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The Nehari manifold for a Navier boundary value problem involving the p-biharmonic

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Abstract

In this paper, we study the Nehari manifold and its application on the following Navier boundary value problem involving the p-biharmonic

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \dfrac{1}{p^*} f(x,u) + \lambda |u|^{q-2}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$. We prove that the problem has at least two nontrivial nonnegtive solutions when the parameter λ belongs to a certain subset of R.

Keywords: Multiple positive solutions; Nehari manifold; critical Sobolev exponent

1. Introduction

The aim of this paper is to establish the existence and multiplicity of nontrivial non-negative solutions to the Navier boundary value problem involving the p-biharmonic

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \dfrac{1}{p^*} f(x,u) + \lambda |u|^{q-2}u, & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset R^N\ (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $f \in C(\Omega \times R^+; R^+)$ is positively homogeneous of degree $p^* - 1$ $(p^* = \frac{pN}{N-2p}$ if $N > 2p)$, that is, $f(x,tu) = t^{p^*-1} f(x,u)$ (t>0) hold for all $(x,u) \in (\Omega, R^+)$ and $\lambda \in R^+$. We assume that $2 < q < p < p^*$, λ>0.

Put

$$F(x,u) = \int_0^u f(x,t)dt \quad (2)$$

for each $(x,u) \in \Omega \times W^{2,p}(\Omega)$.

Through this paper we assume:

(H1) F is homogeneous of degree p^*, that is,
$F(x,tu) = t^{p^*} F(x,u)$, (t > 0), $\forall\ x \in \bar{\Omega}$, $u \in R^+$;

(H2) F(x,0)=f(x,0)=0, where $u \in \mathbb{R}^+$;

(H3) f(x,u) is strictly increasing function respect to u for all $u > 0$.

In addition, using assumption (H1), we have the so-called Euler identity

$$uf(x,u) = p^* F(x,u), \quad (3)$$

and

$$F(x,u) \leq K|u|^{p^*}, \text{ for some constant } K > 0. \quad (4)$$

In recent years, several authors have used the Nehari manifold and fibering maps (i.e., maps of

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the form $t \mapsto J_\lambda(tu)$ where J_λ is the Euler function associated with the equation) to solve semilinear and quasilinear problems, for instance, we cite papers [1-9]. For example, the authors in [10] studied the following subcritical semi-linear elliptic equation with sign-changing weight function

$$\begin{cases} -\Delta u(x) = \lambda\ a(x)\ u + b(x)|u|^{\gamma-2}\ u(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\,\Omega \end{cases} \qquad (5)$$

where γ>2. Also, the authors in [10] by the same arguments considered the following semilinear elliptic problem:

$$\begin{cases} -\Delta u = f_\lambda(x)|u|^{q-2}\ u + g(x)|u|^{p-2}\ u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\,\Omega \end{cases} \qquad (6)$$

where 1<q<2<p. Exploiting the relationship between the Nehari manifold and fibering maps, they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value.

In this work, motivated by the above works we are interested in studying the problem (1) by using a variational method involving the Nehari manifold (see [3, 8, 10]).

Our main result is Theorem 1. Under the hypothesis of the theorem, the Nehari manifold associated with the problem consists of two distinct components. We shall prove that there exists at least one solution on each component.

This paper is organized as follows. In Section 2, we give some notations, preliminaries and properties of the Nehari manifold and set up the variational framework of the problem. In Section 3, we give our main result.

2. Notations and preliminaries

Here, in the sequel, W define the Sobolev space $W = W^{2,p}(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p\, dx \right)^{\frac{1}{p}}.$$

First we give the definition of the weak solution of problem (1).

Definition 1. We say that $u \in W$ is a weak solution of (1) if for all $v \in W$ we have

$$\int_\Omega |\Delta u|^{p-2}\, \Delta u\, \Delta v dx - \frac{1}{p^*}\int_\Omega f\ (x, u)\ v dx - \lambda \int_\Omega |u|^{q-2}\, u v dx.$$

Thus, by (3) the corresponding energy functional of problem (1) is defined by

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*}\int_\Omega F(x,u)\,\mathrm{dx} - \frac{\lambda}{q}\int_\Omega g\,\left|u^q\right| dx,$$

For $u \in W$.

