Subdivisions of the spectra for cesaro, rhaly and weighted mean operators on $c_0$, $c$ and $\ell^p$

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Abstract

There are many different ways to subdivide the spectrum of a bounded linear operator; some of them are motivated by applications to physics (in particular, quantum mechanics). In this study, the relationship between the subdivisions of spectrum which are not required to be disjoint and Goldberg's classification are given. Moreover, these subdivisions for some summability methods are studied.

Keywords: Spectrum, fine spectrum, approximate point spectrum, defect spectrum, compression spectrum, weighted mean operators, Rhaly operators, Cesáro operators.

1. Introduction

Let $X$ be a Banach space and $B(X)$ denote the linear space of all bounded linear operators on $X$. Given an operator $L \in B(X)$, the set

$$\rho(L) := \{ \lambda \in K : \lambda I - L \text{ bijection} \} \quad (1.1)$$

is called the resolvent set of $L$ (where $K = \mathbb{R}$ or $K = \mathbb{C}$), its complement

$$\sigma(L) := K \setminus \rho(L) \quad (1.2)$$

the spectrum of $L$. We denote the operator $R(\lambda; L)$ as follows:

$$R(\lambda; L) := (\lambda I - L)^{-1}. \quad (1.3)$$

By the closed graph theorem, the inverse operator

$$R(\lambda; L) := (\lambda I - L)^{-1} \quad (\lambda \in \rho(L))$$

is always bounded; this operator is usually called resolvent operator of $L$ at $\lambda$.

1.1. Subdivision of the spectrum: The point spectrum, continuous spectrum and residual spectrum

Let $X$ be a Banach space over $K$ and $L \in B(X)$. Recall that a number $\lambda \in K$ is called an eigenvalue of $L$ if the equation $Lx = \lambda x$ has a nontrivial solution $x \in X$. Any such $x$ is then called eigenvector, and the set of all eigenvectors is a subspace of $X$ called eigenspace.

Throughout the following, we will call the set of eigenvalues

$$\sigma_p(L) := \{ \lambda \in K : Lx = \lambda x \text{ for some } x \neq 0 \}. \quad (1.4)$$

We say that $\lambda \in K$ belongs to the continuous spectrum $\sigma_c(L)$ of $L$ if the resolvent operator $(1.3)$ is defined on a dense subspace of $X$ and is unbounded. Furthermore, we say that $\lambda \in K$ belongs to the residual spectrum $\sigma_r(L)$ of $L$ if the resolvent operator $(1.3)$ exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$) is not dense in $X$; in this case $R(\lambda; L)$ may be bounded or unbounded. Together with the point spectrum $(1.4)$, these two subspectra form a disjoint subdivision

$$\sigma(L) = \sigma_p(L) \cup \sigma_c(L) \cup \sigma_r(L) \quad (1.5)$$
of the spectrum of $L$. Loosely speaking, the elements $\lambda$ in the subspectrum $\sigma_p(L)$ characterize some lack of injectivity, those in $\sigma_r(L)$ lack of surjectivity, and those in $\sigma_c(L)$ lack of stability of the operator $\lambda I - L$. We illustrate the subdivision (1.5) in Table 1.

Table 1. Disjoint subdivision of spectrum

<table>
<thead>
<tr>
<th>$R(\lambda; L)$ exists and is bounded</th>
<th>$R(\lambda; L)$ exists and is unbounded</th>
<th>$R(\lambda; L)$ does not exist</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(\lambda I - L) = X$ $\lambda \in \rho(L)$</td>
<td>$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_c(L)$</td>
<td>$\lambda \in \sigma_p(L)$</td>
</tr>
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<td>$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_c(L)$</td>
<td>$\lambda \in \sigma_p(L)$</td>
</tr>
<tr>
<td>$R(\lambda I - L) \neq X$ $\lambda \in \sigma_r(L)$</td>
<td>$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_c(L)$</td>
<td>$\lambda \in \sigma_p(L)$</td>
</tr>
</tbody>
</table>

1.2. The approximate point spectrum, defect spectrum and compression spectrum

Given a bounded linear operator $L$ in a Banach space $X$, we call a sequence $(x_k)_k$ in $X$ a Weyl sequence for $L$ if $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(L) := \left\{ \lambda \in K : \text{there is a Weyl sequence for } \lambda I - L \right\}$$

(1.6)

the approximate point spectrum of $L$. Moreover, the subspectrum

$$\sigma_{d}(L) := \left\{ \lambda \in K : \lambda I - L \text{ is not surjective} \right\}$$

(1.7)

is called defect spectrum of $L$.

