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Regarding the Kähler-Einstein structure on Cartan spaces with Berwald connection

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Abstract

A Cartan manifold is a smooth manifold M whose slit cotangent bundle T^*M_0 is endowed with a regular Hamiltonian K which is positively homogeneous of degree 2 in momenta. The Hamiltonian K defines a (pseudo)-Riemannian metric g_{ij} in the vertical bundle over T^*M_0 and using it, a Sasaki type metric on T^*M_0 is constructed. A natural almost complex structure is also defined by K on T^*M_0 in such a way that pairing it with the Sasaki type metric an almost Kähler structure is obtained. In this paper we deform g_{ij} to a pseudo-Riemannian metric G_{ij} and we define a corresponding almost complex Kähler structure. We determine the Levi-Civita connection of *G* and compute all the components of its curvature. Then we prove that if the structure (T^*M_0, G, J) is Kähler- Einstein, then the Cartan structure given by K reduces to a Riemannian one.

Keywords: Cartan space; Kähler structure; symmetric space; Einstein manifold; Laplace operator; Divergence; Gradient

1. Introduction

The structure of the tangent and cotangent bundles of a differentiable manifold is well studied in Riemannian geometry, Finsler geometry and Physics, and has many applications in Biology too [1-7].

É. Cartan has originally introduced a Cartan space, which is considered a dual of Finsler space [8]. H. Rund [9], F. Brickell [10] and others then studied the relation between these two spaces. The theory of Hamilton spaces was introduced and studied by R. Miron [11, 12]. He proved that Cartan space is a particular case of Hamilton space. Indeed, the geometry of regular Hamiltonians as smooth functions on the cotangent bundle is due to R. Miron and is now systematically described in the monograph [13].

Let us denote the Hamiltonian structure on a manifold *M* by $(M, H(x,P))$. If the fundamental function $H(x,P)$ is 2-homogeneous on the fibres of the cotangent bundle (T^*M, M), then the notion of Cartan space is obtained. The modern formulation of the notion of Cartan spaces is due to R. Miron [14-16]. Based on the studies of E. Cartan, A. Kawaguchi [17], R. Miron [13], [15], [16], S. Vacaru [18, 19], D. Hrimiuc and H. Shimada [20], [21], P.L. Antonelli and M. Anastasiei [22-25], etc., the geometry of Cartan spaces is today an important chapter of differential geometry.

Under Legendre transformation, the Cartan spaces appear as dual of the Finsler spaces [11]. It is remarkable that regular Lagrangian, which is 2-homogeneous in velocities is nothing but the square of a fundamental Finsler function and its geometry is Finsler geometry. Using this duality several important results in the Cartan spaces can be obtained: the canonical nonlinear connection, the canonical metrical connection, the notion of (α, β)-metrics, etc [26]. Therefore, the theory of

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Cartan spaces has the same symmetry and beauty like Finsler geometry. Moreover, it gives a geometrical framework for the Hamiltonian theory of Mechanics or Physical fields.

Let (M, K) be a Cartan space on a manifold M and put $\tau := \frac{1}{2}K^2$. Let us define the symmetric M-tensor field $G_{ij} := \frac{1}{\beta} g_{ij} + \frac{v(\tau)}{\alpha\beta} p_i\, p_j$ on slit cotangent bundle $T^*M_0 = T^*M - \{0\}$, where $v = v(\tau)$ is a real valued smooth function defined on $[0,\infty) \subset R$ and α and β are real constants. Using this, we can define a Riemannian metric and almost complex structure on T^*M_0 as follows

$$G = G_{ij}\, dx^i\, dx^j + G^{ij}\, \delta p_i\, \delta p_j,$$

$$J(\delta_i) = G_{ik}\, \dot{\partial}^k, \quad J(\dot{\partial}^k) = -G^{ik}\delta_k,$$

where G^{ij} is the inverse of G_{ij}.

In this paper, we prove that (T^*M_0, G, J) is an almost Kählerian manifold. We then show that the almost complex structure J on T^*M_0 is integrable if and only if M has constant scalar curvature c and the function v is given by $v = -c\alpha\beta^2$. We conclude that on a Cartan manifold M of negative constant flag curvature, (T^*M_0, G, J) has a Kählerian structure. For Cartan manifolds of positive constant flag curvature, we show that the tube around the zero section has a Kählerian structure (see Theorem 5).

Then we find the Levi-Civita connection ∇ of the metric G. For the connection ∇, we compute the curvature of all of the components. For a Cartan space (M,K) of constant curvature c, we prove that in the following cases it reduces to a Riemannian space: (i) for $c < 0$, (T^*M_0, G, J) became a Kähler Einstein manifold, (ii) for $c > 0$, $(T^*_\beta M_0, G, J)$ became a Kähler Einstein manifold, where $T^*_\beta M_0$, the tube around the zero section in T^*M is defined by the condition $2\tau < \frac{1}{c\beta^2}$. It results that, there is no non-Riemannian Cartan structure such that (T^*M_0, G, J) became a Einstein manifold.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold and $\pi^* : T^*M \to M$ its cotangent bundle. If (x^i) are local coordinates on M, then (x^i, p_i) will be taken as local coordinates on T^*M with the momenta (p_i) provided by $p = p_i dx^i$ where $p \in T^*_x M$, $x = (x^i)$ and (dx^i) is the natural basis of $T^*_x M$. The indices $i,j,k,\ldots$ will run from 1 to n and the Einstein convention on summation will be used.

