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Bivariations and tensor products

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Abstract

The ordinary tensor product of modules is defined using bilinear maps (bimorphisms), that are linear in each component. keeping this in mind, Linton and Banaschewski with Nelson defined and studied the tensor product in an equational category and in a general (concrete) category **K**, respectively, using bimorphisms, that is, defined via the Hom-functor on **K**. Also, the so-called sesquilinear, or one and a half linear maps and the corresponding tensor products generalize these notions for modules and vector spaces. In this paper, taking a concrete category **K** and an arbitrary subfunctor H of the functor $U' = Hom \circ (U^{op} \times U)$ rather than just the Hom-functor, where U is the underlying set functor on **K**, we generalize sesquilinearity to *bivariation* and study the related notions such as functional internal lifts, universal bivariants, tensor products, and their interdependence.

Keywords: Bilinear; bivariance; functional internal lift; tensor product

1. Introduction

Let **Mod** be the category of modules over a ring R (commutative with 1) and Hom_{Mod} : $\mathbf{Mod}^{op} \times \mathbf{Mod} \to \mathbf{Set}$ be the usual Hom-functor to the category Set, of sets and maps, taking any pair (A,B) of R-modules to the set $Hom_{\text{Mod}}(A,B)$ of R-morphisms (linear maps) and any R-morphisms to its underlying set map. Recall that, for R-modules A,B,C, a (set) map $v : A \times B \to C$ is said to be bilinear if for every element $a \in A$ and $b \in B$, the map $v_a = v(a,-) : B \to C$ is in $Hom_{\text{Mod}}(B,C)$ and the map $v^b = v(-,b) : A \to C$ is in $Hom_{\text{Mod}}(A,C)$. In other words, both $v_a = v(a,-)$ and $v^b = v(-,b)$ are Hom_{Mod}-maps, that is, are in Mod rather than just in Set.

Now, recall that the tensor product of R-modules is defined using the above mentioned bilinear maps. In fact, for any two R-modules A and B, their tensor product is an R-module $A \otimes B$ equipped with a universal bilinear map $u : A \times B \to A \otimes B$, which means that for any R-module C and a bilinear map $v : A \times B \to C$, there exists a unique R-morphism $w : A \otimes B \to C$ making the following triangle commutative:

$$\begin{tikzcd} A \times B \arrow[r, "u"] \arrow[dr, "v"'] & A \otimes B \arrow[d, "w"] \\ & C \end{tikzcd}$$

These notions can be one step extended to the cases where we replace objects and morphisms of **Mod** with those of, for example, an equational category **K** of algebras (see Linton [1]), or with those of an arbitrary concrete category **K** over the category **Set** of sets and functions (see Banaschewski and Nelson [2]). In these cases, v_a and v^b mentioned above would be Hom_{K}-maps (morphisms in **K**).

Before stating the main aims of the present paper, let us recall one more generalization of the above notions. Sesquilinear, also called one and a half linear, maps and the corresponding tensor products, studied in multilinear algebra, analysis, geometry, and physics (see, for example, [3], [4], and [5]), further generalize the above notions for modules and vector spaces. For vector spaces A,B,C over a field F and a given automorphism α of the field F, a map $v : A \times B \to C$ is said to be α-sesquilinear if it satisfies:

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(1) $v(a_1 + a_2, b_1 + b_2) = v(a_1, b_1) + v(a_1, b_2) + v(a_2, b_1) + v(a_2, b_2)$

(2) $v(ra, sb) = \alpha(r)sv(a, b)$

for all $a_1, a_2 \in A, b_1, b_2 \in B, r, s \in F$. (Often the field F, and also the vector space C, is the field of complex numbers and α is the complex conjugation). Note that if α is the identity map, α-sesquilinear maps are just the bilinear ones. Of course, one could further generalize the notion of α-sesquilinear maps by taking two automorphisms α and β of the field F and define (α, β)-bivariant maps, replacing the above condition (2) by

$(2)'$ $v(ra, sb) = \alpha(r)\beta(s)v(a, b)$.

Notice that, an (α, id_F)-bivariant map is just an α-sesquilinear map, and $\alpha = \beta = id_F$ gives a bilinear one.

