q),$$

then the following identity is satisfied:

$$\begin{aligned}\partial_x^m\partial_y(uR)=RL(u)&+\sum_{0<i+j<m+1}(-1)^{m+1-(i+j)+1}\partial_x^i\partial_y^j[u(\partial_x^{m-i}\partial_y^{1-j}R+\\&+\sum_{i\le\alpha,j\le\beta,\alpha+\beta<m+1}(-1)^{m+1-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}R)]+\\&+\sum_{i=0}^m\sum_{j=0}^1\sum_{b=0}^i\sum_{l=0}^j K_{i,bl}\partial_x^b\partial_y^l(u)\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R),\end{aligned} \tag{6}$$

where

$$a_{m1}=1,\quad K_{i,bl}=\left[\sum_{\alpha=b}^i(-1)^{i-\alpha}C_\alpha^b\sum_{\beta=l}^j(-1)^{j-\beta}C_\beta^l-M_{bl}\right],$$

$$M_{bl}=1,$$

if $b=l$, $l=j$ and $M_{bl}=0$, on all other cases.

Later, we call (6) the main identity. In order to prove (6), first we study the remainder term

$$\Omega=\sum_{i=0}^m\sum_{j=0}^1\sum_{b=0}^i\sum_{l=0}^j K_{i,bl}\partial_x^b\partial_y^l(u)\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R).$$

For the latter investigation it is easy to rewrite this expression in the form

$$\begin{aligned}\Omega=\sum_{i=0}^m\sum_{j=0}^1\sum_{b=0}^i\sum_{l=0}^j&\left[\sum_{\alpha=b}^i(-1)^{i-\alpha}C_\alpha^b\sum_{\beta=l}^j(-1)^{j-\beta}C_\beta^l-M_{bl}\right]\bullet\\&\bullet\partial_x^b\partial_y^l(u)\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R)=\\&=\sum_{i=0}^m\sum_{j=0}^1\sum_{b=0}^i\sum_{l=0}^j(-1)^{i+j-(b+l)}\partial_x^b\partial_y^l\{u\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R)\}-\\&-\sum_{i=0}^m\sum_{j=0}^1\partial_x^i\partial_y^j(u)\partial_x^0\partial_y^0(a_{ij}R).\end{aligned}$$

The latter relation can be checked directly:

$$\begin{aligned}&\sum_{b=0}^i\sum_{\alpha=b}^i(-1)^{i-\alpha}C_\alpha^b\partial_x^b(u)\partial_x^{i-b}(a_{ij}R)\\&=\sum_{\alpha=0}^i\sum_{b=0}^\alpha(-1)^{i-\alpha}C_\alpha^b\partial_x^b(u)\partial_x^{i-b}(a_{ij}R).\end{aligned}$$

We re-indicate α to b and b to α, then

$$\begin{aligned}&\sum_{b=0}^i\sum_{\alpha=b}^i(-1)^{i-\alpha}C_\alpha^b\partial_x^b(u)\partial_x^{i-b}(a_{ij}R)\\&=\sum_{b=0}^i\sum_{\alpha=0}^b(-1)^{i-b}C_b^\alpha\partial_x^\alpha(u)\partial_x^{i-\alpha}(a_{ij}R)\\&=\sum_{b=0}^i(-1)^{i-b}\partial_x^b(u\partial_x^{i-b}(a_{ij}R)).\end{aligned} \tag{7}$$

Thus, the relation is checked.