By definition, we then have $\|\lambda x - Lx\| \geq \epsilon \|x\|$ for all $x \in X$ if $\lambda \in \sigma_{ap}(L)$; equivalently, this may be stated as

$$\inf \{ \|e - Le\| : e \in S(X) \} > 0 \quad \left( \lambda \in \sigma_{ap}(L) \right)$$

(1.8)

where $S(X) := \{ x \in X : \|x\| = 1 \}$. The two subspectra (1.6) and (1.7) form a (not necessarily disjoint) subdivision

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_{d}(L)$$

(1.9)

of the spectrum. There is another subspectrum,

$$\sigma_{co}(L) := \left\{ \lambda \in K : R(\lambda I - L) \neq X \right\}$$

(1.10)

which is often called compression spectrum in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_{co}(L)$$

(1.11)

of the spectrum. Clearly, $\sigma_{p}(L) \subseteq \sigma_{ap}(L)$ and $\sigma_{co}(L) \subseteq \sigma_{d}(L)$. Moreover, comparing these subspectra with those in (1.5) we note that

$$\sigma_{f}(L) = \sigma_{co}(L) \setminus \sigma_{p}(L)$$

(1.12)

and

$$\sigma_{r}(L) = \sigma(L) \setminus \left[ \sigma_{p}(L) \cup \sigma_{co}(L) \right]$$

(1.13)

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint, building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 1.1. ([1], Proposition 1.3). The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:
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(a) \( \sigma(L^*) = \sigma(L) \), (b) \( \sigma_{ap}(L^*) \subseteq \sigma_{ap}(L) \), (c) \( \sigma_{ap}(L^*) = \sigma_{ap}(L) \), (d) \( \sigma_{\delta}(L^*) = \sigma_{ap}(L) \), (e) \( \sigma_{p}(L^*) = \sigma_{co}(L) \), (f) \( \sigma_{co}(L^*) \supseteq \sigma_{p}(L) \) , (g) \( \sigma(L) = \sigma_{ap}(L) \cup \sigma_{p}(L^*) = \sigma_{p}(L) \cup \sigma_{ap}(L^*) \).

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The last equation (g) implies, in particular, that \( \sigma(L) = \sigma_{ap}(L) \cup \sigma_{p}(L^*) = \sigma_{p}(L) \cup \sigma_{ap}(L^*) \).

The relationship between the fine spectrum of bounded linear operator and fine spectrum of its adjoint is given by Fig. 1.

The last equation (g) implies, in particular, that \( \sigma(L) = \sigma_{ap}(L) \cup \sigma_{p}(L^*) = \sigma_{p}(L) \cup \sigma_{ap}(L^*) \).

1.3. Goldberg’s classification of spectrum

If \( X \) is a Banach space, \( B(X) \) denotes the collection of all bounded linear operators on \( X \) and \( T \in B(X) \), so there are three possibilities for \( R(T) \), the range of \( T \):

(I) \( R(T) = X \), (II) \( \overline{R(T)} = X \), but \( R(T) \neq X \),

(III) \( R(T) \neq X \).

and three possibilities for \( T^{-1} \):

(1) \( T^{-1} \) exists and is continuous, (2) \( T^{-1} \) exists but is discontinuous, (3) \( T^{-1} \) does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: \( I \), \( I_1 \), \( I_2 \), \( I_3 \), \( II \), \( II_1 \), \( II_2 \), \( II_3 \), \( III_1 \), \( III_2 \), \( III_3 \).

If an operator is in state \( III_2 \) for example, then \( R(T) \neq X \) and \( T^{-1} \) exist but are discontinuous (see [2]).

The relationship between the fine spectrum of bounded linear operator and fine spectrum of its adjoint is given by Fig. 1.

Fig. 1. State diagram for \( B(X) \) and \( B(X^*) \) for a non-reflective Banach space \( X \).

If \( \lambda \) is a complex number such that \( T = \lambda I - L \in II_2 \), then \( \lambda \in \rho(L,X) \).