Now, we consider the problem on the Nehari manifold. Define the Nehari manifold

$$N_\lambda = \{u \in W \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0\},$$

where

$$\langle J'_\lambda(u), u\rangle = \|u\|^p - \int_\Omega F(x,u)\, dx - \lambda \int_\Omega g\,\left|u^q\right| dx.$$

Note that N_λ contains every non-zero solution of problem (1). Define

$$\Phi_\lambda(u) = \langle J'_\lambda(u), u\rangle,$$

then for $u \in N_\lambda$,

$$\langle \Phi'_\lambda(u), u\rangle = p\ \|u\|^p - p^* \int_\Omega F(x,u)dx - q\lambda \int_\Omega g\ \left|u^q\right| dx \qquad (7)$$

$$= \text{(p-p}^*)\ \|u\|^p - (q - p^*)\lambda \int_\Omega g\ \left|u^q\right|\ dx \qquad (8)$$

$$= \text{(p-q)}\ \|u\|^p - (p^* - q)\int_\Omega F(x,u)\ dx \qquad (9)$$

$$= \text{(p-p}^*)\int_\Omega F(x,u)dx - (q-p)\ \lambda \int_\Omega g\ \left|u^q\right|\ dx.$$

(10)

We split N_λ into three parts:

$$N_\lambda^+ = \{u \in W \setminus \{0\} : \langle \Phi'_\lambda(u), u\rangle > 0\},$$
$$N_\lambda^0 = \{u \in W \setminus \{0\} : \langle \Phi'_\lambda(u), u\rangle = 0\},$$
$$N_\lambda^- = \{u \in W \setminus \{0\} : \langle \Phi'_\lambda(u), u\rangle < 0\}.$$

We now present some important properties of N_λ^+, N_λ^0 and N_λ^-.

Lemma 1. There exists $\mu > 0$ such that $0 < \lambda < \mu$, we have $N_\lambda^0 = \phi$.

Proof: Suppose otherwise, thus for

$$\mu = \left[\frac{p-q}{KC_1^{p^*}(p^*-q)}\right]^{\frac{p-q}{p^*-p}} \left[\frac{p^*-p}{C_1^q(p^*-q)}\right],$$

where C_1 is the best Sobolev constant for the embedding of $W^{2,p}(\Omega)$ in $L^{p^*}(\Omega)$. There exists $0<\lambda<\mu_0$ such that $N_\lambda^0 \neq \phi$. Then for $u \in N_\lambda^0$, we have

$$0 = \langle \Phi'_\lambda(u), u\rangle = (p-p^*)\ \|u\|^p - (q-p^*)\ \lambda \int_\Omega g\ |u^q|\ dx \tag{11}$$

$$= (p-q)\ \|u\|^p - (p^*-q) \int_\Omega F(x,u)\ dx. \tag{12}$$

By the sobolev embedding theorem,

$$\int_\Omega F(x,u)\ dx \le K \int_\Omega |u|^{p^*}\ dx$$

$$\le KC_1^{p^*} \|u\|^{p^*}, \tag{13}$$

and

$$\int_\Omega g\ |u^q|\ dx \le C_1^q \|u\|^q. \tag{14}$$

By using (13) (14) in (11) and (12) we get

$$\|u\| \ge \left(\frac{p-q}{KC_1^{p^*}(p^*-q)}\right)^{\frac{1}{p^*-p}},$$

and

$$\|u\| \le \left(\frac{\lambda(p^*-q)C_1^q}{p^*-p}\right)^{\frac{1}{p-q}}.$$

This implies $\lambda \ge \mu$, which is a contradiction. Thus, we can conclude that there exists $\mu>0$ such that $0<\lambda<\mu$, we have $N_\lambda^0 = \phi$.

By (8) and (9), It is easy to see that the following lemma holds.

Lemma 2. We have:

(i) if $u \in N_\lambda^+$, then $\int_\Omega g\ |u|^q\,dx > 0;$

(ii) if $u \in N_\lambda^-$, then $\int_\Omega F(x,u)\ dx > 0;$

(iii) if $u \in N_\lambda^0$, then $\int_\Omega g\ |u|^q\,dx > 0$ and $\int_\Omega F(x,u)\,dx > 0.$

Lemma 3. The energy functional J_λ is coercive and bounded below on N_λ.