Now, we come to the lemma statement. We calculate

$$\begin{aligned}\partial_x^m\partial_y(uR)=R&\sum_{0\le i+j\le m+1}a_{ij}\partial_x^i\partial_y^j u+\sum_{0<i+j<m+1}(-1)^{m+1-(i+j)+1}\bullet\\&\bullet\partial_x^i\partial_y^j\left[u(\partial_x^{m-i}\partial_y^{1-j}R+\sum_{i\le\alpha,j\le\beta,\alpha+\beta<m+1}(-1)^{m+1-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}))\right]+\\&+\sum_{i=0}^m\sum_{j=0}^1\sum_{b=0}^i\sum_{l=0}^j(-1)^{i+j-(b+l)}\partial_x^b\partial_y^l\left\{u\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R)\right\}-\sum_{i=0}^m\sum_{j=0}^1\partial_x^i\partial_y^j(u)a_{ij}R.\end{aligned}$$

Using L^*, we multiply both parts by $(-1)^{m+1}$, then

$$(-1)^{m+n}\partial_x^m\partial_y\ \nu+\sum_{0\le i+j<m+1}(-1)^{i+j}\partial_x^i\partial_y^j(a_{ij}\nu)=$$

$$=\sum_{0\le i+j\le m+1}(-1)^{i+j}\partial_x^i\partial_y^j(a_{ij}\nu)=0.$$

Consequently,

$$\partial_x^m\partial_y(uR)=\sum_{0<i+j<m+1}(-1)^{m+1-(i+j)+1}\partial_x^i\partial_y^j\left\{u\partial_x^{m-i}\partial_y^{1-j}R\right\}$$

$$+\sum_{0<i+j<m+1}(-1)^{m+1-(i+j)+1}\partial_x^i\partial_y^j$$

$$\left[u\sum_{i\le\alpha,\,j\le\beta,\,\alpha+\beta<m+1}(-1)^{m+1-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}uR)\right]$$

$$+\sum_{0<i+j<m+1}\ \sum_{0<b+l\le i+j}(-1)^{i+j-(b+l)}\partial_x^b\partial_y^l(u\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R))$$

$$+\sum_{0<b+l\le m+1}(-1)^{m+n-(b+l)}\partial_x^b\partial_y^l(u\partial_x^{m-b}\partial_y^{n-l}(a_{ij}R)).$$

Then

$$\sum_{0<i+j<m+1}(-1)^{m+1-(i+j)+1}\partial_x^i\partial_y^j$$

$$\left\{u\sum_{i\le\alpha,\,j\le\beta,\,\alpha+\beta<m+1}(-1)^{m+1-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}R)\right\}$$

$$+\sum_{0<i+j<m+1}\ \sum_{0<b+l\le i+j}(-1)^{i+j-(b+l)}\partial_x^b\partial_y^l(u\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R))=0.$$

By virtue of (7) the mutual cancellation of the corresponding addends takes place in the later equality. That is why the lemma is proved.

Now, we come directly to the Goursat problem. First, we obtain some auxiliary formulas.

By the direct check we ascertain that

$$\Omega=\sum_{i=2}^{m}\sum_{j=0}^{1}\sum_{b=0}^{i}\sum_{l=0}^{j}K_{i,bl}\partial_x^b\partial_y^l(u)\partial_x^{i-b}\partial_y^{j-l}(a_{ij}R).$$

Based on the analysis of this expression under small m, we obtain the following Hypothesis.

$$\Omega=\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b+1}\left\{\sum_{j=0}^{1}C_{i-1-b}^{i-2-2b}C_1^j\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}\partial_y^0(a_{ij}R)\right\}.\quad(8)$$

This hypothesis is checked and can be checked by the Visual Basic 3.0 program.

The second auxiliary formula is connected with the integral form of the remainder.

$$I(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}\Omega(\xi,\eta)d\eta d\xi.$$

From (8), we calculate

$$I(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b+1}\left\{\sum_{j=0}^{1}C_{i-1-b}^{i-2-2b}C_1^j\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}\partial_y^0(a_{ij}R)\right\}d\eta d\xi$$

$$=\int_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b}\left\{\sum_{j=0}^{1}C_{i-1-b}^{i-2-2b}C_1^j\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}\partial_y^0(a_{ij}R)\right\}(x,\eta)d\eta-$$

$$-\int_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b}\left\{\sum_{j=0}^{1}C_{i-1-b}^{i-2-2b}C_1^j\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}\partial_y^0(a_{ij}R)\right\}(x_0,\eta)d\eta=$$

$$=\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b}\left\{C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}\partial_y^0(a_{i1}R)\right\}(x,y)-$$

$$-\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b}\left\{C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}\partial_y^0(a_{i1}R)\right\}(x,y_0)+$$

$$+\int_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b}\left\{\sum_{j=0}^{1}(-1)^jC_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}\partial_y^j(a_{ij}R)\right\}(x,\eta)d\eta-$$

$$-\int_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^{b}\left\{\sum_{j=0}^{1}C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}\partial_y^0(a_{ij}R)\right\}(x,\eta)d\eta.\quad(9)$$

