\textbf{Abstract}

The concept of $\Gamma$-semihypergroups is a generalization of semigroups, a generalization of semihypergroups and a generalization of $\Gamma$-semigroups. In this paper, we study the concept of semiprime ideals in a $\Gamma$-semihypergroup and prove some results. Also, we introduce the notion of $\Gamma$-hypergroups and closed $\Gamma$-subhypergroups. Finally, we study the concept of $\Gamma$-semihypergroups associated to binary relations and give necessary and sufficient conditions on a set of binary relations $\Gamma$ on a non-empty set $S$ such that $S$ becomes a $\Gamma$-semihypergroup or a $\Gamma$-hypergroup.

\textbf{Keywords}: Hypergroup; semihypergroup; $\Gamma$-semigroup; $\Gamma$-semihypergroup; binary relation

\section{1. Introduction}

The \textit{hyperstructure} theory was born in 1934, when Marty introduced the notion of a \textit{hypergroup} [1]. Since then, hundreds of papers and several books have been written on this topic, see [2-5]. A recent book on hyperstructures [6] points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a non-empty set. More exactly, let $H$ be a non-empty set. Then the map $\circ: H \times H \to \mathcal{P}(H)$ is called a hyperoperation where $\mathcal{P}(H)$ is the family of non-empty subsets of $H$. The couple $(H, \circ)$ is called a hypergroupoid.

In the above definition, if $A$ and $B$ are two non-empty subsets of $H$ and $x \in H$, then we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

A hypergroupoid $(H, \circ)$ is called a \textit{semihypergroup} if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$, and is called a \textit{quasihypergroup} if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a hypergroup if it is a semihypergroup and a quasihypergroup.

The notion of $\Gamma$-\textit{semigroups} was introduced by Sen in [7, 8]. Let $S$ and $\Gamma$ be two non-empty sets. Then $S$ is called a $\Gamma$-semigroup if there exists a mapping $S \times \Gamma \times S \to S$, written $(a, \gamma, b)$ by $a \gamma b$, such that it satisfies the identities $(a \gamma b) \beta c = a \gamma (b \beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Let $S$ be an arbitrary semigroup and $\Gamma$ a non-empty set. Define a mapping $S \times \Gamma \times S \to S$ by $a \gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that $S$ is a $\Gamma$-semigroup. Thus a semigroup can be considered to be a $\Gamma$-semigroup. Many classical notions of semigroups have been extended to $\Gamma$-semigroups, see ([9, 10]).

Let $S$ be a $\Gamma$-semigroup and $\alpha$ be a fixed element in $\Gamma$. We define $a \cdot b = a \alpha b$ for all $a, b \in S$. Then $(S, \cdot)$ is a semigroup and is denoted by $S_\alpha$. 

*Corresponding author

Received: 12 October 2010 / Accepted: 20 February 2011
2. Preliminaries and basic definitions

The concept of $\Gamma$-semihypergroups was introduced by Davvaz et al. [11, 12]. In this section we introduce some preliminaries and basic definitions of $\Gamma$-semihypergroups and give some examples.

**Definition 2.1.** Let $S$ and $\Gamma$ be two non-empty sets. Then $S$ is called a $\Gamma$-semihypergroup if each $\gamma \in \Gamma$ be a hyperoperation on $S$, i.e., $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property that is $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Let $A$ and $B$ be two non-empty subsets of $S$ and $\gamma \in \Gamma$. Then we define:
\[ A\gamma B = \bigcup \{a\gamma b \mid a \in A, b \in B\}, \]
and
\[ A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B = \bigcup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}. \]

A $\Gamma$-semihypergroup $S$ is called commutative if for every $x, y \in S$ and for every $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$. A non-empty subset $A$ of $S$ is called a $\Gamma$-subsemihypergroup of $S$ if $A\Gamma A \subseteq A$.

Let $(S, \cdot)$ be a semihypergroup and let $\Gamma = \{\cdot\}$. Then $S$ is a $\Gamma$-semihypergroup. So every semihypergroup is a $\Gamma$-semihypergroup.