In the present paper we propose the next step of generalization to the above notion of an (α, β)-sesquivariant map. Firstly, as in [2], from now on we consider an arbitrary concrete category **K** having (small) concrete products and pullbacks, that is such that the products and pullbacks are preserved by the underlying-set functor $U : \mathrm{K} \to \mathrm{Set}$. If no confusion arises, we do not distinguish in notation between an object A in **K** and its underlying set $U(A) = \mid A \mid$, and also between a morphism f in **K** and its underlying set map $U(f) = \mid f \mid$.

Secondly, the main generalization in this paper, is to take an arbitrary concrete category **K** having (small) concrete products and pullbacks and replace Hom_K by arbitrary subfunctors, from now on denoted by G, H, of the composite

$$U' : \mathrm{K}^{op} \times \mathrm{K} \xrightarrow{U^{op} \times U} \mathrm{Set}^{op} \times \mathrm{Set} \xrightarrow{Hom_{\mathrm{Set}}} \mathrm{Set}$$

Thus, generalizing [1], [2], [6], [7], we present a generalization of α-sesquilinear maps, and the corresponding tensor products.

These generalizations not only provide a deeper understanding of the real features of these notions, but also show that these classical notions can be studied in a categorical setting. For example, we see that the existence of universal bimorphisms and functional hom-functors survives the passage from the category of modules or vector spaces to many other categories, such as the category of acts over a monoid, see [7]. One can now study these notions for multi modules or algebras (as sheaves of modules or algebras).

2. Bivariations

In this section we study the notion of bivariation which generalizes the usual notions of a bilinear (bimorphism) and an α-sesquilinear morphism used, for example, in multilinear algebra, analysis, geometry, and physics.

Definition 2.1. Let G, H be subfunctors of U', and $A, B, C \in \mathrm{K}$. A map $v : \mid A \mid \times \mid B \mid \to \mid C \mid$ is said to be a (G, H)- bivariant if for every $a \in A$, the map $v_a = v(a, -)$ is an H-map, that is $v_a \in H(B, C)$, and for every element $b \in B$, the map $v^b = v(-, b)$ is a G-map, that is $v^b \in G(A, C)$. Instead of (G, H)-bivariant we also say:
(1) G- bivariant when $G = H$,
(2) H- sesquivariant when $G = Hom_{\mathrm{K}}$,
(3) bilinear when $G = H = Hom_{\mathrm{K}}$.

Notice that the same special cases as in Definition 2.1 apply to the notions and results given throughout the paper which, in particular, gives the cases discussed in [1] and [2].

Also notice that an α-sesquilinear map is the same as an H-sesquivariant map in the sense of Definition 2.1, where H is the subfunctor of $U' = Hom_{\mathrm{Set}} \circ (U^{op} \times U)$ mentioned above (here U is the underlying-set functor of the concrete category **K** of vector spaces over F) which takes a pair (A, B) of vector spaces to the set of all maps $f : A \to B$ satisfying
(1) $f(x + y) = f(x) + f(y)$,
(2) $f(rx) = \alpha(r)f(x)$,

for all $x, y \in A$ and $r \in F$. It is easy to check that this H admits an internal lift (to **K**), that is $H = \mid \mathrm{H} \mid$, where $\mid \mathrm{H} \mid = U \circ \mathrm{H}$, for some functor $\mathrm{H} : \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{K}$.

We need the following assumptions before stating our first proposition:

Remark 2.2. In the rest of the paper we consider the subfunctors

$$H : \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{Set}$$

of U' that satisfy the following condition:

(**Productive**) For any family $\{f_i\}_{i\in I}$ of H-maps between the underlying sets of objects A, B of **K**,

$$f_i \in H(A,B)\ (\forall i \in I) \Rightarrow (f_i)_{i\in I} \in H(A, B^I)$$

where $(f_i)_{i\in I} : A \to B^I$ is the induced map from A to the I-fold product B^I of B with itself. Note that Hom_K clearly satisfies the Condition (Productive).

Also notice that H being a functor is closed under composition with morphisms in **K**; in the sense that, given $f \in H(A,B)$ and morphisms $g : A' \to A$, $h : B \to B'$, we get $h \circ f \circ g \in H(A', B')$.

Notation 2.3. Let H be a subfunctor of U', $A, B \in \mathrm{K}$, and $a \in A$. We use special notations for the following set maps:

(1) $\mathrm{a} : H(A,B) \to B$ defined by $\mathrm{a}(h) = h(a)$, for every $h \in H(A,B)$,

(2) $e_{A,B} : H(A,B) \times A \to B$ defined by $e_{A,B}(h,a) = h(a)$, and

(3) $i_{A,B}$ for the inclusion map $H(A,B) \to U'(A,B) = \mid B \mid^{|A|}$.