Put $\partial_i := \frac{\partial}{\partial x^i}$ and $\dot{\partial}^i := \frac{\partial}{\partial p_i}$. Let $(\partial_i, \dot{\partial}^i)$ be the natural basis in $T_{(x,p)}T^*M$ and (dx^i, dp_i) be the dual basis of it. The kernel $V_{(x,p)}$ of the differential $d\pi^* : T_{(x,p)}T^*M \to T_x M$ is called the *vertical* subspace of $T_{(x,p)}T^*M$ and the mapping $(x,p) \to V_{(x,p)}$ is a regular distribution on T^*M called the *vertical distribution*. This is integrable with the leaves $T^*_x M$, $x \in M$ and is locally spanned by $\dot{\partial}^i$. The vector field $C^* = p_i \dot{\partial}^i$ is called the Liouville vector field and $\omega = p_i dx^i$ is called the Liouville 1-form on T^*M. So $d\omega$ is the canonical symplectic structure on T^*M. For an easer handling of the geometrical objects on T^*M, it is usual to consider a supplementary distribution to the vertical distribution, $(x,p) \to N_{(x,p)}$, called the *horizontal distribution* and to report all geometrical objects on T^*M to the decomposition

$$T_{(x,p)}T^*M = N_{(x,p)} \oplus V_{(x,p)}. \tag{1}$$

The pieces produced by the decomposition (1) are called d-geometrical objects (d is for distinguished) since their local components behave like geometrical objects on M, although they depend on $x = (x^i)$ and momenta $p = (p_i)$.

The horizontal distribution is taken as being locally spanned by the local vector fields

$$\delta_i := \partial_i + N_{ij}(x,p)\dot{\partial}^j \tag{2}$$

The horizontal distribution is also called a nonlinear connection on T^*M and the functions (N_{ij}) are called the local coefficients of this nonlinear connection. It is important to note that any regular Hamiltonian on T^*M determines a nonlinear connection whose local coefficients verify $N_{ij} = N_{ji}$. The basis $(\delta_i, \dot{\partial}^i)$ is adapted to the decomposition (1). Its dual is $(dx^i, \delta p_i)$, for $\delta p_i = dp_i - N_{ji} dx^j$.

A Cartan structure on **M** is a function $K : T^*M \to [0, \infty)$ which has the following properties: (i) K is C^∞ on $T^*M_0 = T^*M - \{0\}$; (ii) $K(x, \lambda p) = \lambda K(x, p)$ for all $\lambda > 0$ and (iii) the $n \times n$ matrix (g^{ij}), where $g^{ij}(x, p) = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j K^2(x, p)$, is positive definite at all points of T^*M_0. We notice that, in fact, $K(x, p) > 0$, whenever $p \neq 0$. The pair (M, K) is called a Cartan space. Using this notation, let us define

$$p^i = \frac{1}{2} \dot{\partial}^i K^2 \quad \textit{and} \quad C^{ijk} = -\frac{1}{4} \dot{\partial}^i \dot{\partial}^j \dot{\partial}^k K^2$$

The properties of K imply that

$$\begin{aligned} &p^i = g^{ij} p_j, \ \ p_i = g_{ij} p^j \\ &g^{ij} p_i p_j = p_i p^i = K^2 \end{aligned} \tag{3}$$

$$C^{ijk} p_k = C^{ikj} p_k = C^{kij} p_k = 0 \tag{4}$$

One considers the *formal Christoffel symbols*

$$\gamma^i_{jk}(x, p) := \frac{1}{2} g^{is} \left(\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{kj} \right), \tag{5}$$

and the contractions $\gamma^o_{jk}(x, p) := \gamma^i_{jk}(x, p) p_i$, $\gamma^o_{jo} := \gamma^i_{jk} p_i p^k$. Then the functions

$$N_{ij}(x, p) = \gamma^\circ_{ij}(x, p) - \frac{1}{2} \gamma^\circ_{h\circ}(x, p) \dot{\partial}^h g_{ij}(x, p), \tag{6}$$

define a nonlinear connection on T^*M. This nonlinear connection was discovered by R. Miron [14]. Thus a decomposition (1) holds. From now on, we shall use only the nonlinear connection given by (6).

A linear connection D on T^*M is said to be an *N-linear connection* if D preserves by parallelism the distribution N and V, also, we have $D\theta = 0$ for $\theta = \delta p_i \wedge dx^i$. One proves that an N-linear connection can be represented in the adapted basis $(\delta_i, \dot{\partial}^i)$ in the form

$$D_{\delta_j} \delta_i = B^k_{ij} \delta_j, \qquad D_{\delta_j} \dot{\partial}^i = -B^i_{kj} \dot{\partial}^k, \tag{7}$$

$$D_{\dot{\partial}^j} \delta_i = V^{kj}_i \delta_k, \qquad D_{\dot{\partial}^j} \dot{\partial}^i = -V^{ij}_k \dot{\partial}^k, \tag{8}$$

where V_i^{kj} is a d-tensor field and $B^k_{ij}(x, p)$ behave like the coefficients of a linear connection on M. The functions B^k_{ij} and V_i^{kj} define operators of h-covariant and v-covariant derivatives in the algebra of d-tensor fields, denoted by $_{|k}$ and $|^k$, respectively. For g^{ij}, these are given by the following equation

$$g^{ij}{}_{|k} = \delta_k g^{ij} + g^{sj} B^i_{sk} + g^{is} B^j_{sk}, \tag{9}$$

$$g^{ij}\big|^k = \dot{\partial}^k g^{ij} + g^{sj} V^{ik}_s + g^{is} V^{jk}_s. \tag{10}$$

An N-linear connection given in the adapted basis $(\delta_i, \dot{\partial}^i)$ as $D\Gamma(N) = (B^i_{jk}, V_j^{ik})$ is called a *Berwald connection* if

$$g^{ij}{}_{|k} = -2L^{ij}_k, \quad g^{ij}\big|^k = -2C^{ijk}, \tag{11}$$

where $L^{ij}_k = C^{ij}_{k|h} p^h$ are components of the Landsberg tensor on M (see [27]).