Let $S$ be a $\Gamma$-semihypergroup and $\alpha \in \Gamma$, if we define $a \cdot b = a\alpha b$ for every $a, b \in S$ then $(S, \cdot)$ becomes a semihypergroup, we denote it by $S_{\alpha}$.

Now, we give some other examples of $\Gamma$-semihypergroups.

**Example 1.** Let $G$ be a group and $\Gamma = \{\alpha, \beta\}$. Then for every $x, y \in G$, we define $x\alpha y = xy$ and $x\beta y = G$. Then $G$ is a $\Gamma$-semihypergroup.

**Example 2.** Let $(S, \leq)$ be a totally ordered set and $\Gamma$ be a non-empty subset of $S$. We define
\[ x\gamma y = \{z \in S \mid z \geq \max\{x, \gamma, y\}\}, \]
for every $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a $\Gamma$-semihypergroup.

**Example 3.** Let $S$ be a $\Gamma$-semigroup and $P$ be a non-empty subset of $S$. Let $\Gamma_P = \{\alpha, x \in \Gamma\}$. If we define $x\alpha_P y = x\alpha P\alpha y$, for every $x, y \in S$ and $\alpha \in \Gamma$, then $S$ is a $\Gamma_P$-semihypergroup.

Let $S$ be a $\Gamma$-semihypergroup. We define a relation $\rho$ on $S \times \Gamma$ as follows:
\[(x, \alpha)\rho(y, \beta) \iff x\alpha s = y\beta s, \forall s \in S.\]

Obviously $\rho$ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing $(x, \alpha)$. Let $M = \{[x, \alpha] \mid x \in S, \alpha \in \Gamma\}$. We define the hyperoperation $\circ$ on $M$ as follows:
\[[x, \alpha] \circ [y, \beta] = \{z, \gamma \mid z = x\alpha y\}, \text{ for all } [x, \alpha], [y, \beta] \in M.\]

Since $(x\alpha y)\beta z = x\alpha(y\beta z)$, for all $x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$, then
\[[x, \alpha] \circ ([y, \beta] \circ [z, \gamma]) = ([x, \alpha] \circ [y, \beta]) \circ [z, \gamma], \text{ for all } [x, \alpha], [y, \beta], [z, \gamma] \in M.\]

Thus the hyperoperation $\circ$ is associative, so $(M, \circ)$ is a semihypergroup. This semihypergroup is called the left operator semihypergroup of $S$.

Let $S$ be a $\Gamma$-semihypergroup. If there exist elements $e \in S$ and $\delta \in \Gamma$ such that $e\delta x = x$ for every $x \in S$, then $S$ is said to have a left partial unity which is denoted by $e_{\delta}$. It is easy to check whether $e_{\delta}$ is a left partial unity of $S$, then $[e, \delta]$ is a left unity of the left operator semihypergroup $M$.

**Example 4.** Consider Example 1 and let $e$ be the identity element of $G$. Then $e_{\delta} = e$ is a left partial unity of the $\Gamma$-semihypergroup $G$.

The concept of $\Gamma$-hyperideals of a $\Gamma$-semihypergroup was defined and studied in [12].

**Definition 2.2.** A non-empty subset $I$ of a $\Gamma$-semihypergroup $S$ is called a left (right) $\Gamma$-hyperideal, "ideal, for short" of $S$, if $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). $S$ is called a left (right) simple $\Gamma$-semihypergroup if it has no proper left (right) ideal. $S$ is simple if $S$ has no proper left and right ideals.
Let $A$ be a non-empty subset of a $\Gamma$-semihypergroup $S$. Then the intersection of all ideals of $S$ containing $A$ is an ideal of $S$ generated by $A$, and denoted by $<A>$. 

**Example 5.** Consider Example 4. Put $S = N$ with natural order. Then the subset $I_n = \{n, n+1, n+2, \cdots\}$ is an ideal of $S$, for every $n \in \mathbb{N}$.