(4) σ for the isomorphism $\mid B \mid \times \mid A \mid \cong \mid A \mid \times \mid B \mid$.

Proposition 2.4. Let G and H be subfunctors of U'. Then for $A, B, C \in \mathrm{K}$ and $v : \mid A \mid \times \mid B \mid \to \mid C \mid$, the following are equivalent:

(i) v is (G,H)-bivariant.

(ii) The exponential adjoint $\overline{v}$ of v, that is $\overline{v} : \mid A \mid \to \mid C \mid^{|B|}$, given by $\overline{v}(a)(b) = v(a,b)$, factors through $i_{B,C} : H(B,C) \to \mid C \mid^{|B|}$, and analogously $\overline{v \circ \sigma} : \mid B \mid \to \mid C \mid^{|A|}$ factors through $i_{A,C} : G(A,C) \to \mid C \mid^{|A|}$.

(iii) $\overline{v} : \mid A \mid \to \mid C \mid^{|B|}$ is a G-map and $\overline{v \circ \sigma} : \mid B \mid \to \mid C \mid^{|A|}$ is an H-map.

(iv) $\overline{v} : \mid A \mid \to \mid C \mid^{|B|}$ is a G-map and factors through $i_{B,C} : H(B,C) \to \mid C \mid^{|B|}$.

(v) $\overline{v \circ \sigma} : \mid B \mid \to \mid C \mid^{|A|}$ is an H-map and factors through $i_{A,C} : G(A,C) \to \mid C \mid^{|A|}$.

Proof: (i)$\Rightarrow$(ii) We have $\overline{v}(a)(b) = v(a,b)$ and so $\overline{v}(a) = v_a$ is an H-map, since v is (G,H)-bivariant. Also, $\overline{(v \circ \sigma)}(b)(a) = v(a,b)$ and so $\overline{(v \circ \sigma)}(b) = v^b$ is a G-map.

(ii)$\Rightarrow$(iii) We have $p_b \circ \overline{v} = v^b = \overline{(v \circ \sigma)}(b)$, where $p_b : \mid C \mid^{|B|} \to \mid C \mid$ is the bth projection map of the product, and is a G-map for all $b \in B$. Since G satisfies Condition (Productive), this implies that $\overline{v} = (p_b \circ \overline{v})_{b\in B}$ is a G-map. Similarly, $\overline{v \circ \sigma}$ is an H-map.

(iii)$\Rightarrow$(iv) By (iii), $\overline{v}$ is a G-map. Also, $\overline{v}(a)(-) = v_a(-) = \overline{(v \circ \sigma)}(-)(a)$ which, is an H-map by (iii).

(iv)$\Rightarrow$(v) We have $\overline{(v \circ \sigma)}(-)(a) = \overline{v}(a)(-)$ which is an H-map, since $\overline{v}$ factors through $i_{B,C}$. Also, $\overline{(v \circ \sigma)}(b)(-) = \overline{v}(-)(b)$ is a G-map and so $\overline{v \circ \sigma}$ factors through $i_{A,C}$.

(v)$\Rightarrow$(i) This is true because $\overline{(v \circ \sigma)}(-)(a) = v_a(-)$ is an H-map, since $\overline{v \circ \sigma}$ is an H-map. Also $v^b = \overline{v \circ \sigma}(b)$ is a G-map, since $\overline{v \circ \sigma}$ factors through $i_{A,C}$.

Since it is easily seen that the composite of a (G,H)-bivariant with a morphism in **K** is again a (G,H)-bivariant map, the above notion of variation gives rise to the following **Set**-valued functor:

Definition 2.5. Let G and H be subfunctors of U'. The functor

$$GH\mathsf{B} : \mathrm{K}^{op} \times \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{Set}$$

defined on objects by

$$GH\mathsf{B}(A,B,C) = \{v : A \times B \to C \mid v \text{ is a } (G,H) - \text{bivariant map}\}.$$

and naturally on morphisms is called the functor of (G,H)-bivariant maps.