In (5) we put that $\xi=x$. Then

$$\nu+\sum_{0\le j\le m-1}(-1)^{1-j}\partial_x^{m-m}\partial_y^{j-1}(a_{mj}\nu)=1,$$

$$\nu-\partial_x^0\partial_y^{-1}(a_{m0}\nu)=1.$$

Differentiating with respect to y we obtain:

$$\partial_y(\nu)(x,\eta)-(a_{m0}\nu)(x,\eta)=0.$$

Differentiating (5) once with respect to x and once with respect to y we have

$$\partial_x\partial_y(\nu)+\sum_{0\le i+j<m+1}(-1)^{m+1-(i-j)}\partial_x^{i+1-m}\partial_y^j(a_{ij}\nu)=0.$$

We put $\xi=x$. Then we have the following

$$\left(\partial_x\partial_y(\nu)-\partial_x^0\partial_y(a_{m-1,1}\nu)+\partial_x^0\partial_y^0(a_{m-1,0}\nu)-\partial_x\partial_y^0(a_{m,0}\nu)\right)(x,\eta)=0,$$

$$\left(\partial_x\partial_y(\nu)-\partial_x\partial_y^0(a_{m,0}\nu)\right)(x,\eta)=\left(\partial_y(a_{m-1,1}\nu)-(a_{m-1,0}\nu)\right)(x,\eta).$$

Consequently,

$$\left(\partial_y(a_{m-1,1}\nu)-(a_{m-1,0}\nu)\right)(x,\eta)=0$$

If we differentiate (5) twice with respect to x and once with respect to y we obtain:

$$\partial_x^2\partial_y(\nu)+\sum_{0\le i+j<m+1}(-1)^{m+1-(i-j)}\partial_x^{i+2-m}\partial_y^j(a_{ij}\nu)=0.$$

We put $\xi=x$. Then

$$\left(\partial_x^2\partial_y(\nu)-\partial_x^2\partial_y^0(a_{m,0}\nu)+\partial_x\partial_y^0(a_{m-1,0}\nu)-\partial_x\partial_y(a_{m-1,1}\nu)\right)(x,\eta)= \\ =\left(\partial_x^0\partial_y^0(a_{m-2,0}\nu)-\partial_x^0\partial_y(a_{m-2,1}\nu)\right)(x,\eta)=0.$$

By virtue of the previous

$$\left(\partial_x^0\partial_y^0(a_{m-2,0}\nu)-\partial_x^0\partial_y(a_{m-2,1}\nu)\right)(x,\eta)=0.$$

Then in (9), we have

$$\int\limits_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{\sum_{j=0}^{1}(-1)^j C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}\partial_y^j(a_{ij}R)\right\} \\ (x,\eta)d\eta\equiv 0.$$

We return again to (5). We write it renaming i to α and j to β:

$$\nu+\sum_{0\le\alpha+\beta<m+1}(-1)^{m+1-(\alpha+\beta)}\partial_x^{\alpha-m}\partial_y^{\beta-1}(a_{\alpha\beta}\nu)=1.$$

Differentiating $\partial_x^{m-i}\partial_y^{1-j}$:

$$\partial_x^{m-i}\partial_y^{1-j}(\nu)+\sum_{0\le\alpha+\beta<m+1}(-1)^{m+1-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}\nu)=0.$$