The following lemmas and theorem were proved in [12].

**Lemma 2.3.** Let $S$ be a $\Gamma$-semihypergroup. If $A$ is a non-empty subset of $S$, then

$$<A> = A \cup \Gamma A S \cup S T A \cup S T A \Gamma S.$$ 

One can see that, if $S$ is a commutative $\Gamma$-semihypergroup and $\phi \neq A \subseteq S$, then $<A> = A \cup \Gamma A S$. If $S$ is a commutative $\Gamma$-semihypergroup with left partial unity, then $<A> = \Gamma A S$.

**Lemma 2.4.** Let $S$ be a $\Gamma$-semihypergroup and $\Lambda$ be a non-empty set such that for every $\lambda \in \Lambda$, $I_\lambda$ is an ideal of $S$. Then the following assertions hold: 

1. $\bigcup_{\lambda \in \Lambda} I_\lambda$ is an ideal of $S$;
2. $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an ideal of $S$.

**Definition 2.5.** A proper ideal $P$ of a $\Gamma$-semihypergroup $S$ is called a prime ideal, if for every ideal $I$ and $J$ of $S$, $I J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If $S$ is a commutative, then a proper ideal $P$ is prime if and only if $a \Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a, b \in S$.

**Example 6.** Consider Example 2. Put $S = \Gamma = \{1, 2, \cdots, n\}$ for some natural number $n$.

Then, all ideals of $S$ have the form $I_i = \{i, i+1, \cdots, n\}$, for every $i \in S$ and $I_2$ is a prime ideal of $S$.

**Theorem 2.6.** Let $S$ be a $\Gamma$-semihypergroup and $P$ be a left ideal of $S$. Then $P$ is a prime ideal of $S$ if and only if for all $x, y \in S$, $x \Gamma S y \subseteq P$ implies that $x \in P$ or $y \in P$.

**Lemma 2.7.** Let $S$ be a commutative $\Gamma$-semihypergroup with a left partial unity and $M$ be a maximal ideal of $S$. Then $M$ is a prime ideal of $S$.

**Proof:** Suppose that $M$ is a maximal ideal and $e_\delta$ is the left partial unity of $S$. Let $x, y \in S$ such that $x \Gamma y \subseteq M$. Then we prove that $x \in M$ or $y \in M$. If $x \not\in M$, then $M = <M, x>$, so by maximality of $M$ we have $S = <M, x>$. Since $e_\delta \not\in M$, it follows that there exist $s \in S$ and $y \in \Gamma$ such that $e_\delta = x\gamma S$. Then, we have

$$y = e_\delta \gamma y = (x\gamma S)\gamma y \subseteq x\Gamma y \subseteq M.$$ 

Similarly, if $y \not\in M$, then one proves that $x \not\in M$. Therefore, $M$ is a prime ideal of $S$.

**Proposition 2.8.** Let $S$ be a $\Gamma$-semihypergroup with a left partial unity and $I$ be a proper ideal of $S$. Then there exists a maximal ideal of $S$ containing $I$.

**Proof:** By Lemma 2.4 and Zorn's lemma the proof is obvious.

Let $S$ be a $\Gamma$-semihypergroup and $M$ be the left operator semihypergroup of $S$. Then for $A \subseteq M$, Davvaz et al. in [12] defined $A^+$ as follows:

$$A^+ = \{x \in S : [x, \alpha] \in A \text{ for all } \alpha \in \Gamma\}.$$ 

Similarly, for $I \subseteq S$, they defined $I^+$ as follows:

$$I^+ = \{[x, \alpha] \in M : x\alpha s \subseteq I \text{ for all } s \in S\}.$$ 

If $I$ is an ideal of $S$ and $A$ is a hyperideal of $M$, then $I \subseteq (I^+)^+$ and $A \subseteq (A^+)^+.$

We recall the following theorems from [12].