All scalar values of \( \lambda \) not in \( \rho(L,X) \) comprise the spectrum of \( L \). The further classification of \( \sigma(L,X) \) gives rise to the fine spectrum of \( L \). That is, \( \sigma(L,X) \) can be divided into the subsets \( \lambda, \sigma(L,X) = II_2, \sigma(L,X) = II_3, \sigma(L,X) = III_2, \sigma(L,X) = III_3 \).

For example, if \( T = \lambda I - L \) is in a given state, \( III_2 \) (say), then we write \( \lambda \in II_2, \sigma(L,X) \).

By the definitions given above, Table 2 can be generalized as shown below.
Table 2. The relationship between subdivisions of the spectrum and Golberg’s classification

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$R(\lambda; L) = X$ and is bounded</td>
<td>$\lambda \in \rho(L)$</td>
<td>$\lambda \in \sigma_p(L)$, $\lambda \in \sigma_{ap}(L)$</td>
</tr>
<tr>
<td>II</td>
<td>$R(\lambda; L) = X$</td>
<td>$\lambda \in \rho(L)$, $\lambda \in \sigma_\delta(L)$</td>
<td>$\lambda \in \sigma_p(L)$, $\lambda \in \sigma_{ap}(L)$, $\lambda \in \sigma_\delta(L)$</td>
</tr>
<tr>
<td>III</td>
<td>$R(\lambda; L) \neq X$</td>
<td>$\lambda \in \sigma_\delta(L)$, $\lambda \in \sigma_{co}(L)$</td>
<td>$\lambda \in \sigma_p(L)$, $\lambda \in \sigma_{ap}(L)$, $\lambda \in \sigma_\delta(L)$</td>
</tr>
</tbody>
</table>

Let $w; c_0; c; \ell^p$ denote the set of all sequences; the space of all null sequences; convergent sequences; sequences such that $\sum |x_i| < \infty$, respectively.

An infinite matrix $A$ is said to be conservative if it is a selfmap of $c$, the space of convergent sequences. Necessary and sufficient conditions for $A$ to be conservative are the well-known Kojima-Schur conditions; i.e.,

(i). $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$;

(ii). $\lim_n a_{nk} = \alpha_k$, exists for each $k$, and

(iii). $t = \lim_n \sum_{k=0}^{\infty} a_{nk} < \infty$ exists.

Associated with each conservative matrix $A$ is a function $\chi$ defined by $\chi(A) = t - \sum \alpha_k$. If $\chi(A) \neq 0$, $A$ is called coregular, and, if $\chi(A) = 0$ then $A$ is called connul. A matrix $A = (a_{nk})$ is said to be regular if $\lim_{\theta} x = \lim x$ for each $x \in c$. If $\alpha_k = 0$ for each $k$ and $t = 1$ in (iii), then the operator $A$ is called regular (see [3]).

2. The approximate point spectrum, defect spectrum and compression spectrum of $C_1$

In this section we developed the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $C_1 = (c_{nk})$, where otherwise $c_{nk} = 1/(n+1), \; k \leq n$ and $c_{nk} = 0, \; n, k$. Reade [4] and in 1975, Wenger [5] determined spectra and the fine spectra of Cesaro operator $C_1$ on $c$, the space of convergent sequences, respectively.

2.1. Subdivision of the spectrum of $C_1$ on $c$.

Theorem 2.1.

(a) $\sigma_{ap}(C_1, c) = \{\lambda \in C : |\lambda - 1/2| = 1/2\}$,

(b) $\sigma_\delta(C_1, c) = \{\lambda \in C : |\lambda - 1/2| \leq 1/2\}$,

(c) $\sigma_{co}(C_1, c) = \{\lambda \in C : |\lambda - 1/2| > 1/2\}$.