Proof: If $u \in N_\lambda$, then by the Sobolev embedding theorem

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*}\int_\Omega F(x,u)\ dx - \frac{\lambda}{q}\int_\Omega g\ |u^q|\ dx$$
$$= (\frac{1}{p}-\frac{1}{p^*})\ \|u\|^p - \lambda\ (\frac{1}{q}-\frac{1}{p^*})\int_\Omega g\ |u^q|\ dx$$
$$\ge (\frac{1}{p}-\frac{1}{p^*})\ \|u\|^p - \lambda C_1^q\ (\frac{1}{q}-\frac{1}{p^*})\|u\|^q.$$

Since $q<p<p^*$, we see that J_λ is coercive and bounded below on N_λ.

Lemma 4. Suppose that u_0 is a local minimizer for J_λ on N_λ and $u_0 \notin N_\lambda^0$, then $J'_\lambda(u_0)=0$ in W^{-1} (the dual space of sobolev space W).

Proof: The proof is standard (cf. We [7]). If u_0 is a local minimizer for J_λ on N_λ, then u_0 is a solution of the optimization problem minimizing $J_\lambda(u)$ subject to $\Phi_\lambda(u)=0$.

Hence, by the theory of Lagrange multiplies, there exists $\theta \in R$, such that

$$J'_\lambda(u_0) = \theta\ \Phi'_\lambda(u_0)\ \text{ in }\ W^{-1}(\Omega).$$

Thus

$$\langle J'_\lambda(u_0), u_0\rangle = \theta\ \langle \Phi'_\lambda(u_0), u_0\rangle\ \text{ in }\ W^{-1}(\Omega).$$

Since $u_0 \in N_\lambda$, we have $\langle J'_\lambda(u_0), u_0\rangle = 0.$ Moreover $\langle \Phi'_\lambda(u_0), u_0\rangle \neq 0,$ then $\theta = 0$. This completes the proof.

By Lemma 1, we let

$$\Theta_{\mu_0} = \{\lambda \in R^+ \setminus \{0\} : 0 < \lambda < \mu_0\},$$

Where $\mu_0 = \frac{q}{p}\mu$. If $\lambda \in \Theta_{\mu_0}$, we have $N_\lambda = N_\lambda^+ \cup N_\lambda^-$. Define

$$\theta_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u), \qquad \theta_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u).$$

Then, we have the following result.

Lemma 5. There exists a positive number μ_0 such that if $\lambda \in \Theta_{\mu_0}$, then:

(i) $\theta_\lambda^+ < 0$;

(ii) $\theta_\lambda^- > k_0$, for some $k_0 = k_0(p^*, p, q, C_1) > 0$.

Proof: (i) for $u \in N_\lambda^+$, we have

$$\lambda \int_\Omega g\ |u|^q\, dx > \frac{p^* - p}{p^* - q} \|u\|^p,$$

and so

$$J_\lambda(u) = (\frac{1}{p} - \frac{1}{p^*})\ \|u\|^p - \lambda(\frac{1}{q} - \frac{1}{p^*}) \int_\Omega g\ |u|^q dx$$

$$\leq \frac{p^* - p}{pp^*} \|u\|^p - \frac{p^* - p}{p^* q} \|u\|^p < 0.$$

Thus, $\theta_\lambda^+ < 0$.

(ii) for $u \in N_\lambda^+$, by Lemma 2, we have

$$\int_\Omega F(x,u)\ dx > \frac{p - q}{p^* - q} \|u\|^p > 0,$$

and by Lemma 1

$$\|u\| \geq \left(\frac{p - q}{K C_1^{p^*} (p^* - q)} \right)^{\frac{1}{p^* - p}}. \qquad (15)$$

By Lemma 3, we have

$$J_\lambda(u) \geq (\frac{1}{p} - \frac{1}{p^*}) \|u\|^p - \lambda C_1^q (\frac{1}{q} - \frac{1}{p^*}) \|u\|^q$$
$$= \|u\|^q \left[\frac{p^* - p}{pp^*} \|u\|^{p-q} - \lambda C_1^q \frac{p^* - q}{p^* q} \right].$$

Thus, if $0 < \lambda < \mu_0$ then

$$J_\lambda(u) > k_0 \quad \text{for all} \quad u \in N_\lambda^-,$$

for some $k_0 = k_0(p^*, p, q, C_1) > 0$. This completes the proof.