**Theorem 2.9.** [12] Let $S$ be a $\Gamma$-semihypergroup and $M$ be its left operator semihypergroup. Then the following assertions hold: 

1. If $A$ is a right hyperideal of $M$, then $A^+$ is a right ideal of $S$. 

(2) If $I$ is a right ideal of $S$ then, $I^*$ is a right hyperideal of $M$.

**Theorem 2.10.** [12] Let $S$ be a $\Gamma$-semihypergroup with a left partial unity and $M$ be its left operator semihypergroup. If $I$ is a right ideal of $S$, then $I = (I^*)^*$.

### 3. Semiprime ideals of $\Gamma$-semihypergroups

In this section, we introduce the concept of semiprime ideals of a $\Gamma$-semihypergroup and prove some results.

**Definition 3.1.** Let $S$ be a $\Gamma$-semihypergroup. Then a proper left (right) ideal $P$ of $S$ is said to be a left (right) semiprime ideal, if for any left (right) ideal $A$ of $S$, $A\Gamma A \subseteq P$ implies that $A \subseteq P$. A proper ideal $P$ is called semiprime ideal if $P$ is both left and right semiprime ideal of $S$.

**Example 7.** Let $S = \Gamma = \{1, 2, 3, \cdots, n\}$ for some $n \in \mathbb{N}$. For every $x, y \in S$ and $\alpha \in \Gamma$ we define the following hyperoperation on $S$

$$x\alpha y = \{s \in S \mid s \geq \max\{x, \alpha, y\}\}.$$ 

Then $S$ is a $\Gamma$-semihypergroup and $I_i = \{i, i+1, \cdots, n\}$ is a semiprime ideal of $S$ for $1 \leq i \leq n$.

**Lemma 3.2** Let $S$ be a $\Gamma$-semihypergroup with a left partial unity and $P$ be a left ideal of $S$. Then $P$ is a left semiprime ideal of $S$ if and only if for every $x, y \in S$ we have

$$x\Gamma S\Gamma x \subseteq P \Rightarrow x \in P.$$

**Proof:** Suppose that $P$ is a left semiprime ideal of $S$ and $x\Gamma S\Gamma x \subseteq P$ for $x \in S$. Then $S\Gamma x\Gamma S\Gamma x \subseteq S\Gamma P \subseteq P$. Since $P$ is a left semiprime ideal and $S\Gamma x$ is a left ideal of $S$, it follows that $x \in S\Gamma x \subseteq P$.

Conversely, let $A$ be an ideal of $S$ such that $A\Gamma A \subseteq P$. If $a \in A$, then $a\Gamma S\Gamma a \subseteq A\Gamma A \subseteq P$. So, by the above implication $a \in P$ thus $A \subseteq P$.

**Lemma 3.3.** Let $S$ be a $\Gamma$-semihypergroup and $M$ be its left operator semihypergroup. Then the following statements hold:

1. If $P$ is a semiprime ideal of $M$, then $P^\prime$ is a semiprime ideal of $S$.
2. If $S$ has a left partial unity and $Q$ is a semiprime ideal of $S$, then $Q^\prime$ is a semiprime ideal of $M$.

**Proof:**
(1) Suppose that $P$ is a semiprime ideal of $M$ and $A$ is an ideal of $S$ such that $A\Gamma A \subseteq P^\prime$. Then $[A\Gamma A, \Gamma] \subseteq P$ so $[A, \Gamma] \cap [A, \Gamma] \subseteq P$. Since $[A, \Gamma]$ is an ideal of $M$ and $P$ is a semiprime ideal of $M$, it follows that $[A, \Gamma] \subseteq P$ hence $A \subseteq P^\prime$. Thus $P^\prime$ is a semiprime ideal of $S$.