We put $i=m-1$, $j=1$ and $\xi=x$, $\eta=y$, then

$$\left(\partial_x(\nu)-a_{m-1,1}\nu\right)(x,y)=0,\quad \partial_x(\nu)(x,y) \\ =(a_{m-1,1}\nu)(x,y).$$

Let $i=m-2$, $j=1$ and $\xi=x$, $\eta=y$, then

$$\left(\partial_x^2(\nu)+\partial_x^0(a_{m-2,1}\nu)-\partial_x(a_{m-1,1}\nu)\right)(x,y)=0,$$

hence

$$\left(a_{m-2,1}\nu\right)(x,y)=\left(-\partial_x^2(\nu)+\partial_x(a_{m-1,1}\nu)\right)(x,y)=0.$$

Continuing to do so, we obtain

$$\left(a_{m-3,1}\nu\right)(x,y)=0,\qquad \left(a_{m-4,1}\nu\right)(x,y)=0,...$$

Then

$$\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}\partial_y^0(a_{i1}R)\right\}(x,y)= \\ =\sum_{i=m-1}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}\partial_y^0(a_{i1}R)\right\}(x,y).$$

Let m be odd, then

$$\mathrm{int}[\frac{m}{2}]-1=\frac{m-1}{2}-1\quad,$$

where $\mathrm{int}[a]$ is integer part a.
Consequently,

$$\int\limits_{x_0}^{x}\int\limits_{y_0}^{y}\Omega(\xi,\eta)d\eta d\xi=\sum_{b=0}^{\frac{m-1}{2}-1}\partial_x^b\left\{C_{m-1-b}^{m-2-2b}\partial_x^{m-2-2b}(u)\partial_x^{b+1}(R)\right\}(x,y)+ \\ +\sum_{b=0}^{\frac{m-1}{2}-1}\partial_x^b\left\{C_{m-2-b}^{m-3-2b}\partial_x^{m-3-2b}(u)\partial_x^{b+2}(R)\right\}(x,y)- \\ -\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}\partial_y^0(a_{i1}R)\right\}(x,y_0)- \\ -\int\limits_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{\sum_{j=0}^{1}C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}\partial_y^0(a_{ij}R)\right\}(x_0,y)d\eta.$$

The two latter addends are completely defined. That is why we consider only the first two.
By direct calculation, we can make sure that for any $j\le m-2$, the coefficient under $\partial_x^j(u)\partial_x^{m-1-j}(R)$ is

$$\sum_{k-\text{even}=0}^{j}C^{k}_{\frac{m-1}{2}+\frac{k}{2}}C^{j-k}_{\frac{m-1}{2}-(\frac{k}{2}+1)} \\ +\sum_{k-\text{odd}=1}^{j}C^{k}_{\frac{m-1}{2}+\frac{k+1}{2}}C^{j-k}_{\frac{m-1}{2}-(\frac{k+1}{2})} \tag{10}$$

Really:
1) Let $j=0$. Then the coefficient is defined from the second addend Ω, if

$$b=\frac{m-1}{2}-1.$$

The coefficient is 1 and in (10) the coefficient is also 1.
2) When $j=1$. Available both in the first and the second addend Ω:

$$C^{m-3-2(\frac{m-1}{2}-1)}_{m-2-(\frac{m-1}{2}-1)} C^{1}_{\frac{m-1}{2}-1} + C^{m-2-2(\frac{m-1}{2}-1)}_{m-1-(\frac{m-1}{2}-1)} C^{0}_{\frac{m-1}{2}-1}$$
$$= \frac{m-1}{2} - 1 + \frac{m+1}{2} = m-1.$$

3) Degree $j=2$. Available both in the first and the second addend Ω:

$$C^{m-3-2(\frac{m-1}{2}-1)}_{m-2-(\frac{m-1}{2}-1)} C^{2}_{\frac{m-1}{2}-1} + C^{m-3-2(\frac{m-1}{2}-2)}_{m-2-(\frac{m-1}{2}-2)} C^{0}_{\frac{m-1}{2}-1}$$
$$= \left(\frac{m-1}{2}\right)^2 - \frac{m-1}{2} + 1,$$

in the first addend

$$C^{m-2-2(\frac{m-1}{2}-1)}_{m-1-(\frac{m-1}{2}-1)} C^{1}_{\frac{m-1}{2}-1} = \left(\frac{m-1}{2}\right)^2 - 1.$$