The Berwald connection $B\Gamma(N) = (\dot{\partial}^i N_{jk}, 0)$ of the Cartan spaces has the torsions d-tensors as follows

$$T^i_{jk} = 0, \quad S_i^{jk} = 0, \quad V_i^{jk} = 0, P^i_{jk} = 0, \tag{12}$$

$$R_{ijk} = \delta_k N_{ij} - \delta_j N_{ik}. \tag{13}$$

The d-tensors of the curvature of $B\Gamma(N)$ are given by

$$\begin{aligned} R^i_{jkj} &= \delta_h B^i_{jk} - \delta_k B^i_{jh} + B^s_{jk} B^i_{sh} \\ &- B^s_{jh} B^i_{sk}, \quad P^{ih}_{jk} = \dot{\partial}^h B^i_{jk}, \quad S_j^{ikh} = 0 \end{aligned}$$

where $B^i_{jk} = \dot{\partial}^i N_{jk}$ are the coefficients of the $B\Gamma(N)$-connection. It also has the following properties

$$K^2_{|j} = \delta_j K^2 = 0, \quad K^2 \Big|^j = 2p^j, \tag{14}$$

$$p_{i|j} = p^i_{|j} = 0, \quad p_i \Big|^j = \delta^j_i, \qquad p^i \Big|^j = g^{ij}, \quad R_{kij} p^k = 0 \tag{15}$$

$$\delta_i g_{ik} = B^s_{ji} g_{sk} + B^s_{ki} g_{js}. \tag{16}$$

3. Kähler structures on cotangent bundle

Suppose that

$$\tau := \frac{1}{2} K^2 = \frac{1}{2} g^{ij}(x, p)\ p_i\ p_j. \tag{17}$$

We consider a real valued smooth function υ defined on $[0,\infty) \subset \mathbb{R}$ and real constants α and β. We define the following symmetric *M*-tensor field of type (0,2) on T^*M_0 having the components

$$G_{ij} := \frac{1}{\beta} g_{ij} + \frac{v(\tau)}{\alpha\beta} p_i p_j. \tag{18}$$

It follows easily that the matrix (G_{ij}) is positive definite if and only if $\alpha, \beta > 0,\ \alpha + 2\tau\upsilon > 0.$

The inverse of this matrix has the entries

$$G^{kl} = \beta g^{kl} - \frac{v\beta}{\alpha + 2\tau v} p^k p^l. \tag{19}$$

The components G^{kl} define symmetric *M*-tensor field of type (0,2) on T^*M_0. It is easy to see that if the matrix (G_{ij}) is positive definite, then matrix (G^{kl}) is positive definite too.

Using (G_{ij}) and (G^{ij}), the following Riemannian metric on T^*M_0 is defined

$$G = G_{ij}\ dx^i\ dx^j + G^{ij}\ \delta p_i\ \delta p_j \tag{20}$$

Now, we define an almost complex structure *J* on T^*M_0 by

$$J(\delta_i) = G_{ik}\dot{\partial}^k, J(\dot{\partial}^i) = -G^{ik}\delta_k \tag{21}$$

It is easy to check that $J^2 = -I.$

Theorem 1. (T^*M_0, G, J) is an almost Kählerian manifold.

Proof: Since the matrix (G^{kl}) is the inverse of the matrix (G_{ij}) we have

$$G\left(J\delta_i, J\delta_j\right) = G_{ik}G_{jr}G\left(\dot{\partial}^k, \dot{\partial}^r\right)$$
$$= G_{ik}G_{jr}G^{kr} = G_{ij} = G\left(\delta_i, \delta_j\right).$$

The relations

$$G\left(J\dot{\partial}^i, J\dot{\partial}^j\right) = G\left(\dot{\partial}^i, \dot{\partial}^j\right),$$
$$G\left(J\delta_i, J\dot{\partial}^k\right) = G\left(\delta_i, \dot{\partial}^k\right) = 0,$$

may be obtained in a similar way, thus

$$G(JX, JY) = G(X, Y), \quad \forall X, Y \in \Gamma\left(T^*M_0\right).$$

It means that *G* is almost Hermitian with respect to *J*. The fundamental 2-form associated with this almost Kähler structure is θ, defined by

$$\theta(X, Y) := G(X, JY), \quad \forall X, Y \in \left(T^*M_0\right)$$

Then we get

$$\theta\left(\dot{\partial}^i, \delta_j\right) = G\left(\dot{\partial}^i, J\delta_j\right) = G\left(\dot{\partial}^i, G_{jk}\dot{\partial}^k\right)$$
$$= G^{ik}G_{jk} = \delta^i_j, \quad \theta\left(\delta_i, \delta_j\right) = \theta\left(\dot{\partial}^i, \dot{\partial}^j\right) = 0$$

Hence, we have

$$\theta = \delta p_i \wedge dx^i. \tag{22}$$

that is the canonical symplectic form of T^*M .

Here, we study the integrability of the almost complex structure defined by *J* on *TM*. To do this, we need the following lemma [13].

Lemma 2. Let *(M,K)* be a Cartan space. Then we have

(1) $\left[\delta_i, \delta_j\right] = R_{kij}\dot{\partial}^k,$

(2) $\left[\delta_i, \dot{\partial}^j\right] = -\left(\dot{\partial}^j N_{ik}\right)\dot{\partial}^k,$

(3) $\left[\dot{\partial}^i, \dot{\partial}^j\right] = 0.$

Lemma 3. Let *(M, K)* be a Cartan space. Then *J* is a complex structure on T^*M_0 if and only if

$A_{kij} = 0$ and

$$R_{kij} = \frac{v}{\alpha\beta^2}\left(g_{ik}p_j - g_{jk}p_i\right), \tag{23}$$

where

$$A_{kij} = \delta_i G_{jk} - \delta_j G_{ik} + G_{ir}\dot{\partial}^r N_{jk} - G_{jr}\dot{\partial}^r N_{ik}.$$

Proof: Using the definition of the Nijenhuis tensor field N_J of *J*, that is,

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y], \quad \forall X,Y \in \Gamma(T^*M)$$

we get

$$N_J(\delta_i,\delta_j) = A_{hij}G^{hk}\delta_k + \left(M_{kij} - R_{kij}\right)\dot{\partial}^k, \tag{24}$$

where $M_{kij} = G_{ir}\dot{\partial}^r G_{jk} - G_{jr}\dot{\partial}^r G_{ik}$. Let $C^{r}_{jk} := g_{jl}g_{sk}C^{rls}$, then we have

$$\dot{\partial}^r g_{jk} = -g_{jl}g_{sk}\dot{\partial}^r g^{ls} = 2g_{jl}g_{sk}C^{rls} = 2C^r_{jk}.$$