Proof:

(a) $\sigma_{ap}(C_1, c) = \sigma(C_1, c) \setminus III, \sigma(C_1, c)$ is obtained from Table 2. By [[5], Theorem 1-3], we have

\[
\sigma_{ap}(C_1, c) = \left\{ \lambda \in C : \Re \frac{1}{\lambda} \geq 1 \right\} \setminus \left\{ \lambda \in C : \Re \frac{1}{\lambda} > 1 \right\} = \left\{ \lambda \in C : \Re \frac{1}{\lambda} = 1 \right\} = \{\lambda \in C : |\lambda - 1/2| = 1/2\}.
\]

(b) $\sigma_\delta(C_1, c) = \sigma(C_1, c) \setminus I, \sigma(C_1, c)$ is obtained from Table 2. Moreover, since the equality $\sigma(C_1, c) = III, \sigma(C_1, c) \cup II, \sigma(C_1, c) \cup III, \sigma(C_1, c)$ holds by [[5], Theorem 1-5] and the subdivisions in Goldberg’s classification are disjoint, then the equalities $I, \sigma(C_1, c) = \emptyset$, $II, \sigma(C_1, c) = \emptyset$, $III, \sigma(C_1, c) = \emptyset$ are valid. Hence $\sigma_\delta(C_1, c) = \sigma(C_1, c)$.

(c) From Table 2,
\[ \sigma_{\infty}(C_1, c) = \sigma(C_1, c) \setminus \left( I_3 \sigma(C_1, c) \cup I_2 \sigma(C_1, c) \right) \]

and by
\[ I_3 \sigma(C_1, c) = \emptyset, \quad I_2 \sigma(C_1, c) = \emptyset, \]
we have
\[ \sigma_{\infty}(C_1, c) = \left\{ \lambda \in \mathbb{C} : \text{Re} \frac{1}{\lambda} \geq 1 \right\} \cup \left\{ \lambda \in \mathbb{C} : \text{Re} \frac{1}{\lambda} = 1, \lambda \neq 1 \right\} \]
\[ = \left\{ \lambda \in \mathbb{C} : \text{Re} \frac{1}{\lambda} > 1 \right\} \cup \{1\} \cup \left( \lambda \in \mathbb{C} : |\lambda - 1/2| < 1/2 \right) \cup \{1\}. \]

The following corollary can be obtained by Proposition 1.1.

**Corollary 2.1.**
(a) \( \sigma_{ap}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : |\lambda - q/2| \leq q/2 \right\} \)
(b) \( \sigma_{b}(C_1^*, \ell^p) = \left( \lambda \in \mathbb{C} : \text{Re} \frac{1}{\lambda} = 1, \lambda \neq 1 \right) \)
(c) \( \sigma_{co}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : \text{Re} \frac{1}{\lambda} > 1 \right\} \cup \{1\} \cup \left( \lambda \in \mathbb{C} : |\lambda - 1/2| < 1/2 \right) \cup \{1\}. \]

2.2. **Subdivision of the spectrum of \( C_1 \) on \( \ell^p \).**

In 1985, M. Gonzales [6] determined the fine spectra of Cesaro operator \( C_1 \) on \( \ell^p \).

**Theorem 2.2.** [6] Let \( 1 < p < \infty, \quad p^{-1} + q^{-1} = 1, \) and \( C_1 \) acting on \( \ell^p \).
(a) For each \( z \in \text{int} \sigma(C_1, \ell^p) = \left\{ \lambda : |\lambda - q/2| < q/2 \right\} \),
\( zI - C_1 \in \Pi_1 \).
(b) For each \( z \in \partial \sigma(C_1, \ell^p) = \left\{ \lambda : |\lambda - q/2| = q/2 \right\}, \) \( zI - C_1 \) is injective with dense range, that is, \( zI - C_1 \in \Pi_2 \).

**Theorem 2.3.** Let \( p > 1 \) and \( p^{-1} + q^{-1} = 1, \) then
(a) \( \sigma_{ap}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : |\lambda - q/2| = q/2 \right\} \)
(b) \( \sigma_{b}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : |\lambda - q/2| \leq q/2 \right\} \)
(c) \( \sigma_{co}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : |\lambda - q/2| < q/2 \right\} \)

**Proof:** The equality \( I_3 \sigma(C_1, \ell^p) = \emptyset \) is clear with Theorem 2.2. Therefore, the proof is taken by Theorem 2.2.

The following corollary can be obtained by Proposition 1.1.