For each $u \in W \setminus \{0\}$ such that $\int_\Omega F(x,u)\ dx > 0$, let

$$t_{\max} = \left(\frac{(p - q)\ \|u\|^p}{(p^* - q) \int_\Omega F(x,u)\ dx} \right)^{\frac{1}{p^* - p}} > 0.$$

Then, we have the following lemma:

Lemma 6. For each $u \in N_\lambda^-$ we have:

(i) if $\lambda \int_\Omega g\ |u|^q\ dx \leq 0$, then there is unique $t^- > t_{\max}$ such that $t^- u \in N_\lambda^-$ and

$$J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu);$$

(ii) if $\lambda \int_\Omega g\ |u|^q\ dx \geq 0$, then there are unique $0 < t^+ = t^+(u) < t_{\max} < t^-$ such that $t^+ u \in N_\lambda^+, t^- u \in N_\lambda^-$ and

$$J_\lambda(t^+ u) = \sup_{0 \leq t \leq t_{\max}} J_\lambda(tu), \qquad J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu).$$

Proof: Fix $u \in W$ with $\int_\Omega F(x,u)\ dx > 0$. Let

$$S(t) = t^{p-q} \|u\|^p - t^{p^* - q} \int_\Omega F(x,u)\ dx.$$

Clearly, $S(0) = 0, S(t) \to -\infty$ as $t \to \infty$. Since

$$S'(t) = (p-q)t^{p-q-1} \|u\|^p - (p^* - q) t^{p^* - q - 1} \int_\Omega F(x,u)\ dx,$$

we have $S'(t) = 0$ at $t = t_{\max}, S'(t) > 0$ for $[0, t_{\max})$ and $S'(t) < 0$ for $t \in (t_{\max}, +\infty)$. Then S(t) achieves its maximum at $t_{\max}$, increasing for

$t \in [0, t_{max})$ and decreasing for $t \in (t_{max}, +\infty)$ Moreover,

$$S(t_{max}) = \left(\frac{(p-q)\ \|u\|^p}{(p^*-q)\int_\Omega F(x,u)\ dx}\right)^{\frac{p-q}{p^*-p}} \|u\|^p$$

$$-\left(\frac{(p-q)\ \|u\|^p}{(p^*-q)\int_\Omega F(x,u)\ dx}\right)^{\frac{p^*-q}{p^*-p}} \int_\Omega F(x,u)\ dx$$

$$= \|u\|^q \left[\left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} - \left(\frac{p-q}{p^*-q}\right)^{\frac{p^*-q}{p^*-p}}\right]\left(\frac{\|u\|^p}{\int_\Omega F(x,u)\ dx}\right)^{\frac{p-q}{p^*-p}}$$

$$\geq \|u\|^q \left(\frac{p-q}{p^*-q}\cdot\frac{1}{KC_1^{p^*}}\right)^{\frac{p-q}{p^*-p}}\left(\frac{p^*-p}{p^*-q}\right).$$

(i) $\lambda\int_\Omega g\ |u|^q\ dx \leq 0,$ there is a unique $t^- > t_{max}$ such that $S(t^-) = \lambda\int_\Omega g\ |u|^q\ dx$ and $S'(t) < 0$. Now,

$$(p-q)(t^-)^p \|u\|^p - (p^*-q)(t^-)^{p^*}\int_\Omega F(x,u)dx = (t^-)^{p+1}S'(t) < 0,$$

and

$$\langle J'_\lambda(t^-u), t^-u\rangle = (t^-)^q\left[S'(t^-) - \lambda\int_\Omega g\ |u|^q\ dx\right] = 0.$$

Thus $t^-u \in N_\lambda^-$. Since for $t > t_{max}$, we have

$$(p-q)\|tu\|^p - (p^*-q)\int_\Omega F(x,tu)dx < 0, \quad \frac{d^2}{dt^2}J_\lambda(tu) < 0$$

and

$$\frac{d}{dt}J_\lambda(tu) = t^{p-1}\|u\|^p - t^{p^*-1}\int_\Omega F(x,tu)dx$$
$$-t^{q-1}\lambda\int_\Omega g\ |u|^q\ dx = 0, \quad \text{for } t = t^-.$$