(2) Suppose that $Q$ is a semiprime ideal of $S$ and $A$ is an ideal of $M$ such that $A \cap A \subseteq Q^\prime$. First, we show that $A^\prime \cap A^\prime \subseteq (A \cap A)^\prime$. Let $t \in A^\prime \cap A^\prime$. Then there exist $x, y \in A^\prime$ and $\gamma \in \Gamma$ such that $t = x\gamma y$. So $[t, \alpha] \in [x, \gamma] \cap [y, \alpha] \subseteq A \cap A$ for every $\alpha \in \Gamma$. Thus $t \in (A \cap A)^\prime$, so $A^\prime \cap A^\prime \subseteq (A \cap A)^\prime$. Now, from $A \cap A \subseteq Q^\prime$ and Theorem 2.10 we have

$$A^\prime \cap A^\prime \subseteq (A \cap A)^\prime \subseteq (Q^\prime)^\prime = Q.$$

Since $Q$ is a semiprime ideal and $A^\prime$ is an ideal of $S$, it follows that $A^\prime \subseteq Q$. Thus $A \subseteq (A^\prime)^\prime \subseteq Q^\prime$. Therefore, $Q^\prime$ is a semiprime ideal of $M$.

**Lemma 3.4.** Let $P_i$ be a prime ideal of a $\Gamma$-semihypergroup $S$ for every $i \in I$ and let $P = \bigcap_{i \in I} P_i$. Then if $P \neq \emptyset$, then $P$ is a semiprime ideal of $S$.

**Proof:** It is immediate.

**Lemma 3.5.** Let $T$ be a $\Gamma$-subsemihypergroup and $I$ be an ideal of the $\Gamma$-semihypergroup $S$ such that $I \cap T = \emptyset$. Then $T$ is contained in a $\Gamma$-subsemihypergroup that is maximal with respect to the property of not meeting $I$. 


**Proof:** Since the set \( A = \{ K | T \leq K \leq S \text{ and } K \cap I = \emptyset \} \) is non-empty, it follows that by Zorn’s lemma, \( A \) has a maximal element that satisfies the theorem.

**Lemma 3.6.** Let \( T \) be a commutative \( \Gamma \)-subsemihypergroup and \( I \) be an ideal of the \( \Gamma \)-semihypergroup \( S \) such that \( I \cap T = \emptyset \). Then there exists a prime ideal of \( S \), say \( P \), such that \( I \subseteq P \) and \( P \cap T = \emptyset \).

**Proof:** By Zorn’s lemma, there exists an ideal \( P \) such that \( P \) is maximal with respect to properties of \( I \subseteq P \) and \( P \cap T = \emptyset \). We claim that \( x, y \in S \setminus P \). Then, we show that \( x \mathcal{I} \mathcal{G} \mathcal{T} \mathcal{Y} \subseteq P \). Since \( x, y \notin P \) and \( P \) is maximal, it follows that \( < P, x > \cap T \neq \emptyset \) and \( < P, y > \cap T \neq \emptyset \).

Thus, there exist \( s, t \in S \) such that \( s < P, x > \cap T \) and \( t < P, y > \cap T \). From the property \( P \cap T = \emptyset \), we have only four cases: (i) \( s = s, x \alpha \) and \( t = t, \beta \gamma \) for some \( s, t, \alpha \in S \) and \( \alpha, \beta \in \Gamma \), (ii) \( s = s, x \alpha \) and \( t = t, \beta \gamma \) for some \( s, t, \alpha \in S \) and \( \alpha, \beta \in \Gamma \), (iii) \( s = x \) and \( t = t, \beta \gamma \) for some \( t, \beta \in S \) and \( \alpha \in \Gamma \), and (iv) \( s = x \) and \( t = y \). If (i) holds, then \( s \mathcal{I} \mathcal{G} \mathcal{I} \mathcal{Y} \subseteq (s, x \alpha) \Gamma (t, \beta \gamma) \subseteq x \mathcal{I} \mathcal{G} \mathcal{I} \mathcal{Y} \).