Then

$$\left(\frac{m-1}{2}\right)^2 - \frac{m-1}{2} + 1 + \left(\frac{m-1}{2}\right)^2 - 1$$
$$= \frac{(m-1)^2}{2} - \frac{m-1}{2}.$$

4) Let $j=3$. In the second addend:

$$C^{m-3-2(\frac{m-1}{2}-1)}_{m-2-(\frac{m-1}{2}-1)} C^{3}_{\frac{m-1}{2}-1} + C^{m-3-2(\frac{m-1}{2}-2)}_{m-2-(\frac{m-1}{2}-2)} C^{1}_{\frac{m-1}{2}-2}$$
$$= C^{0}_{\frac{m-1}{2}} C^{3}_{\frac{m-1}{2}-1} + C^{2}_{\frac{m-1}{2}+1} C^{1}_{\frac{m-1}{2}}.$$

In the first addend the following is available

$$C^{1}_{\frac{m-1}{2}} C^{2}_{\frac{m-1}{2}-1} + C^{3}_{\frac{m-1}{2}+2} C^{0}_{\frac{m-1}{2}-2}.$$

5) Let $j=4$, then

$$C^{0}_{\frac{m-1}{2}} C^{4}_{\frac{m-1}{2}-1} + C^{2}_{\frac{m-1}{2}+1} C^{2}_{\frac{m-1}{2}-2} + C^{4}_{\frac{m-1}{2}+2} C^{0}_{\frac{m-1}{2}-3}$$
$$= C^{1}_{\frac{m-1}{2}+1} C^{3}_{\frac{m-1}{2}-1} + C^{3}_{\frac{m-1}{2}+2} C^{1}_{\frac{m-1}{2}-2}.$$

and so on. Thus, (9) is true.

If $j=m-1$, then the coefficient is zero since

$$\frac{m-1}{2} - \left(\frac{k}{2}+1\right) = \frac{m-1}{2} - \left(\frac{m-1}{2}+1\right) = -1,$$

the second addend is not available since $(m-1)$ is even.

Besides, for small m it is directly checked that for $j \le m-2$, the following formula takes place:

$$\sum_{k-even=0}^{j} C^{k}_{\frac{m-1}{2}+\frac{k}{2}} C^{j-k}_{\frac{m-1}{2}-(\frac{k}{2}+1)} + \sum_{k-odd=1}^{j} C^{k}_{\frac{m-1}{2}+\frac{k+1}{2}} C^{j-k}_{\frac{m-1}{2}+(\frac{k+1}{2})} = C^{j}_{m-1}.$$

We consider it as the hypothesis which is checked and can be checked with the help of Foxpro programs.

Let *m* be even, then

$$\mathrm{int}[\frac{m-1}{2}] - 1 = \frac{m}{2} - 2,$$

$$\sum_{b=0}^{\frac{m}{2}-1} \partial_x^b \left\{C^{m--2-2b}_{m-1-b} \, \partial_x^{m-2-2b}(u) \, \partial_x^{b+1}(R)\right\}(x,y) +$$
$$+ \sum_{b=0}^{\frac{m-2}{2}-1} \partial_x^b \left\{C^{m--3-2b}_{m-2-b} \, \partial_x^{m-3-2b}(u) \, \partial_x^{b+2}(R)\right\}(x,y) =$$
$$= \sum_{j=0}^{m-1} \left\{ \sum_{k-even=0}^{j} C^{k}_{\frac{m}{2}+\frac{k}{2}} C^{j-k}_{\frac{m}{2}-(\frac{k}{2}+1)} + \sum_{k-odd=1}^{j} C^{k}_{\frac{m}{2}+\frac{k-1}{2}} C^{j-k}_{\frac{m}{2}-(\frac{k-1}{2})-2} \right\}$$
$$\partial_x^j(u) \partial_x^{m-1-j}(R).$$

Here, likely for *m* odd the coincidence of the corresponding coefficients with C^{j}_{m-1} take place except in the case $j=m-1$, when the coefficient is zero.