Thus, $J_\lambda(t^-u) = \sup_{t\geq 0} J_\lambda(tu).$

(ii) $\lambda\int_\Omega g\ |u|^q dx \geq 0.$

For $0 < \lambda < \mu_0 < \mu,$ we have

$$\begin{aligned} S(0) = 0 &< \lambda\int_\Omega g\ |u|^q\ dx \\ &\leq \lambda C_1^q \|u\|^q \\ &< \|u\|^q\left(\frac{p-q}{p^*-q}\cdot\frac{1}{KC_1^{p^*}}\right)^{\frac{p-q}{p^*-p}}\left(\frac{p^*-p}{p^*-q}\right) \\ &\leq S(t_{max}). \end{aligned}$$

There are unique t^+ and t^- such that $0 < t^+ < t_{max} < t^-$,

$$S(t^-) = \lambda\int_\Omega g\ |u|^q\ dx = S(t^+),$$

and

$$S'(t^+) > 0 > S'(t^-).$$

We have $t^+u \in N_\lambda^+, t^-u \in N_\lambda^-$ and

$$J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u), \quad t \in [t^+, t^-],$$

and

$$J_\lambda(t^+u) \leq J_\lambda(tu), \quad t \in [0, t_{max}].$$

Thus,

$$J_\lambda(t^+u) \leq \sup_{0\leq t\leq t_{max}} J_\lambda(tu), \quad {}_\lambda(t^-u) \leq \sup_{t\geq 0} J_\lambda(tu).$$

This completes the proof.

3. Main result and proof

Our main result is as follows.

Theorem 1. If the parameter λ satisfies $0 < \lambda < \mu_0$, then problem (1) has at least two solutions u_0^+ and u_0^- such that $u_0^\pm \geq 0$ in Ω and $u_0^\pm \neq 0$.

The proof of this theorem will be a consequence of the next two theorems.

Theorem 2. If $0 < \lambda < \mu_0$ then the functional J_λ has a minimizer u_0^+ in N_λ^+ and satisfies:

(i) $J_\lambda(u_0^+) = \theta_\lambda^+;$

(ii) u_0^+ is a solution of problem (1) such that $u_0^+ \geq 0$ in Ω.

Proof: By Lemma 3, J_λ is coercive and bounded below on N_λ. Let $\{u_n\}$ be a minimizing sequence for J_λ on N_λ^+ i.e.,

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^+} J_\lambda(u).$$

Then by Lemma 3 and the compact embedding theorem, there exist a subsequence (we call again $\{u_n\}$) and $u_0^+ \in W$ such that $u_0^+ \in W$ is the solution of problem (1) and

$$u_n \to u_0^+, \text{ weakly in W,}$$
$$u_n \to u_0^+, \text{ strongly in } L^q(\Omega).$$

This implies

$$\lambda\int_\Omega g\ |u_n|^q\,dx \to \lambda\int_\Omega g\ |u_0^+|^q\,dx \qquad \text{as } n\to\infty.$$

Since

$$J_\lambda(u_n) = (\frac{1}{p}-\frac{1}{p^*})\ \|u_n\|^q - (\frac{1}{q}-\frac{1}{p^*})\ \lambda\int_\Omega |u_n|^q\,dx,$$

and by Lemma 5 (i)

$$J_\lambda(u_n) \to \theta_\lambda^+ < 0 \qquad \text{as } n\to\infty.$$

Letting $n\to\infty$, we see that $\lambda\int_\Omega |u_0^+|\ dx > 0$.

Thus u_0^+ is a nontrivial solution of problem (1).

Now, it follows that $u_n \to u_0^+$ strongly in $w^{2,p}(\Omega)$ and $J_\lambda(u_0^+) = \theta_\lambda^+$.

By $u_0^+ \in N_\lambda$ and applying Fatou's Lemma, we get

$$\begin{aligned}\theta_\lambda^+ \le J_\lambda(u_0^+) &= (\frac{1}{p}-\frac{1}{p^*})\ \|u_0^+\|^p - (\frac{1}{q}-\frac{1}{p^*})\ \lambda\int_\Omega |u_0^+|^q\,dx \\ &\le \liminf_{n\to\infty}\left((\frac{1}{p}-\frac{1}{p^*})\ \|u_n\|^p - (\frac{1}{q}-\frac{1}{p^*})\ \lambda\int_\Omega |u_n|^q\,dx \right) \\ &\le \liminf_{n\to\infty} J_\lambda(u_n) = \theta_\lambda^+.\end{aligned}$$

This implies that

$$J_\lambda(u_0^+) = \theta_\lambda^+ \quad \text{and} \quad \lim_{n\to\infty}\|u_n\|^p = \|u_0^+\|^p.$$

Let $\tilde{u}_n = u_n - u_0^+$, then by Brezis-Lieb lemma [11], we have

$$\|\tilde{u}_n\|^p = \|u_n\|^p - \|u_0^+\|^p.$$

Therefore, $u_n \to u_0^+$ strongly in $w^{2,p}(\Omega)$.