From the above proposition we have

Proposition 2.6. Let G and H be subfunctors of U'. Then $GH\mathsf{B}(A,B,C)$ is the following pullback

$$\begin{array}{ccc} GH\mathsf{B}(A,B,C) & \rightarrow & G(A,C^{|B|}) \\ \downarrow & & \downarrow\ i_{A,C^{|B|}} \\ & (i_{B,C})^{|A|} & \\ H(B,C)^{|A|} & \rightarrow & (|\ C\ |^{|B|})^{|A|} \end{array}$$

Proof: Since $i_{A,C^{|B|}}$ is the inclusion map, the pullback of i and $i_{B,C}$ is

$$P = [(i_{B,C})^{|A|}]^{-1}G(A,C^{|B|}) =$$
$$= \{g :|\ A\ |\rightarrow H(B,C) \mid (i_{B,C})^{|A|}(g)$$
$$= i_{B,C} \circ g \in G(A,C^{|B|})\}$$

Now, taking $v \in GH\mathsf{B}(A,B,C)$, by part (i)$\Rightarrow$(iv) of Proposition 2.4, $\overline{v} \in P$. Conversely, $g \in P$ implies that $\hat{g} : A \times B \rightarrow C$, given by $\hat{g}(a,b) = g(a)(b)$, belongs to $GH\mathsf{B}(A,B,C)$, since $\overline{\hat{g}} = g$ and using (iv)$\Rightarrow$(i) in Proposition 2.4.

Remark 2.7. Notice that a map $v :|\ A\ | \times |\ B\ |\rightarrow|\ C\ |$ is (G,H)-bivariant if and only if $\tilde{v} : \{0\} \rightarrow|\ C\ |^{|A|\times|B|}$, $0 \mapsto v$, factors through $i_{A,B,C} : GH\mathsf{B}(A,B,C) \rightarrow|\ C\ |^{|A|\times|B|}$.

Definition 2.8. A (G,H)-bivariant map $u_{A,B} : A \times B \rightarrow U(A,B)$ is said to be (G,H)-universal (for A and B) if any (G,H)-bivariant map $v : A \times B \rightarrow C$ factors through it by a unique morphism in **K** (that is, there exists a unique **K**-morphism $f : U(A,B) \rightarrow C$ with $f \circ u_{A,B} = v$.)

Remark 2.9. If a universal (G,H)-bivariant map $u_{A,B} : A \times B \rightarrow U(A,B)$ exists for each $A,B \in \mathrm{K}$, then we naturally get a functor

$$GH\mathsf{U} : \mathrm{K} \times \mathrm{K} \rightarrow \mathrm{K}$$

defined on objects by $GH\mathsf{B}(A,B) = U(A,B)$, and for any two morphisms $h : A \rightarrow A'$, $k : B \rightarrow B'$ in **K**, the morphism $U(h,k) : U(A,B) \rightarrow U(A',B')$ is the unique morphism in **K** induced by the universal property of universal (G,H)-bivariant:

$$\begin{array}{cccc} & A \times B & \xrightarrow{u_{A,B}} & U(A,B) \\ h \times k & \downarrow & & \downarrow\ U(h,k) \\ & A' \times B' & \xrightarrow{u_{A',B'}} & U(A',B') \end{array}$$

where $u_{A',B'} : A' \times B' \rightarrow U(A',B')$ is a universal (G,H)-bivariant map for A' and B' (note that since $u_{A',B'}$ is (G,H)-bivariant, so is the composite $u_{A',B'} \circ (h \times k)$). Also, notice that the existence of the universal (G,H)-bivariant maps clearly gives $GH\mathsf{B}(A,B,C) \cong Hom_{\mathrm{K}}(U(A,B),C))$.

3. Functional internal lifts

It is well known that even in the case where **K** is the category of R-modules or the category of M-sets (of sets with actions of a monoid M on them, see [7]), the Hom-functor Hom_{K} is, in general, not functional; that is, for objects A and B in **K**, $Hom_{\mathrm{K}}(A,B)$ is not in general a subobject (submodule) of the $|\ A\ |$-fold product $B^{|A|}$.

Some conditions equivalent to Hom_{K} being functional are given in [2] and [6]. In this section we study this notion for any subfunctor H of U' for the general concrete category we have been working with in this paper, and give some necessary and sufficient conditions under which H is functional.