The proof is conducted with the help of the FoxPro program, is checked and can be checked.

Now, we are ready for the integration of the main identity.

$$\begin{aligned} \partial_x^m \partial_y (uR) = RL(u) + \sum_{0<i+j<m+1} (-1)^{m+1-(i+j)+1} \partial_x^i \partial_y^j \Big[u(\partial_x^{m-i} \partial_y^{1-j} R + \\ + \sum_{i\le\alpha,\, j\le\beta, \alpha+\beta<m+1} (-1)^{m+1-(\alpha+\beta)} \partial_x^{\alpha-i} \partial_y^{\beta-j} (a_{\alpha\beta}R)) \Big] + \\ + \sum_{i=2}^{m} \sum_{b=0}^{\frac{i}{2}-1} \sum_{j=0}^{1} \left\{ C^{i-2-2b}_{i-1-b} \, C^{j}_{1} \, \partial_x^{i-2-2b} \partial_x^j (u) \partial_x^{b+1} \partial_y^0 (a_{ij}R) \right\}. \end{aligned} \tag{11}$$

Integrating this identity once with respect to x, and once with respect to y, we obtain:

$$\int_{x_0}^{x}\int_{y_0}^{y} \left(\partial_\xi^m \partial_\eta (uR)\right)(\xi,\eta) d\eta d\xi = \partial_x^{m-1} \partial_y^{l-1} (uR)(x,y) -$$
$$- \partial_x^{m-1} \partial_y^{l-1} (uR)(x_0,y) - \partial_x^{m-1} \partial_y^{l-1} (uR)(x,y_0) + \partial_x^{m-1} \partial_y^{l-1} (uR)(x_0,y_o).$$

Since the addends that depend on $(x_0\,, y)\,, (x, y_0)$ or (x_0, y_0) are known we dwell only on the first.

$$\partial_x^{m-1}\partial_y^0(uR)(x,y)=\sum_{i=0}^{m} C_{m-1}^i\left[\partial_x^i(u)\partial_x^{m-1-i}(R)\ \right](x,y)=$$
$$=\sum_{i=0}^{m-2} C_{m-1}^i\left[\partial_x^i(u)\partial_x^{m-1-i}(R)\ \right](x,y)+C_{m-1}^{m-1}\partial_x^{m-1}(u)R(x,y).$$

By virtue of the property of the Riemann function, $R(x,y;x,y)=1$,

$$\partial_x^{m-1}\partial_y^0(uR)(x,y)=\sum_{i=0}^{m-2} C_{m-1}^i\left[\partial_x^i(u)\partial_x^{m-1-i}(R)\ \right](x,y)+\partial_x^{m-1}(u)(x,y).$$

Let us now consider the right part of (11):

$$\int_{x_0}^{x}\int_{y_0}^{y} RL(u)d\eta d\xi+\int_{x_0}^{x}\int_{y_0}^{y}\sum_{0<i+j<m+1}(-1)^{m+l-(i+j)+1}\partial_x^i\partial_y^j$$
$$\left[u(\partial_x^{m-i}\partial_y^{l-j}+\right.$$
$$\left.+\sum_{i\le\alpha,\, j\le\beta,\,\alpha+\beta<m+1}(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}))\ \right]d\eta d\xi$$
$$+\sum_{i=0}^{m-2} C_{m-1}^i\left\{\partial_x^i(u)\partial_x^{m-1-i}(R)\ \right\}(x,y)-$$
$$-\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left[C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}(a_{i1}R)\ \right](x,y_0)-$$
$$-\int_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{\sum_{j=0}^{1} C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}(a_{i1}R)\ \right\}(x_0,\eta)d\eta.$$