By above equation, we obtain

$$G_{ir}\dot{\partial}^r G_{jk} = \frac{2}{\beta^2}C_{ijk} + \frac{v}{\alpha\beta^2}\left(g_{ji}p_k + g_{ik}p_j\right) + \left(\frac{v'}{\alpha\beta^2} + \frac{2vv'\tau}{\alpha\beta} + \frac{2v^2}{\alpha^2\beta^2}\right)p_i p_j p_k. \tag{25}$$

where $C_{ijk} = g_{ir}C^r_{jk}$. From (25) we get

$$M_{kij} = \frac{v}{\alpha\beta^2}\left(g_{ik}p_j - g_{jk}p_i\right). \tag{26}$$

By a straightforward computation, it follows that $N_J(\dot{\partial}^i,\dot{\partial}^j) = 0, N_J(\dot{\partial}^i,\delta_j) = 0,$ whenever $N_J(\delta_i,\delta_j) = 0$. Therefore, from relations (24) and (26) we conclude that the necessary and sufficient conditions for the Nijenhuis tensor field N_J to vanish, so that J is a complex structure, are that $A_{kij} = 0$ and (23) hold.

In equation (23), we put $-\frac{\upsilon}{\alpha\beta^2} = c$, where c is constant. Then we get

$$R_{kij} = c\left(g_{jk}p_i - g_{ik}p_j\right). \tag{27}$$

Theorem 4. Let *(M,K)* be a Cartan space of dimension $n \geq 3$. Then the almost complex structure **J** on T^*M_0 is integrable if and only if (27) is held and the function υ is given by

$$v = -c\alpha\beta^2. \tag{28}$$

Proof: From equation $p_{i|k} = 0$ of relation (15), we conclude that $\delta_i p_k = N_{ik}$. Hence we obtain

$$\begin{aligned} A_{kij} &= \delta_i g_{jk} - \delta_j g_{ik} + g_{ir}\dot{\partial}^r N_{jk} - g_{jr}\dot{\partial}^r N_{ik} \\ &= \delta_i g_{jk} - \delta_j g_{ik} + g_{ir}B^r_{jk} - g_{jr}B^r_{ik} \\ &= g_{jk|i} - g_{ik|j} \\ &= 2L_{jki} - 2L_{ikj} = 0 \end{aligned} \tag{29}$$

Now we suppose that $\upsilon = -c\alpha\beta^2$. Thus from equation $A_{kij} = 0$ and Lemma 3, we conclude that **J** is integrable if and only if (27) is held.

A Cartan space K^n is of constant scalar curvature *c* if

$$H_{hijk}p^i p^j X^h X^k = c\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right)p^i p^j X^h X^k, \tag{30}$$

for every $(x,p) \in T_0^*M$ and $X = (X^i) \in T_xM$. Here H_{hijk} is the (hh)h-curvature of the linear Cartan connection of K^n. We replace H_{hijk} in (30) with $g_{is}H^s_{hjk}$ and so it reduces to the following

$$p_s H^s_{hjk}p^j X^h X^k = c\left(p_h p_k - K^2 g_{hk}\right)X^h X^k. \tag{31}$$

By part (ii) of Proposition 5.1 in chapter 7 of [13], $p_s H^s_{hjk} = -R_{hjk}$, hence we get

$$R_{hjk}p^j X^h X^k = c\left(K^2 g_{hk} - p_h p_k\right)X^h X^k,$$

or equivalently

$$R_{hjk}p^j = c\left(K^2 g_{hk} - p_h p_k\right), \tag{32}$$

because (X^h) and (X^k) are arbitrary vector fields on *M*. It is easy to check that (31) follows from (27). Similarly it can be shown that if Cartan space K^n has the constant scalar curvature *c*, then the equation (27) is held (see [28]).

Theorem 3.5. Let *(M,K)* be a Cartan space with constant curvature *c*. Suppose that υ is given by (28). Then
(i) for negative constant *c*, structure (T^*M_0, G, J) is a Kähler manifold;
(ii) for positive constant *c*, the tube around the zero section in T^*M, defined by the condition $2\tau = K^2 < \frac{1}{c\beta^2}$, is a Kähler manifold.

Proof: The function v must satisfy in the following condition

$$a + 2\tau v = a\left(1 + 2(-c)\beta^2\tau\right) > 0, \quad \alpha, \beta > 0 \quad (33)$$

By using the above relation and Theorem 4, we complete the proof.

By attention to the Theorem 5, the components of the Kähler metric G on T^*M_0 are

$$\begin{cases} G_{ij} = \dfrac{1}{\beta} g_{ij} - c\beta p_i p_j, \\ G^{ij} = \beta g_{ij} + \dfrac{c\beta^3}{1 - 2c\beta^2} p_i p_j. \end{cases} \quad (34)$$

4. A Kähler Einstein structure on cotangent bundle

In this section, we study the property of $\left(T^*M_0, G\right)$ to be Einstein. We find the expression of the Levi-Civita connection ∇ of the metric G on T^*M_0, then we get the curvature tensor field of ∇. Then, by computing the corresponding traces, we find the components of Ricci tensor field of ∇.