**Corollary 2.2.** Let \( p > 1 \) and \( p^{-1} + q^{-1} = 1, \) then
(a) \( \sigma_{ap}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : |\lambda - q/2| \leq q/2 \right\} \)
(b) \( \sigma_{b}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : |\lambda - q/2| = q/2 \right\} \)
(c) \( \sigma_{co}(C_1^*, \ell^p) = \left\{ \lambda \in \mathbb{C} : |\lambda - q/2| < q/2 \right\} \)

3. **The approximate point spectrum, defect spectrum and compression spectrum of rhaly operator**

We assume that, given a scalar sequence of \( a = (a_n), \) a Rhaly matrix \( R_n = (a_{nk}) \) is the lower triangular matrix where \( a_{nk} = a_n, \quad k \leq n \) and \( a_{nk} = 0 \) otherwise.
(a) \( L = \lim_n (n+1)a_n \) exists, finite,
(b) \( a_n > 0 \) for all \( n, \) and
(c) \( a_i \neq a_j \) for \( i \neq j. \) Let \( S \) denote the set \( \{ a_n : n = 0, 1, 2, \ldots \}. \)
(d) \( a = (a_n) \) is monotone decreasing.

In [7], the spectrum of the Rhaly operators on \( c_0 \) and \( c, \) under the assumption that \( L = \lim_n (n+1)a_n \neq 0 \) has been determined. Also, in [8-12] the spectrum of the Rhaly operator over some kinds of spaces has been determined.

3.1. **Subdivision of the spectrum of \( R_n \) on \( c_0 \) for \( L = 0. \)**

**Theorem 3.1.** If \( L = \lim_n (n+1)a_n = 0, \) then
(a) \( \sigma_{ap}(R_n, c_0) = S \cup \{0\}, \)
(b) \( \sigma_{b}(R_n, c_0) = S \cup \{0\}, \)
(c) \( \sigma_{co}(R_n, c_0) = S. \)

**Proof:** The proof is taken by [[9], Theorem 5-7].

The following corollary can be obtained by Proposition 1.1.

**Corollary 3.1.** (a) \( \sigma_{ap}(R_n^*, \ell^1) = S \cup \{0\}, \) (b) \( \sigma_{b}(R_n^*, \ell^1) = S \cup \{0\}. \)

3.2. **Subdivision of the spectrum of \( R_n \) on \( c \) for \( L = 0. \)**

**Theorem 3.2.** If \( L = \lim_n (n+1)a_n = 0, \) then
(a) \( \sigma_{ap}(R_n, c) = S \cup \{0\}, \)
(b) $\sigma_{\delta}(R_a, c) = S \cup \{0\}$,
(c) $\sigma_{co}(R_a, c) = S \cup \{0\}$.

**Proof:** The proof is taken by [[9], Theorem 12, 14, 15].

The following corollary can be obtained by Proposition 1.1.

**Corollary 3.2.** (a) $\sigma_{ap}(R_a^*, \ell^1) = S \cup \{0\}$, (b) $\sigma_{\delta}(R_a^*, \ell^1) = S \cup \{0\}$.

3.3. **Subdivision of the spectrum of $R_a$ on $\ell^p$ for $L = 0$.**

Leibowitz [[13], Proposition 3.1] shows that
(a) If $\{(n+1)a_n\}$ is bounded, then $R_a$ acts boundedly on $\ell^p$ for $p > 1$, and
$\|R_a\| \leq (p/(p-1))\sup_n |(n+1)a_n|.$
(b) If $\lim_n \|(n+1)a_n\| = 0$, then $R_a$ is compact operator on $\ell^p$ for every $p > 1$.
(c) If $\lim_n \|(n+1)a_n\| = \infty$, then $R_a$ is not bounded on $\ell^p$ for every $p > 1$.

**Theorem 3.3.** If $L = \lim_n (n+1)a_n = 0$, then (a) $\sigma_{ap}(R_a, \ell^p) = S \cup \{0\}$ for $p \geq 2$,
(b) $\sigma_{\delta}(R_a, \ell^p) = S \cup \{0\}$ for $p \geq 2$, (c) $\sigma_{co}(R_a, \ell^p) = S$ for $p \geq 2$.

**Proof:** The proof is taken by [[10], Theorem 2.3-2.5].

The following corollary can be obtained by Proposition 1.1.

**Corollary 3.3.** If $L = \lim_n (n+1)a_n = 0$, then (a) $\sigma_{ap}(R_a^*, \ell^1) = S \cup \{0\}$ for $p \geq 2$, $p^{-1} + q^{-1} = 1$,
(b) $\sigma_{\delta}(R_a^*, \ell^1) = S \cup \{0\}$ for $p \geq 2$, $p^{-1} + q^{-1} = 1$.