Now, since \( T \) is a \( \Gamma \)-subsemihypergroup, it follows that \( x \mathcal{I} \mathcal{G} \mathcal{I} \mathcal{Y} \subseteq T \). Thus \( x \mathcal{I} \mathcal{G} \mathcal{I} \mathcal{Y} \subseteq P \). Similarly, in the other cases we conclude that \( x \mathcal{I} \mathcal{G} \mathcal{I} \mathcal{Y} \subseteq P \). Therefore, \( P \) is a prime ideal of \( S \).

Let \( S \) be a \( \Gamma \)-semihypergroup and \( I \) be an ideal of \( S \). A prime ideal \( P \) of \( S \) is called a minimal prime ideal belonging to \( I \) if \( I \subseteq P \) and there is no prime ideal containing \( I \) and properly contained in \( P \).

**Corollary 3.7.** If \( Q \) is a prime ideal containing an ideal \( I \), then there exists a minimal prime ideal belonging to \( I \) which is contained in \( Q \).

**Definition 3.8.** Let \( S \) be a \( \Gamma \)-semihypergroup and \( I \) be an ideal of \( S \). Then the prime radical of \( I \) is defined as the intersection of all prime ideals of \( S \) containing \( I \) and is denoted by \( \sqrt{I} \).

**Proposition 3.9.** Let \( S \) be a \( \Gamma \)-semihypergroup and \( I \) be an ideal of \( S \). Then the following statements hold:

1. \( \sqrt{I} \) is a semiprime ideal of \( S \);
2. \( \sqrt{I} \cap \mathcal{P}(I) \) is a minimal prime ideal belonging to \( I \).

**Proof:** (1) It is straightforward.

(2) It is taken from Corollary 3.7.

4. \( \Gamma \)-hypergroups

In this section we study the concept of \( \Gamma \)-hypergroups and give some examples. Also, we introduce the concept of closed \( \Gamma \)-subhypergroups of a \( \Gamma \)-hypergroup.

**Definition 4.1.** A \( \Gamma \)-semihypergroup \( S \) is called a \( \Gamma \)-hypergroup if \( (S, \alpha, \beta) \) is a hypergroup for every \( \alpha, \beta \in \Gamma \).

**Example 8.** Let \( S = \{a, b, c, d\} \) and \( \Gamma = \{a, b\} \). We define the hyperoperations \( \alpha \) and \( \beta \) as follows:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>{a, b}</td>
<td>{b, c}</td>
<td>{c, d}</td>
<td>{a, d}</td>
</tr>
<tr>
<td>( b )</td>
<td>{b, c}</td>
<td>{c, d}</td>
<td>{a, d}</td>
<td>{a, b}</td>
</tr>
<tr>
<td>( c )</td>
<td>{c, d}</td>
<td>{a, d}</td>
<td>{a, b}</td>
<td>{b, c}</td>
</tr>
<tr>
<td>( d )</td>
<td>{a, d}</td>
<td>{a, b}</td>
<td>{b, c}</td>
<td>{c, d}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>{b, c}</td>
<td>{c, d}</td>
<td>{a, d}</td>
<td>{a, b}</td>
</tr>
<tr>
<td>( b )</td>
<td>{c, d}</td>
<td>{a, d}</td>
<td>{a, b}</td>
<td>{b, c}</td>
</tr>
<tr>
<td>( c )</td>
<td>{a, d}</td>
<td>{a, b}</td>
<td>{b, c}</td>
<td>{c, d}</td>
</tr>
<tr>
<td>( d )</td>
<td>{a, b}</td>
<td>{b, c}</td>
<td>{c, d}</td>
<td>{a, d}</td>
</tr>
</tbody>
</table>

Then \( S \) is a \( \Gamma \)-hypergroup.

**Example 9.** Let \( S \) be a non-empty set and \( \Gamma = \{a, \beta\} \). Then for every \( x, y \in S \) and \( a, \beta \in \Gamma \) we define \( x \alpha y = S \) and \( x \beta y = \{x, y\} \). Then \( S \) is a \( \Gamma \)-hypergroup.