4.1. The Levi-Civita Connection

Lemma 1. The Levi-Civita connection of the Kähler metric G are given by the following

$$\nabla_{\dot\partial^i}\dot\partial^j = \left(\beta^2 L^{ijs}\right)\delta_s + \left(-C_s^{ij} + c\beta G^{ij} p_s\right)\dot\partial^s, \quad (35)$$

$$\nabla_{\delta_i}\dot\partial^j = \left(C_i^{js} - c\beta G^{js} p_i\right)\delta_s - \left(L_{is}^j + B_{is}^j\right)\dot\partial^s, \quad (36)$$

$$\nabla_{\dot\partial^i}\delta_j = \left(C_j^{is} - c\beta G^{is} p_j\right)\delta_s - L_{js}^i\dot\partial^s, \quad (37)$$

$$\nabla_{\delta_i}\delta_j = \left(L_{ij}^s + B_{ij}^s\right)\delta_s + \left(-\frac{1}{\beta^2}C_{ijs} + c\beta G_{js} p_i\right)\dot\partial^s, \quad (38)$$

Proof: Recall that for Cartan space with a Berwald connection, the relation $B_{jk}^i = \dot\partial^j N_{ik}$ is held, and so we have $\left[\delta_i, \dot\partial^j\right] = B_{ik}^j \dot\partial^k$. Let $\nabla_{\dot\partial^j}\dot\partial^j = \Gamma^{ijh}\delta_h + \Gamma_h^{ij}\dot\partial^h$. Then by using the Koszul formula we get

$$\Gamma^{ijs} = \frac{1}{2}\begin{bmatrix} -\delta_k\left(\beta g^{ij} + \dfrac{c\beta^3}{1-2c\beta^2\tau} p^i p^j\right) \\ -B_{km}^i\left(\beta g^{mj} + \dfrac{c\beta^3}{1-2c\beta^2\tau} p^m p^j\right) \\ -B_{km}^j\left(\beta g^{mi} + \dfrac{c\beta^3}{1-2c\beta^2\tau} p^m p^i\right) \end{bmatrix} \left(\beta g^{ks} + \frac{c\beta^3}{1-2c\beta^2\tau} p^k p^s\right) = \beta^2 g^{ks} L_k^{ij}. \quad (39)$$

Similarly we obtain

$$\Gamma_s^{ij} = \frac{1}{2}\begin{bmatrix} \dot\partial^i\left(\beta g^{jk} + \dfrac{c\beta^3}{1-2c\beta^2\tau} p^j p^k\right) \\ +\dot\partial^j\left(\beta g^{ik} + \dfrac{c\beta^3}{1-2c\beta^2\tau} p^i p^k\right) \\ -\dot\partial^k\left(\beta g^{ij} + \dfrac{c\beta^3}{1-2c\beta^2\tau} p^i p^j\right) \end{bmatrix} \left(\frac{1}{\beta} g_{ks} - c\beta p_k p_s\right) = -C_s^{ij} + c\beta G^{ij} p_s. \quad (40)$$

Using the two above equation we have (35). In a similar way, we obtain (36), (37) and (38).

We say that the vertical distribution VT^*M_0 is totally geodesic (resp. minimal) in TT^*M_0 if $H\nabla_{\dot\partial^i}\dot\partial^j = 0$ (resp. $g_{ij}H\nabla_{\dot\partial^i}\dot\partial^j = 0$), where H denotes the horizontal projection. Similarly, if we denote by V the vertical projection, then we say that the horizontal distribution HT^*M_0 is totally geodesic (resp. minimal) in TT^*M_0 if $V\nabla_{\delta_i}\delta_j = 0$ (resp. $g^{ij}V\nabla_{\delta_i}\delta_j = 0$). By using (35), we obtain

$$H\nabla_{\dot\partial^i}\dot\partial^j = \beta^2 L^{ijs}\delta_s, \quad (41)$$

and

$$g_{ij} H \nabla_{\dot{\partial}^i} \dot{\partial}^j = \beta^2 g_{ij} L^{ijs} \delta_s = \beta^2 J^s \delta_s, \tag{42}$$

where J^s is the mean Landsberg tensor. Hence, we have the following.

Corollary 2. Let (M, K) be a Cartan space with Berwald connection. Then we have

(i) K is Landsberg metric if and only if the vertical distribution VT^*M_0 is totally geodesic in TT^*M_0;

(ii) K is weakly Landsberg metric if and only if the vertical distribution VT^*M_0 is minimal in TT^*M_0.

Corollary 3. The horizontal distribution HT^*M_0 cannot be totally geodesic or minimal in TT^*M_0.

Proof: By (38), we have

$$V\nabla_{\delta_i} \delta_j = \left(-\frac{1}{\beta^2} C_{ijs} + c\beta G_{js} p_i \right) \dot{\partial}^s .$$

If HT^*M_0 is totally geodesic, then we have $c\beta G_{js} p_i p^j = 0$. Therefore, we obtain $cp_i p_s \left(1 - 2c\beta^2 \tau\right) = 0$, which cannot be true.

4. 2. The Curvature Tensors

Theorem 4. The coefficients of the curvature tensor of Kähler metric G are as follows

$$\begin{aligned} K\left(\dot{\partial}^i, \dot{\partial}^j\right)\dot{\partial}^k &= \left[\beta^2 \left(L^{jkh}\big|^i - L^{ikh}\big|^j\right)\right]\delta_h \\ &+ \begin{bmatrix} C_h^{ik,j} - C_h^{jk,i} \\ +c\beta G^{jk}\delta_h^i - c\beta G^{ik}\delta_h^j \\ +C_s^{jk} C_h^{is} - C_s^{ik} C_h^{js} \\ +\beta^2\left(L_{sh}^j L^{sik} - L_{sh}^i L^{sjk}\right) \end{bmatrix} \dot{\partial}^h, \end{aligned} \tag{43}$$

$$\begin{aligned} K\left(\delta_i, \dot{\partial}^j\right)\dot{\partial}^k &= \begin{bmatrix} c\beta G^{kh}\delta_i^j - C_i^{kh,j} \\ -C_s^{ih} C_i^{ks} - C_s^{jk} C_i^{hs} + \\ \beta^2\left(L^{sjk} L_{is}^h + L^{hjk}{}_{|i} + L_{si}^k L^{hjs}\right) \end{bmatrix} \delta_h \\ &+ \begin{bmatrix} B_{ih}^{k,j} - C_{h\ |i}^{jk} - c\beta^2 L_i^{jk} p_h \\ -C_{ish} L^{jsk} + C_k^{js} L_{ih}^s + C_i^{sk} L_{sh}^j \\ -C_h^{js} L_{is}^k + L_{hi}^{k,j} \end{bmatrix} \dot{\partial}^h, \end{aligned} \tag{44}$$