Definition 3.1. Let $H : \mathrm{K}^{op} \times \mathrm{K} \rightarrow \mathrm{Set}$ be a subfunctor of U'. Then:

(1) A functor $\mathrm{H} : \mathrm{K}^{op} \times \mathrm{K} \rightarrow \mathrm{K}$ is said to be an internal lift (IL) of H (to **K**) if $U \circ \mathrm{H} = H$; that is, the following diagram is commutative:

$$\begin{tikzcd} \mathrm{K}^{op} \times \mathrm{K} \arrow[r, "H"] \arrow[dr, "\mathrm{H}"'] & \mathrm{Set} \\ & \mathrm{K} \arrow[u, "U"'] \end{tikzcd}$$

(2) The internal lift H of H is called a strong internal lift (SIL) of H if it satisfies the condition

(S) For any $A,B,C \in \mathrm{K}$, a set map $f : A \rightarrow \mathrm{H}(B,C)$ is an H-map whenever for each $b \in B$, $\mathrm{b} \circ f$ is an H-map.

Definition 3.2. Let H be an internal lift of H. A monomorphism $f : A \rightarrow B$ in **K** is called an H-embedding if any map $g : C \rightarrow A$ is in $H(C,A)$ whenever $f \circ g$ is in $H(C,B)$.

Definition 3.3. An internal lift H of H is said to be:
(1) A functional internal lift (FIL) if for every two objects $A, B \in \mathrm{K}$, the inclusion map $i_{A,B} : \mathrm{H}(A,B) \to B^{|A|}$ is a morphism (and hence a monomorphism) in **K**.
(2) A strong functional internal lift (SFIL) if it is a functional internal lift (FIL) and, further, each $i_{A,B} : \mathrm{H}(A,B) \to B^{|A|}$ is an H-embedding.

Notice that the $|A|$-fold product functor $(-)^{|-|} : \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{K}$ is not only a strong internal lift (SIL) of the functor $U' : \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{Set}$, but is also a strong functional internal lift (SFIL) of it.

Theorem 3.4. Let H be an internal lift of H.
Then, the following are equivalent:
(i) H is a functional internal lift (FIL) of H.
(ii) For any $A, B \in \mathrm{K}, a \in A$, the set map $\mathrm{a} : \mathrm{H}(A,B) \to B$ is a morphism in **K**.
(iii) The evaluation maps $e_{A,B} : \mathrm{H}(A,B) \times A \to B$ are H-sesquivariant.

Proof: (i)$\Leftrightarrow$(ii) Notice that the inclusion map $i_{A,B} : \mathrm{H}(A,B) \to B^{|A|}$ is in fact the map which exists by the universal property of products and makes the following diagram commutative:

$$\begin{tikzcd} H(A,B) \arrow[r, "i_{A,B}"] \arrow[d, "\mathrm{a}"'] & B^{|A|} \arrow[dl, "p_a"] \\ B & \end{tikzcd}$$

Now, if $i_{A,B}$ is a morphism in **K** then, since each p_a is a morphism in **K**, we get that $\mathrm{a} = p_a \circ i_{A,B}$ is a morphism in **K**. Conversely, if each a is a morphism in **K**, then $i_{A,B}$, which exists by the universal property of products in **K**, is a morphism in **K**.
(ii)$\Rightarrow$(iii) By definition, we have $e_{AB}(v,-) = v$ is an H-map. Also, $e_{AB}(-,a) = \mathrm{a}$ which is in **K** by (ii).
(iii)$\Rightarrow$(i) Note that $\mathrm{a} = e_{AB}(-,a)$, which is in **K**, by the definition of a H-sesquivariant.

Proposition 3.5. Let H be a subfunctor of U'. Then, an internal lift (IL) of H is a strong functional internal lift (SFIL) if and only if it is a strong internal lift (SIL) as well as a functional internal lift (FIL).