In the latter formula, we calculate the second addend:

$$\int_{x_0}^{x}\int_{y_0}^{y}\sum_{0<i+j<m+1}(-1)^{m+l-(i+j)+1}\partial_x^i\partial_y^j\left[u(\partial_x^{m-i}\partial_y^{l-j}R+\right.$$
$$\left.+\sum_{i\le\alpha,\, j\le\beta,\,\alpha+\beta<m+1}(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}R))\ \right]d\eta d\xi.$$

If we indicate the integrand expression Q, then

$$\int_{x_0}^{x}\int_{y_0}^{y}\sum_{0<i+j<m+1}Q_{i,j;i,j}(\xi,\eta)d\eta d\xi=\int_{x_0}^{x}\int_{y_0}^{y}\sum_{0<i+j<m+1,\, i\le1,\, j\le1}Q_{i,j;i,j}(\xi,\eta)d\eta d\xi+$$
$$+\int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=0,\, j=1}^{m}Q_{i,j;i,j}(\xi,\eta)d\eta d\xi+\int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1,\, j=0}^{m}Q_{i,j;i,j}(\xi,\eta)d\eta d\xi.$$

Differentiating (5) $\partial_x^{m-i}\partial^{l-j}$:

$$\partial_x^{m-i}\partial^{l-j}(\nu)+\sum_{0\le\alpha+\beta<m+1,\, i\le\alpha,\, j\le\beta}(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}\nu)+$$
$$+\sum_{0\le\alpha+\beta<m+1,\, i>\alpha,\, j>\beta}(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}(a_{\alpha\beta}\nu)=0.$$

We put $\xi=x$, $\eta=y$, then

$$\partial_x^{m-i}\partial^{l-j}R(x,y;x,y)+\sum_{0\le\alpha+\beta<m+1,\, i\le\alpha,\, j\le\beta}(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta-j}$$
$$(a_{\alpha\beta}R)(x,y)=0.$$

If $i=0$, then

$$\partial_x^{m}\partial^{l-j}(\nu)+\sum_{0\le\alpha+\beta<m+1,\, i\le\alpha,\, j\le\beta}(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha}\partial_y^{\beta-j}(a_{\alpha\beta}\nu)+$$
$$+\sum_{0\le\alpha+\beta<m+1,\, j>\beta}(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha}\partial_y^{\beta-j}(a_{\alpha\beta}\nu)=0.$$

Putting $\eta=y$, then

$$\partial_x^{m}\partial^{l-j}(\nu)(\xi,y;x,y)+\sum_{0\le\alpha+\beta<m+1,\, i\le\alpha,\, j\le\beta}$$
$$(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha}\partial_y^{\beta-j}(a_{\alpha\beta}\nu)(\xi,y)=0.$$

Similarly, if $j=0$, putting $\xi=x$, we obtain

$$\partial_x^{m-i}\partial_y(\nu)(\xi,\eta;x,y)+\sum_{0\le\alpha+\beta<m+1,\, 0\le\alpha,\, 0\le\beta}$$
$$(-1)^{m+l-(\alpha+\beta)}\partial_x^{\alpha-i}\partial_y^{\beta}(a_{\alpha\beta}\nu)(x,\eta)=0.$$

That is why

$$\int_{x_0}^{x}\int_{y_0}^{y}\sum_{0<i+j<m+1}Q_{i,j;i,j}(\xi,\eta)d\eta d\xi=-\sum_{0<i+j<m+1,\, i\ge1,\, j\ge1}Q_{i-1,j-1;i,j}(x,y_0)-$$
$$-\sum_{0<i+j<m+1,\, i\ge1,\, j\ge1}Q_{i-1,j-1;i,j}(x_0,y)+\sum_{0<i+j<m+1,\, i\ge1,\, j\ge1}Q_{i-1,j-1;i,j}(x_0,y_0)-$$
$$-\int_{x_0}^{x}\sum_{i=0,\, j=1}Q_{0,j-1;i,j}(\xi,y_0)d\xi+\int_{y_0}^{y}\sum_{i=1,\, j=0}Q_{i-1,0;i,j}(x_0,\eta)d\eta.$$