**Example 10.** Let \( (S, \cdot) \) be a group. Let \( \Gamma \subseteq \mathcal{P}(S) \). We define \( x \alpha y = x \cdot \alpha \cdot y \) for every
$x, y \in S$ and $\alpha \in \Gamma$. Then $S$ is a $\Gamma$-hypergroup.

**Example 11.** Let $(S, \circ)$ be a hypergroup and $\emptyset \neq \Gamma \subseteq S$. We define $x \alpha y = x \circ \alpha \circ y$ for every $x, y \in S$ and $\alpha \in \Gamma$. Then $S$ is a $\Gamma$-hypergroup.

**Example 12.** Let $(G, \ast)$ be a group and $\{A_\gamma\}_{\gamma \in G}$ be a collection of disjoint sets. Consider $\bigcup_{\gamma \in G} A_\gamma$ and $\Gamma = G$. For $x, y \in S$, there exist $g_x, g_y \in G$ such that $x \in A_{g_x}$ and $y \in A_{g_y}$. We define $x \alpha y = A_{g_x \ast \alpha \ast g_y}$. Then $S$ is a $\Gamma$-hypergroup.

**Theorem 4.2.** [12] Let $S$ be a $\Gamma$-semihypergroup. Then $S$ is a simple $\Gamma$-semihypergroup if and only if $S_\alpha$ is a hypergroup for every $\alpha \in \Gamma$.

**Theorem 4.3.** Let $S$ be a $\Gamma$-semihypergroup. Then for every $\alpha \in \Gamma$, $S_\alpha$ is a hypergroup if and only if $S$ is left and right simple.

**Proof:** Suppose that $S_\alpha$ is a hypergroup and $I$ is a left (right) ideal of $S$. If $x \in I$, then the reproduction axiom implies that $x \alpha S = S = S \alpha x$. On the other hand, we have $S \alpha x \subseteq I$ ($x \alpha S \subseteq I$). Therefore, $I = S$.

Conversely, suppose that $S$ is left and right simple. Then for every $x \in S$ and $\alpha \in \Gamma$, put $I = x \alpha S$. Thus, $I$ is a right ideal of $S$, for

$I \Gamma S = (x \alpha S) \Gamma S = x \alpha (S \Gamma S) \subseteq x \alpha S = I$

so $x \alpha S = S$. Similarly, we have $S = S \alpha x$. Therefore, $S$ is a $\Gamma$-hypergroup.

**Corollary 4.4.** If $S_\alpha$ is a hypergroup for some $\alpha \in \Gamma$, then for every $\alpha \in \Gamma$, $S_\alpha$ is a hypergroup.

**Definition 4.5.** A subset $H$ of a $\Gamma$-hypergroup is called a $\Gamma$-subhypergroup if for every $h, k \in H$ and $\alpha \in \Gamma$ we have $h \alpha k \subseteq H$ and $h \alpha H = H = H \alpha h$.

**Definition 4.6.** Let $S$ be a $\Gamma$-hypergroup. Then a subset $H$ of $S$ is called closed if for every $h, k \in H$, $x \in S$ and $\alpha \in \Gamma$ we have the following implication

$h \in x \alpha H \Rightarrow x \in H$.

**Example 13.** Let $S$ be a $\Gamma$-group and $P$ be a $\Gamma$-subgroup of $S$. Let $\Gamma' = \{\gamma' | \gamma \in \Gamma\}$. Now, for every $x, y \in S$ and $\alpha' \in \Gamma$ we define $x \alpha' y = x \alpha y \cup P$. Then, $S$ is a $\Gamma'$-hypergroup.

**Theorem 4.4.** [2] Let $S$ be a $\Gamma$-hypergroup, and $\alpha$, $\beta$ are non-empty subsets of $\Gamma$. Then $S$ is a $\Gamma$-hypergroup if and only if $S_\alpha \cap S_\beta$ is a right ideal of $S$.