Proof: ($\Rightarrow$) Let H be an internal lift of H which is a SFIL. Then it is a FIL, by definition. To prove that it is a SIL, let $f : A \to \mathrm{H}(B,C)$ be a map such that $\mathrm{b} \circ f$ is an H-map

for every $b \in B$. But, as the proof of Theorem 3.4 shows, for every $b \in B$, $\mathrm{b} = p_b \circ i_{B,C}$, so $p_b \circ i_{B,C} \circ f = \mathrm{b} \circ f$ is an H-map. Now, since H satisfies Condition (Productive), this implies that $i_{B,C} \circ f$ is an H-map, because $i_{B,C} \circ f = (p_b \circ i_{B,C} \circ f)_{b \in B}$. Finally, using the fact that $i_{B,C}$ is an H-embedding, it is concluded that f is an H-map.
($\Leftarrow$) Let $g : C \to \mathrm{H}(A,B)$ be a map such that $i_{A,B} \circ g$ is an H-map. Applying Condition (S), we show that $\mathrm{a} \circ g$ is an H-map for each $a \in A$, and then get that g is an H-map. Since H is closed under composition with morphisms in **K**, we get that $p_a \circ i_{A,B} \circ g$ is an H-map. But, as we stated in the proof of Theorem 3.4, $\mathrm{a} = p_a \circ i_{A,B}$, and so $\mathrm{a} \circ g$ is an H-map.

Also, it is clear that:

Proposition 3.6. If all monomorphisms in **K** are H-embeddings, then an internal lift (IL) of H is a strong functional internal lift (SFIL) if and only if it is a functional internal lift (FIL).

Proposition 3.7. Let $G, H : \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{Set}$ be subfunctors of U', and H be a functional internal lift of H. Then, there is a natural equivalence

$$G(A, \mathrm{H}(B,C)) \to GH\mathsf{B}(A,B,C)$$

such that $h \mapsto \hat{h}$ is defined by $\hat{h}(a,b) = h(a)(b)$ for all $a \in A$ and $b \in B$ if and only if for each $B, C \in \mathrm{K}$, $i_{B,C} : \mathrm{H}(B,C) \to C^{|B|}$ is a G-embedding.

Proof: ($\Leftarrow$) Recalling Proposition 2.6, the given assignment is a bijection between $P = \{g : |A| \to H(B,C) \mid i_{B,C} \circ g \in G(A, C^{|B|})\}$ and $GH\mathsf{B}(A,B,C)$. On the other hand, each member g of P belongs to $G(A, \mathrm{H}(B,C))$. This is because $i_{B,C} \circ g$, and hence g, is a G-map, since

$i_{B,C}$ is a G-embedding. Conversely, each element of $G(A,\mathrm{H}(B,C))$ is clearly in P. Therefore, $G(A,\mathrm{H}(B,C))$ and $GH\mathsf{B}(A,B,C)$ are both pullbacks of $G(A,C^{|B|})$ and $\mathrm{H}(B,C)^{|A|}$. Hence, we get the desired equivalence. The proof of the naturality is straightforward.
$(\Rightarrow)$ Let $f: A \to \mathrm{H}(B,C)$ be such that $i_{B,C} \circ f$ is a G-map. Then f is also a G-map, since $\hat{f}: A \times B \to C$ is in $GH\boldsymbol{B}(A,B,C)$. In fact, $\hat{f}(a,-) = f(a)$ is an H-map and $\hat{f}(-,b) = \mathrm{b} \circ f = p_b \circ i_{B,C} \circ f$ is a G-map.

Corollary 3.8. Let $\mathrm{H}: \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{K}$ be a functional internal lift (FIL) of $H: \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{Set}$. Then, there is a natural equivalence

$$H(A,\mathrm{H}(B,C)) \to HH\mathsf{B}(A,B,C)$$

such that $h \mapsto \hat{h}$ is defined by $\hat{h}(a,b) = h(a)(b)$ for all $a \in A$ and $b \in B$ if and only if H is a strong functional internal lift of H.

Corollary 3.9. Let G and H be subfunctors of U' with functional internal lifts G and H, respectively. If each $i_{B,C}: \mathrm{H}(B,C) \to C^{|B|}$ is a G-embedding and each $i_{A,C}: \mathrm{G}(A,C) \to C^{|A|}$ is an H-embedding, then

$$G(A,\mathrm{H}(B,C)) \cong GH\mathsf{B}(A,B,C) \cong H(B,\mathrm{G}(A,C))$$

Proof: Applying Proposition 3.7, we get that

$$G(A,\mathrm{H}(B,C)) \cong GH\mathsf{B}(A,B,C)$$

and $H(B,\mathrm{G}(A,C)) \cong HG\mathsf{B}(B,A,C)$. But there is, clearly, a natural equivalence between $GH\boldsymbol{B}(A,B,C)$ and $HG\boldsymbol{B}(B,A,C)$. Hence, the desired equivalences are obtained.