Moreover, we have $u_0^+ \in N_\lambda^+$. In fact, if $u_0^+ \in N_\lambda^-$, by Lemma 6, there are unique t_0^+ and t_0^- such that $t_0^+ u_0^+ \in N_\lambda^+$ and $t_0^- u_0^+ \in N_\lambda^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_\lambda(t_0^+u_0^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}J_\lambda(t_0^+u_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that

$$J_\lambda(t_0^+u_0^+) < J_\lambda(\bar{t}u_0^+) \le J_\lambda(t_0^-u_0^+) = J_\lambda(u_0^+),$$

which contradicts $J_\lambda(u_0^+) = \theta_\lambda^+$.

It follows from the maximum principle that u_0^+ is a positive solution of problem (1).
This completes the proof.

Next, we establish the existence of a local minimum for J_λ on N_λ^-.

Theorem 3. If λ satisfies $0 < \lambda < \mu_0$, then J_λ has minimize u_0^- in N_λ^- which satisfies:

(i) $J_\lambda(u_0^-) = \theta_\lambda^-$.

(ii) u_0^- is a solution of problem (1) such that $u_0^- \ge 0$ in Ω.

Proof: Let $\{u_n\}$ be a minimizing sequence for J_λ on N_λ^-, i.e.,

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^-} J_\lambda(u).$$

Then by Lemma 3 and the compact embedding Theorem, there exist a subsequence (we call again $\{u_n\}$) and $u_0^- \in W$ such that u_0^- is a solution of problem (1) and

$$u_n \to u_0^+, \text{ weakly in W,}$$
$$u_n \to u_0^+, \text{ strongly in } L^q(\Omega).$$

This implies

$$\lambda\int_\Omega g\ |u|^q\,dx \to \lambda\int_\Omega g\ |u_0|^q\,dx, \qquad \text{as } n\to\infty$$

and by (4)

$$\int_\Omega F(x,u_n)\ dx \to \lambda\int_\Omega F(x,u_0)\ dx, \qquad \text{as } n\to\infty.$$

Moreover, by (9) we obtain

$$\int_{\Omega} F(x,u_n)\,dx > \frac{p-q}{p^*-q}\|u_n\|^p. \qquad (16)$$

By (15) and (16) there exists a positive number η_0 such that

$$\int_{\Omega} F(x,u_n)\,dx > \eta_0$$

This implies

$$\int_{\Omega} F(x,u_0^-)\,dx > \eta_0. \qquad (17)$$

Now, we prove that $u_n \to u_0^-$ strongly in W. Suppose otherwise, then

$$\|u_0^-\| < \liminf_{n\to\infty} \|u_n\|$$

By Lemma 6, there is unique t_0^- such that $t_0^- u_0^- \in N_\lambda^-$. Since $\{u_n\} \in N_\lambda^-$, $J_\lambda(u_n) \geq J_\lambda(tu_n)$ *for all* $t \geq 0$, *we have*

$$J_\lambda(t_0^- u_0^-) < \lim_{n\to\infty} J_\lambda(t_0^- u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = \theta^-_\lambda$$

which is a contradiction.

It follows from maximum principle that u_0^- is a positive solution of problem (1). This completes the proof.

Now, we complete the proof of Theorem 1. By Theorem 2, we obtain that for all $\lambda > 0$ and $0 < \lambda < \mu_0$, (1) has a positive solution $u_0^+ \in N_\lambda^+$. On the other hand, from Theorem 3, we get the second positive solution $u_0^- \in N_\lambda^-$ for all $\lambda > 0$ and $0 < \lambda < \mu_0$. Since $N_\lambda^+ \cap N_\lambda^- = \phi$, this implies that u_0^+ that u_0^- are distinct.

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