$$\begin{aligned} K\left(\delta_i, \delta_j\right)\delta_k &= \begin{bmatrix} R_{kji}^h + \frac{1}{\beta^2}\left(C_{iks} C_j^{hs} - C_{jks} C_i^{hs}\right) \\ +c^2\beta^2\left(p_i\delta_j^h - p_j\delta_i^h\right)p_k \\ +\left(L_{kj}^s L_{is}^h - L_{ki}^s L_{js}^h\right) + \left(L_{kj|i}^h - L_{ki|j}^h\right) \end{bmatrix} \delta_h \\ &+ \begin{bmatrix} \frac{1}{\beta^2}\left(C_{ikh|j} - C_{jkh|i}\right) + 2R_{sij} L_{hk}^s \\ + \frac{1}{\beta^2}\left(C_{jks} L_{ih}^s - C_{iks} L_{jh}^s\right) \\ +C_{jhs} L_{ki}^s - C_{ihs} L_{jk}^s \end{bmatrix} \dot{\partial}^h, \end{aligned} \tag{45}$$

$$\begin{aligned} K\left(\delta_i, \delta_j\right)\dot{\partial}^k &= \begin{bmatrix} C_{j\ |i}^{kh} - C_{i\ |j}^{kh} + c\beta^2\left(p_j L_i^{kh} - p_i L_j^{kh}\right) \\ +C_j^{ks} L_{si}^h - C_i^{ks} L_{sj}^h \\ +C_j^{sh} L_{si}^k + C_i^{sh} L_{sj}^k \end{bmatrix} \delta_h \\ &+ \begin{bmatrix} -R_{hji}^k + \frac{1}{\beta^2}\left(C_i^{ks} C_{jhs} - C_j^{ks} C_{ihs}\right) \\ +c^2\beta^2 p_h\left(p_j\delta_i^k - p_i\delta_j^k\right) \\ +L_{hi|j}^k - L_{hj|i}^k + L_{sj}^k L_{hi}^s - L_{si}^k L_{hj}^s \end{bmatrix} \dot{\partial}^h, \end{aligned} \tag{46}$$

$$\begin{aligned} K\left(\dot{\partial}^i, \dot{\partial}^j\right)\delta_k &= \begin{bmatrix} C_k^{jh,i} - C_k^{ih,j} + C_k^{js} C_s^{ih} - C_k^{is} C_s^{jh} \\ +c\beta\left(G^{ih}\delta_k^j - G^{jh}\delta_k^i\right) \\ +\beta^2\left(L^{jsh} L_{sk}^i - L^{ish} L_{sk}^j\right) \end{bmatrix} \delta_h \\ &+ \left[L_{kh}^i\big|^j - L_{kh}^j\big|^i\right]\dot{\partial}^h, \end{aligned} \tag{47}$$

$$\begin{aligned} K\left(\delta_i, \dot{\partial}^j\right)\delta_k &= \begin{bmatrix} C_{k|i}^{jh} + c\beta^2 L_i^{jh} p_k - L_{ki}^{h,j} \\ -B_{ik}^{h,j} + C_k^{js} L_{si}^h \\ -C_s^{jh} L_{ki}^s - C_i^{sh} L_{sk}^j + C_{iks} L^{hjs} \end{bmatrix} \delta_h \\ &+ \begin{bmatrix} \frac{1}{\beta^2}\left(C_{ikh}^{\ ,j} - C_{ish} C_k^{js} - C_{iks} C_h^{js}\right) \\ +cp_h C_{ik}^j + cp_k C_{ih}^j \\ -c\beta G_{kh}\delta_i^j + L_{sk}^j L_{hi}^s \\ +L_{sh}^j L_{ki}^s - L_{hk|i}^j \end{bmatrix} \dot{\partial}^h. \end{aligned} \tag{48}$$

Proof: Recall that the curvature K of ∇ is obtained from the following

$$K(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z \\ -\nabla_{[X,Y]}Z,\quad \forall X,Y,Z\in\Gamma\left(T^*M_0\right). \tag{49}$$

Using (49) we have

$$K\left(\dot{\partial}^i,\dot{\partial}^j\right)\dot{\partial}^k=\nabla_{\dot{\partial}^i}\nabla_{\dot{\partial}^j}\dot{\partial}^k-\nabla_{\dot{\partial}^j}\nabla_{\dot{\partial}^i}\dot{\partial}^k \tag{50}$$

By (35), it follows that

$$\begin{aligned}\nabla_{\dot{\partial}^i}\nabla_{\dot{\partial}^j}\dot{\partial}^k&=\dot{\partial}^i\left(\beta^2 g^{ms}L^{kj}_m\right)\delta_s\\&+\left(\beta^2 g^{ms}L^{jk}_m\right)\nabla_{\dot{\partial}^i}\delta_s\\&+\dot{\partial}^i\left(-C_s^{jk}+c\beta G^{jk}p_s\right)\dot{\partial}^s\\&+\left(-C_s^{jk}+c\beta G^{jk}p_s\right)\nabla_{\dot{\partial}^i}\dot{\partial}^s\end{aligned} \tag{51}$$