Remark 3.10. The equivalence $G(A,\mathrm{H}(B,C)) \cong H(B,\mathrm{G}(A,C))$ obtained in Corollary 3.9 is in fact given by $f \mapsto f^{\#}, f^{\#}(b)(a) = f(a)(b)$ for all $a \in A$ and $b \in B$.

Finally, we consider Proposition 3.7 in the case where $G = Hom_{\mathrm{K}}$.

Proposition 3.11. Let H be an internal lift (IL) of $H: \mathrm{K}^{op} \times \mathrm{K} \to \mathrm{Set}$. Then,

$$Hom_K(A,\mathrm{H}(B,C)) \to Hom_K H\mathsf{B}(A,B,C),$$

given by $h \mapsto \hat{h}$ with $\hat{h}(a,b) = h(a)(b)$ for all $a \in A$ and $b \in B$, is a natural equivalence if and only if H is a functional internal lift (FIL) of H.

Proof: $(\Rightarrow)$ Consider the identity map $id: \mathrm{H}(B,C) \to \mathrm{H}(B,C)$. Then $\hat{id}: \mathrm{H}(B,C) \times B \to C$ belongs to $Hom_{\mathrm{K}} H\mathsf{B}(A,B,C)$. In particular, for $b \in B$, $\mathrm{b} = \hat{id}^b$ belongs to **K**. So, by Theorem 3.4, H is a FIL of H.
$(\Leftarrow)$ Applying Proposition 3.7 for $G = Hom_{\mathrm{K}}$, since each $i_{B,C}: \mathrm{H}(B,C) \to C^{|B|}$ is clearly a G-embedding, we get the desired equivalence.

Remark 3.12 The equivalence obtained in Proposition 3.7 will be an isomorphism in **K** if we add the assumption that G admits an internal lift G. This is because, following the proof of Proposition 3.7, $\mathrm{G}(A,\mathrm{H}(B,C))$ and $GH\boldsymbol{B}(A,B,C)$ are both pullbacks of subobjects $\mathrm{G}(A,C^{|B|})$ and $\mathrm{H}(B,C)^{|A|}$ of $(C^{|B|})^{|A|}$. So, we also get that the equivalences given in Corollaries 3.8, 3.9 are in fact isomorphisms in **K**. A similar result is true for Proposition 3.11 if we assume that Hom_{K} has an internal lift.

4. Tensor products

In this last section we define tensor products with respect to an arbitrary subfunctor H of U' and study the interdependence between the following notions: tensor product, bivariance, and functionality of an internal lift. These notions generalize the ordinary ones.

Definition 4.1 Let G, H be subfunctors of U' and H be an internal lift of H. A functor $\otimes: \mathrm{K} \times \mathrm{K} \to \mathrm{K}$ is said to be a (G,H)- tensor product if there is a natural equivalence

$$Hom_{\mathrm{K}}(\otimes(-,-),-) \to G(-,\mathrm{H}(-,-)).$$

Instead of (G,H)-tensor product we also say:
(1) G - tensor product when $G = H$,
(2) H - sesquitensor product when $G = Hom_{\mathrm{K}}$,

(3) tensor product when $G = H = Hom_{\mathrm{K}}$.

Theorem 4.2. Let G, H be subfunctors of U', H be a functional internal lift of H, and $\otimes : \mathrm{K} \times \mathrm{K} \to \mathrm{K}$ be any functor. Then any two of the following statements imply the third one:
(T) $\otimes$ is a (G,H)-tensor product.
(E) $i_{B,C} : \mathrm{H}(B,C) \to C^{|B|}$ is a G-embedding for each $B, C \in \mathrm{K}$.
(U) $\otimes$ is the functor GHU.

Proof: (T)+(E)$\Rightarrow$(U) Let $\eta : Hom_{\mathrm{K}}(\otimes(-,-),-) \to G(-,\mathrm{H}(-,-))$ be a natural equivalence. Take $A, B \in \mathrm{K}$ and consider l as the image of $id : A \otimes B \to A \otimes B$ under $\eta_{A,B,A\otimes B}$. Using (E) and applying the equivalence given in Proposition 3.7, we get a (G,H)-bivariant map $\hat{l} : A \times B \to A \otimes B$. We show that this map is a universal (G,H)-bivariant. Let $v : A \times B \to C$ be a (G,H)-bivariant. Then again, applying the equivalence given in Proposition 3.7, we get $\overline{v} \in G(A, \mathrm{H}(B,C))$. Now, $\eta^{-1}_{A,B,C}(\overline{v}) : A \otimes B \to C$ is a unique morphism in **K** with $\eta^{-1}_{A,B,C}(\overline{v}) \circ \hat{l} = v$. This is because, by the naturality of η, $\overline{v} = \overline{\eta^{-1}_{A,B,C}(\overline{v}) \circ \hat{l}}$.
(U)+(E)$\Rightarrow$(T) Let $A, B, C \in \mathrm{K}$. Define