3.4. **Subdivision of the spectrum of $R_a$ for $0 < L < \infty$.**

**Theorem 3.4.** Let $0 < L < \infty$, then

(a) $\sigma_{ap}(R_a^*, \ell^1) = \{\lambda : |\lambda - L/2| \leq L/2\} \cup \{a_i \in S : a_i \geq L\},$
(b) $\sigma_{\delta}(R_a, c) = S \cup \{0\}$, (c) $\sigma_{co}(R_a, c) = \{\lambda : |\lambda - L/2| \leq L/2\} \cup S \cup \{L\}.$

**Proof:** (a) Since $\sigma_{ap}(R_a^*, c) = \sigma(R_a^*, c) \backslash \{\lambda : |\lambda - L/2| \leq L/2\}$, $\sigma_{ap}(R_a, c) = \left[\left\{\lambda : |\lambda - L/2| \leq L/2\right\} \cup S \cup \{0\}\right]$ 
$\cup \{a_i \in S : a_i \geq L\}$
$= \{\lambda : |\lambda - L/2| \leq L/2\} \cup \{\lambda : \lambda \neq L\} \cup \{a_i \in S : a_i \geq L\}$
$= \{\lambda : |\lambda - L/2| \leq L/2\} \cup \{\lambda : \lambda \neq L\} \cup \{a_i \in S : a_i \geq L\}$

**Theorem 3.5.** Let $0 < L < \infty$, $p > 1$ and $p^{-1} + q^{-1} = 1$, then

(a) $\sigma_{ap}(R_a, \ell^p) = \{\lambda : |\lambda - qL/2| \leq qL/2\} \cup S,$
(b) $\sigma_{\delta}(R_a, \ell^p) = \{\lambda : |\lambda - qL/2| \leq qL/2\} \cup S,$
(c) $\sigma_{co}(R_a, \ell^p) = \{\lambda : |\lambda - qL/2| < qL/2\} \cup S.$

**Proof:** The proof is taken by [[10], Theorem 3.3] and [[15], Theorem 6-8].
The following corollary can be obtained by Proposition 1.1.

**Corollary 3.5.**
Let $0 < L < \infty$, $p > 1$ and $p^{-1} + q^{-1} = 1$, then

(a) $\sigma_{ap}(R_n^*, \ell^q) = \{ \lambda : |\lambda - qL/2| = qL/2 \} \cup S$,

(b) $\sigma_s(R_n^*, \ell^q) = \{ \lambda : \Re \lambda - qL/2 \leq qL/2 \} \cup S$.

4. The approximate point spectrum, defect spectrum and compression spectrum of weight mean operator

A weight mean matrix $A$ is a lower triangular matrix with entries $a_{nk} = p_k / P_n$, where $p_k > 0$, $p_n \geq 0$ for $n > 0$, and $P_n = \sum_{k=0}^n p_k$.

The necessary and sufficient condition for the regularity of $A$ is that $\lim P_n = \infty$.

In [16-20] the spectrum and fine spectrum of weight mean matrix over some kinds of spaces has been determined.

The necessary and sufficient condition for the regularity of $A$ is that $\lim P_n = \infty$.

4.1. Subdivision of the spectrum of $A$ on $c$.

We shall consider those regular weighted mean methods for which $\delta = \gamma$, i.e., for which the main diagonal entries converge.

**Theorem 4.1.** Let $A$ be a regular weighted mean method such that $\lim p_n / P_n = \gamma > 0$, $p_n / P_n \geq \gamma$ for all $n$ sufficiently large and suppose no diagonal entry of $A$ occurs an infinite number of times, then

(a) $\sigma_{ap}(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| \leq (1-\gamma)/(2-\gamma) \} \cup E$,

(b) $\sigma_s(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma) \} \cup S$,

(c) $\sigma_{ap}(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma) \} \cup S$.