Since $p_h L^h_{tl}=0$, $p^l L^h_{rl}=p^r L^h_{rl}=0$ and $\dot{\partial}^k g^{ij}=-2C^{kij}$, then by (51) the following relation yields

$$\nabla_{\dot{\partial}^i}\nabla_{\dot{\partial}^j}\dot{\partial}^k=\begin{bmatrix}\beta^2 L^{hjk,i}\\+\beta^2\left(C^{mih}L^{jk}_m-C_s^{jk}L^{ihs}\right)\end{bmatrix}\delta_h \\ +\begin{bmatrix}-\beta^2 g^{ms}g_{nh}L^{jk}_m L^{in}_s+C_s^{jk}C_h^{is}\\+c^2\beta^2 G^{jk}G^{is}p_s p_h-c\beta^2 C^{ijk}p_h\\+\dot{\partial}^i\left(-C_h^{jk}+c\beta G^{jk}p_h\right)\end{bmatrix}\dot{\partial}^h, \tag{52}$$

where $L^{hjk,i}=\dot{\partial}^i L^{hjk}$. Since $\dot{\partial}^i\tau=p^i$ and $\dot{\partial}^i p^j=g^{ij}$, we obtain

$$\begin{aligned}&c\beta G^{jk,i}p_s-c\beta G^{ik,j}p_s\\&+c^2\beta^2 G^{jk}G^{ih}p_h p_s\\&-c^2\beta^2 G^{ik}G^{jh}p_h p_s\\&=\frac{c^2\beta^4}{1-2c\beta^2\tau}\begin{pmatrix}g^{ik}p^j-g^{jk}p^i\\+g^{jk}p^i-g^{ik}p^j\end{pmatrix}=0.\end{aligned} \tag{53}$$

By replacing i,j in (52) and setting these equations in (50), also by attention to (53), we get

$$K\left(\dot{\partial}^i,\dot{\partial}^j\right)\dot{\partial}^k=\left[\beta^2\left(L^{jkh|i}-L^{ikh|j}\right)\right]\delta_h \\ +\begin{bmatrix}C_h^{ik,j}-C_h^{jk,i}+c\beta G^{jk}\delta^i_h\\-c\beta G^{ik}\delta^j_h+C_s^{jk}C_h^{is}\end{bmatrix} \\ -C_s^{ik}C_h^{js}+\beta^2\left(L^j_{sh}L^{sik}-L^i_{sh}L^{sjk}\right)\dot{\partial}^h. \tag{54}$$

Similarly we can obtain the other components of curvature tensor.

By using (45), we have

$$\left(K\left(\delta_i,\delta_j\right)\delta_k\right)^H=\begin{bmatrix}R^h_{kji}+\frac{1}{\beta^2}\left(C_{iks}C_j^{hs}-C_{jks}C_i^{hs}\right)\\+c^2\beta^2\left(p_i\delta^h_j-p_j\delta^h_i\right)p_k\\+\left(L^s_{kj}L^h_{is}-L^s_{ki}L^h_{js}\right)+\left(L^h_{kj|i}-L^h_{ki|j}\right)\end{bmatrix}\delta_h.$$

Contracting the above equation by p_h gives us

$$p_h\left(K\left(\delta_i,\delta_j\right)\delta_k\right)^H=-p_h R^h_{kji}=R_{kji}.$$

If $\left(T^*M_0,G\right)$ is locally flat, then we have $K(X,Y)Z=0$ for all $X,Y,Z\in\chi\left(T^*M_0\right)$. Hence, by using the above equation we infer that $R_{kji}=0$ or $R_{kji}p^j=0$. Therefore, we have the following.

Theorem 5. Let (M,K) be a Cartan space and G be the Riemannian metric on T^*M_0 defined by (34). If $\left(T^*M_0,G\right)$ is locally flat, then (M,K) has the zero constant curvature.

Theorem 6. Let (M,K) be a Cartan space of constant curvature c and the components of the metric G are given by (34). Then the following are held if and only if (M,K) is reduce to a Riemannian space.

(i) for $c<0$, $\left(T^*M_0,G,J\right)$ is a Kähler Einstein manifold.

(ii) for $c>0$, $\left(T^*_\beta M_0,G,J\right)$ is a Kähler Einstein manifold, where $T^*_\beta M_0$ is the tube around the zero section in TM, defined by the condition $2\tau<\frac{1}{c\beta^2}$.

Proof: Let (M,K) be a Riemannian space. Then C^{hi}_k and P^h_{ik} vanish and H^i_{jk} is a function of $\left(x^h\right)$. Therefore (45) reduces to the following

$$K\left(\delta_i,\delta_j\right)\delta_k=\left[R^s_{kji}+c^2\beta^2\left(p_i\delta^s_j-p_j\delta^s_i\right)p_k\right]\delta_s. \quad (55)$$

From Proposition 10.2 in chapter 4 of [13], we have $R_{kji}=-p_hR^h_{kji}$. Then we have

$$p_hR^h_{kji}=c\left(g_{kj}\delta^h_i-g_{ki}\delta^h_j\right)p_h. \quad (56)$$

Differentiating (56) with respect to p_s and taking $p=0$, it follows that

$$R^s_{kji}=c\left(g_{kj}\delta^s_i-g_{ki}\delta^s_j\right). \quad (57)$$

By putting (57) in (55), one can obtain

$$\begin{aligned}K\left(\delta_i,\delta_j\right)\delta_k&=c\beta\begin{bmatrix}\left(\frac{1}{\beta}g_{kj}-c\beta p_kp_j\right)\delta^s_i\\-\left(\frac{1}{\beta}g_{ki}-c\beta p_kp_i\right)\delta^s_j\end{bmatrix}\delta_s\\&=c\beta\left(G_{kj}\delta^s_i-G_{ki}\delta^s_j\right)\delta_s\end{aligned} \quad (58)$$

Also from (48), we get

$$K\left(\dot\partial^i,\delta_j\right)\delta_k=c\beta G_{sk}\delta^i_j\dot\partial^s. \quad (59)$$

From (58) and (59), we conclude that

$$\begin{aligned}&Ric\left(\delta_j,\delta_k\right)=G^{hi}G\left(K\left(\delta_i,\delta_j\right)\delta_k,\delta_h\right)\\&+G_{hi}G\left(K\left(\dot\partial^i,\delta_j\right)\delta_k,\dot\partial^h\right),\\&=c\beta\left(G_{kj}\delta^s_i-G_{ki}\delta^s_j\right)G^{hi}G_{sh}\\&+c\beta G_{sk}\delta^i_jG_{hi}G^{sh}\\&=cn\beta G_{jk}=cn\beta G\left(\delta_i,\delta_k\right).\end{aligned} \quad (60)$$