Consequently

$$\partial_x^{m-1}u=\partial_x^{m-1}(uR)(x_0,y)+\partial_x^{m-1}(uR)(x,y_0)-\partial_x^{m-1}(uR)(x_0,y_0)+$$
$$+\int_{x_0}^{x}\int_{y_0}^{y}RL(u)d\eta d\xi-\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^0(u)\partial_x^{b+1}(a_{i1}R)\right\}(x,y_0)-$$
$$-\int_{y_0}^{y}\sum_{i=2}^{m}\sum_{b=0}^{\frac{i}{2}-1}\partial_x^b\left\{\sum_{j=0}^{1} C_{i-1-b}^{i-2-2b}\partial_x^{i-2-2b}\partial_y^j(u)\partial_x^{b+1}(a_{ij}R)\ \right\}(x_0,\eta)d\eta-$$
$$-\sum_{i\ge1,\, j\ge1}Q_{i-1,j-1;i,j}(x_0,y)-\sum_{i\ge1,\, j\ge1}Q_{i-1,j-1;i,j}(x,y_0)+$$
$$+\sum_{i\ge1,\, j\ge1,\, 0<i+j<m+1}Q_{i-1,j-1;i,j}(x_0,y_0)-\int_{x_0}^{x}\sum_{i=0,\, j=1}^{m}Q_{0,j-1;i,j}(\xi,y_0)d\xi-$$
$$-\int_{y_0}^{y}\sum_{i=1,\, j=0}^{m}Q_{i-1,0;i,j}(x_0,\eta)d\eta.$$

ω is the sum of all the addends depending on

(x, y_0), (x_0, y), (x_0, y_0), (x_0, η) and (ξ, y_0)" (sic). Differentiating the latter formula $(m-1)$ times with respect to x, we find

$$u(x,y) = u(x_0,y) + \sum_{i=1}^{m-2} \partial_x^{-i} \begin{bmatrix} \vartheta_i(y) \;] + \partial_x^{-m}\partial_y^{-1} \\ (RL(u)) + \partial_x^{-(m-1)}(\omega), \end{bmatrix}$$

$$u(x,y) = \vartheta(y) + \sum_{i=1}^{m-2} \partial_x^{-i}[\vartheta_i(y) \;] + \partial_x^{-m}\partial_y^{-1}(Rf) + \partial_x^{-(m-1)}(\omega). \quad (12)$$

Hence, we obtain the solution of the Goursat problem in the case under consideration (1) - (3) in the following form

$$u(x,y) = \sum_{k=0}^{m+1} u_k(x,y), \quad (13)$$

where $u_k(x,y)$ is the solution of the homogeneous version of equation (1) subject to the boundary conditions $(\partial_x^i u)(x_0, y) = 0$ for $y \in P$ and $i = 0,1,...,k-1,k,k+1,...,m-1$,$(\partial_x^i u)(x_0, y) = \vartheta_k(y)$ for $y \in P$ and $u(x, y_0) = 0$ for $x \in q$ for $k = 0,1,...,m-1$, where u_m is the solution of the homogeneous version of equation (1) subject to the boundary conditions $(\partial_x^i u)(x_0, y) = 0$ for $y \in P$ for $i = 0,1,...,m-1$, and $u(x, y_0) = \psi_0(x)$ for $x \in q$, and where u_{m+1} is the solution of the inhomogeneous equation (1) subject to homogeneous boundary conditions, i.e. $(\partial_x^i u)(x_0, y) = 0$ for $y \in P$ and $i = 0,1,...,m-1$, and $u(x, y_0) = 0$ for $x \in q$.

Therefore, our results can be formulated as follows:

Theorem: The Goursat problem for the equation (1) with the boundary conditions (2), (3) and (4) can be formulated according to the formula (13) .

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