**Proof:** (a) Since the relation

$$III \sigma(A,c) = \left\{ \lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma) \right\} \cup S$$

holds by [[18] Theorem 1-2], use [[17] Corollary 2] to get

$$\sigma_{ap}(A,c) = \sigma(A,c) \setminus III \sigma(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| = (1-\gamma)/(2-\gamma) \} \cup E.$$  

(b) $\sigma_s(A,c) = III \sigma(A,c) \cup II \sigma(A,c) \cup III \sigma(A,c)$

is easily seen by [[17] Corollary 2] and [[18] Theorem 1-4]. Therefore,

$$III \sigma(A,c) = I \sigma(A,c) = II \sigma(A,c) = \emptyset$$

and hence,

$$\sigma_{ap}(A,c) = \sigma(A,c) \setminus I \sigma(A,c) = \sigma(A,c).$$

(c) Since, $\sigma_{ap}(A,c) = III \sigma(A,c) \cup II \sigma(A,c) \cup III \sigma(A,c)$

the result is taken by [[17] Corollary 2] and [[18] Theorem 1-4].

The following corollary can be obtained by Proposition 1.1.

**Corollary 4.1.** Let $A$ be a regular weighted mean method such that $\lim p_n / P_n = \gamma > 0$, $p_n / P_n \geq \gamma$ for all $n$ sufficiently large and suppose no diagonal entry of $A$ occurs an infinite number of times, then

(a) $\sigma_{ap}(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| \leq (1-\gamma)/(2-\gamma) \} \cup E$,

(b) $\sigma_s(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma) \} \cup S$,

(c) $\sigma_{ap}(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| < (1-\gamma)/(2-\gamma) \} \cup S$.

4.2. Subdivision of the spectrum of $A$ on $\ell_p$.

In [16] it was shown that, if

$$\lim p_n / P_n = \gamma > 0, (4.1)$$

then $A \in B(\ell_p)$ and

$$\sigma(A,\ell_p) = \{ \lambda : |\lambda - 1/(2-\gamma)| \leq (1-\gamma)/(2-\gamma) \} \cup S. (4.2)$$

In Theorem 4.2 and Corollary 4.2 $A$ is a weighted mean matrix satisfying $\gamma < 1$, since $\gamma = 1$ implies $\sigma(A,\ell_p) = S$.

**Theorem 4.2.** Let $A$ be a regular weighted mean method such that $\gamma = \lim p_n / P_n$ exists and $p_n / P_n \geq \gamma$ for all $n$ sufficiently large. Suppose that no main diagonal entry of $A$ occurs an infinite number of times, then

$$\sigma_{ap}(A,c) = \{ \lambda : |\lambda - 1/(2-\gamma)| \leq (1-\gamma)/(2-\gamma) \} \cup S.$$
Theorem 4.3. Let \( A \) be a weighted mean method such that \( \lim_n n \Delta (p_n/P_n) = \alpha > 1/p \). Suppose no diagonal entry of \( A \) occurs an infinite number of times, if
\[
\lim_{n} n \Delta (p_n/P_n) = \lim_{n} n (n c_{n,1} + n c_{n,2}) = 0,
\]
then
\[
\sigma_{\omega}(A, \ell^p) = [\lambda : \beta = \alpha p/2(\alpha p - 1) ] \cup \{0\},
\]
where \( c_{n} = p_n/P_n \).

Proof: (a) The proof is taken by (4.4) and [20] Theorem 6,8.

(b) From [20] Theorem 6-9, the following corollary can be obtained by Proposition 1.1, such that \( \lambda \).
\[
\lim_n n \Delta \left( np_n / P_n \right) = \lim_n \left( nc_n - (n+1)c_{n+1} \right) = 0,
\]
then
(a) \[ \sigma_0(A^*, \ell_p) = \left\{ \lambda : |\lambda - \alpha p / 2(\alpha p - 1)| \leq \alpha p / 2(\alpha p - 1) \right\} \cup S \cup \{0\}, \]
(b) \[ \sigma_0(A^*, \ell_p) = \left\{ \lambda : |\lambda - \alpha p / 2(\alpha p - 1)| = \alpha p / 2(\alpha p - 1) \right\}, \]
(c) If \( A \) is a triangle, then
\[
\sigma_p(A^*, \ell_p) = \left\{ \lambda : |\lambda - \alpha p / 2(\alpha p - 1)| \leq \alpha p / 2(\alpha p - 1) \right\} \cup S \cup \{0\},
\]
where \( c_n = p_n / P_n \).

References