Similarly from (43) and (44), respectively, it follows that

$$\begin{aligned}&K\left(\dot\partial^i,\dot\partial^j\right)\dot\partial^k=c\beta\left(G^{jk}\delta^i_s-G^{ik}\delta^j_s\right)\dot\partial^s,\\&K\left(\delta_i,\dot\partial^j\right)\dot\partial^k=c\beta G^{ks}\delta^j_i\delta_s\end{aligned} \quad (61)$$

By using (61), we obtain

$$\begin{aligned}&Ric\left(\dot\partial^j,\dot\partial^k\right)=G^{ih}G\left(K\left(\delta_i,\dot\partial^j\right)\dot\partial^k,\delta_h\right)\\&+G_{ih}G\left(K\left(\dot\partial^i,\dot\partial^j\right)\dot\partial^k,\dot\partial^h\right),\\&=c\beta G^{ks}\delta^j_iG^{ih}G_{hs}+c\beta\left(G^{jk}\delta^i_s-G^{ik}\delta^j_s\right)G_{hi}G^{hs}\\&=cn\beta G^{jk}=cn\beta G\left(\dot\partial^j,\dot\partial^k\right).\end{aligned} \quad (62)$$

From (44) and (46), we have, respectively

$$\begin{aligned}&K\left(\delta_i,\delta_j\right)\dot\partial^k=\left(R^k_{sij}+c\beta R_{hij}G^{hk}p_s\right)\dot\partial^s,\\&K\left(\dot\partial^i,\delta_j\right)\dot\partial^k=-c\beta G^{ks}\delta^i_j\delta_s.\end{aligned} \quad (63)$$

By using (63), we obtain

$$\begin{aligned}&Ric\left(\delta_j,\dot\partial^k\right)=G^{ih}G\left(K\left(\delta_i,\delta_j\right)\dot\partial^k,\delta_h\right)\\&+G_{ih}G\left(K\left(\dot\partial^i,\delta_j\right)\dot\partial^k,\dot\partial^h\right)=0.\end{aligned} \quad (64)$$

From (47), we get

$$K\left(\dot\partial^i,\dot\partial^j\right)\delta_k=c\beta\left(G^{is}\delta^j_k-G^{js}\delta^i_k\right)\delta_s. \quad (65)$$

By attention to (59) and (65), one can yield

$$\begin{aligned}&Ric\left(\dot\partial^j,\delta_k\right)=G^{ih}G\left(K\left(\delta_i,\dot\partial^j\right)\delta_k,\delta_h\right)\\&+G_{ih}G\left(K\left(\dot\partial^i,\dot\partial^j\right)\delta_k,\dot\partial^h\right)=0.\end{aligned} \quad (66)$$

From (60), (62), (64) and (66), it follows that $Ric\left(X,Y\right)=cn\beta G\left(X,Y\right)$, $\forall X,Y\in\chi\left(T^*M\right)$. This means that $\left(T^*M,G\right)$ is an Einstein manifold. Conversely, let $\left(i\right),\left(ii\right)$ are held. Then there exist constant λ such that $Ric\left(X,Y\right)=\lambda G\left(X,Y\right)$. We consider the following cases:

Case (1). If $\lambda=0$ (i.e., $\left(T^*M,G\right)$ is Ricci flat), then we have $Ric\left(\dot\partial^j,\dot\partial^k\right)=0$. By using (43) and (44) we get

$$\begin{aligned}&p_kG_{ih}G\left(K\left(\dot\partial^i,\dot\partial^j\right)\dot\partial^k,\dot\partial^h\right)=p_kC_h^{hk,j}\\&-p_kC_h^{jk,h}+\left(n-1\right)c\beta p_kG^{jk}\end{aligned}$$

and

$$\begin{aligned}&G^{ih}G\left(K\left(\delta_i,\dot\partial^j\right)\dot\partial^k,\delta_h\right)=c\beta p_kG^{jk}\\&-p_kC_h^{kh,j}+\beta^2p_kL^{hjk}{}_{|h}\end{aligned}$$

By using the two above equations, it results that

$$\begin{aligned}&0=p_kRic\left(\dot\partial^j,\dot\partial^k\right)=cn\beta p_kG^{jk}\\&-p_kC_h^{jk,h}+\beta^2p_kL^{hjk}{}_{|h}.\end{aligned} \quad (67)$$

With a simple calculation, one can obtain

$$cn\beta p_k G^{jk} = cn\beta p_k\left(\beta g^{jk} + \frac{c\beta^3}{1-2c\beta^2\tau}p^j p^k\right) \quad (68)$$
$$= \frac{cn\beta^2}{1-2c\beta^2\tau}p^j,$$

and

$$p_k C_h^{jk,h} = -p_k^{;h} C_h^{jk} = -\delta_k^h C_h^{jk} = -C_h^{jh} = -I^j, \quad (69)$$

$$p_k L^{hjk}{}_{|h} = -p_{k|h} L^{hjk} = 0. \quad (70)$$

By using (67),(70), we obtain

$$\frac{cn\beta^2}{1-2c\beta^2\tau}p^j + I^j = 0. \quad (71)$$

Since $p_j I^j = 0$, by contracting (71) with p_j we have $\frac{2cn\beta^2\tau}{1-2c\beta^2\tau} = 0$. Thus we get $\beta = 0$, which is a contradiction.

Case (2). If $\lambda \neq 0$, then we have $p_k Ric\left(\dot{\partial}^j, \dot{\partial}^k\right) = \lambda G^{ik} p_k$. By using (44), (43), (68) and (70), we obtain

$$I^j = \left(\lambda - cn\beta\right)\frac{\beta}{1-2c\beta^2\tau}p^j. \quad (72)$$

Contracting (72) with p_j yields

$$\left(\lambda - cn\beta\right)\frac{2\beta\tau}{1-2c\beta^2\tau} = 0, \quad (73)$$

i.e., $\lambda = cn\beta$. Thus by (72), we conclude that $I^j = 0$, i.e., $\left(M, K\right)$ is reduced to a Riemannian space.

Corollary 7. There is no non-Riemannian Cartan structure such that $\left(T^* M_0, G, J\right)$ becomes an Einstein manifold.

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