$$\eta_{A,B,C} : Hom_{\mathrm{K}}(A \otimes B, C) \to G(A, \mathrm{H}(B,C))$$

by $f \mapsto \eta_{A,B,C}(f)$, where $\eta_{A,B,C}(f)(a)(b) = fu(a,b)$, $a \in A$, $b \in B$, and $u_{A,B} : A \times B \to A \otimes B$ is a universal (G,H)-bivariant map which exists by (U). Notice that $\eta_{A,B,C}$ is injective, since by the universality of u, $f \circ u = g \circ u$ implies $f = g$. To see that it is also surjective, let $h : A \to \mathrm{H}(B,C)$ be a G-map. Then, by the equivalence given in Proposition 3.7, we get a (G,H)-bivariant $\hat{h} : A \times B \to C$. Using the universality of u, we get a unique morphism $f_g : A \otimes B \to C$ in K with $f_g \circ u = \hat{g}$, which means that $\eta_{A,B,C}(f_g) = g$ as required. The naturality of η also follows from the universality of each $u_{A,B}$.

(T)+(U)$\Rightarrow$(E) Applying (T), we have an equivalence

$$Hom_{\mathrm{K}}(A \otimes B), C) \cong G(A, \mathrm{H}(B,C))$$

and, by (U), we get an equivalence $GH\mathsf{B}(A,B,C) \cong Hom_{\mathrm{K}}(A \otimes B, C)$ (see Remark 2.9). Consequently, we have an equivalence $GH\mathsf{B}(A,B,C) \cong G(A, \mathrm{H}(B,C))$. So, by Proposition 3.7, each $i_{B,C} : \mathrm{H}(B,C) \to C^{|B|}$ is a G-embedding.

Corollary 4.3. Let G, H be subfunctors of U' and H be a functional internal lift (FIL) of H, and each $i_{B,C} : \mathrm{H}(B,C) \to C^{|B|}$ be a G-embedding. Then, a functor $\otimes : \mathrm{K} \times \mathrm{K} \to \mathrm{K}$ is a (G,H)-tensor product if and only if $\otimes$ is a functor of (G,H)-universal bivariants.

Finally, using Proposition 3.11, we get Theorem 4.2 for the special case where $G = Hom_{\mathrm{K}}$.

Theorem 4.4. Let H be an internal lift of H and $\otimes : \mathrm{K} \times \mathrm{K} \to \mathrm{K}$ be a functor. Then, any two of the following conditions imply the third:
(T) ($\otimes$ is an H-sesquitensor product) There is a bijection

$$Hom_{\mathrm{K}}(A \otimes B, C) \cong Hom_{\mathrm{K}}(A, \mathrm{H}(B,C))$$

natural in all three arguments; in particular, this gives that for every object B in **K**, the functor $- \otimes B$ is a left adjoint to the functor $\mathrm{H}(B,-)$.
(E) (H is a functional internal lift of H) For any two objects A and B in **K**, the inclusion map $i_{A,B} : \mathrm{H}(A,B) \to B^{|A|}$ is a morphism in **K** (here $B^{|A|}$ denotes the $|A|$-fold product in **K** of B with itself).
(U) ($\otimes$ is the functor of universal (Hom_K, H)-bivariants) There is a family of maps

$$u_{A,B} : A \times B \to A \otimes B, \quad A, B \in \mathrm{K}$$

which forms a natural transformation $u : - \times - \to - \otimes -$, such that for any two objects A and B in **K**, the map $u_{A,B}$ is the universal H-sesquivariant map for A and B; that is $u_{A,B}$ is H-sesquivariant and for any H-sesquivariant map $v : A \times B \to C$, there exists exactly one morphism $f : A \otimes B \to C$ in **K** such that $v = f \circ u_{A,B}$.

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