**Proof:** Suppose that $S_\alpha \cap S_\beta$ is a right ideal of $S$. Then for every $x \in S$ and $\alpha \in \Gamma$, put $I = x \alpha S$. Thus, $I$ is a right ideal of $S$, for

$I \Gamma S = (x \alpha S) \Gamma S = x \alpha (S \Gamma S) \subseteq x \alpha S = I$

so $x \alpha S = S$. Similarly, we have $S = S \alpha x$. Therefore, $S$ is a $\Gamma$-hypergroup.

**Corollary 4.4.** If $S_\alpha$ is a hypergroup for some $\alpha \in \Gamma$, then for every $\alpha \in \Gamma$, $S_\alpha$ is a hypergroup.

**Definition 4.5.** A subset $H$ of a $\Gamma$-hypergroup is called a $\Gamma$-subhypergroup if for every $h, k \in H$ and $\alpha \in \Gamma$ we have $h \alpha k \subseteq H$ and $h \alpha H = H = H \alpha h$.

**Definition 4.6.** Let $S$ be a $\Gamma$-hypergroup. Then a subset $H$ of $S$ is called closed if for every $h, k \in H$, $x \in S$ and $\alpha \in \Gamma$ we have the following implication

$h \in x \alpha H \Rightarrow x \in H$.

**Example 14.** Consider $(\mathbb{Z}, +)$ and let $\Gamma = \{\alpha, \beta\}$ where $\alpha = \{-1,1\}$ and $\beta = \{-2,+2\}$. If for every $x, y \in \mathbb{Z}$ we define:

$x \alpha y = \{x + y - 1, x + y + 1\}, x \beta y = \{x + y - 2, x + y + 2\}$.

Then, $\mathbb{Z}$ is a $\Gamma$-hypergroup and $H = 2 \mathbb{Z}$ is a closed subset of $\mathbb{Z}$.

**Example 15.** Consider $(\mathbb{Z}, +)$ and let $\Gamma = \{\alpha, \beta\}$ where $\alpha = \{-2,2\}$ and $\beta = \{-4,4\}$. If for every $x, y \in \mathbb{Z}$ we define:

$x \alpha y = \{x + y - 2, x + y + 2\}, x \beta y = \{x + y - 4, x + y + 4\}$.

Then $\mathbb{Z}$ is a $\Gamma$-hypergroup and $H = 2 \mathbb{Z}$ is a closed $\Gamma$-subhypergroup of $\mathbb{Z}$.

Let $S$ be a $\Gamma$-hypergroup. Then two new hyperoperations may be defined on $S$ as follows:

$a / b = \{x \in S | a \in x \alpha b, \alpha \in \Gamma\}$ and $a \setminus b = \{x \in S | a \in b \alpha x, \alpha \in \Gamma\}$.

If $A$ and $B$ are non-empty subsets of $S$, then

$A / B = \bigcup_{a \in A, b \in B} a / b$ and $A \setminus B = \bigcup_{a \in A, b \in B} a \setminus b$.

**Lemma 4.7.** Let $S$ be a $\Gamma$-hypergroup, $A, B, C$ and $D$ be non-empty subsets of $S$ and $x, y \in S$. Then the following assertions hold:

1. If $A \subseteq B$ and $C \subseteq D$, then $A / C \subseteq B / D$;
2. $(A / B) / C = A / (B \setminus C)$;
3. $(A \setminus B) \setminus C = A \setminus (B \setminus C)$;
4. $y \in x \setminus (x / y)$;
(5) \( y \in x/(x \setminus y); \)

(6) If \( A \) is a closed subset of \( S \), then \( A / A \subseteq A \);

(7) \( A \subseteq (A \Gamma B)/B; \)

(8) If \( H \) is a \( \Gamma \)-subhypergroup, then \( H \subseteq H/H \).

**Proof:** (1) It is immediate.

(2) Suppose that \( x \in (A/B)/C \). Then, there exist \( a \in A, b \in B \) and \( c \in C \) such that \( x \in (a/b)/c \). So, we have

\[
x \in (a/b)/c \quad \Rightarrow \exists y \in a/b, x \in y/c
\]

\[
\Rightarrow a \in y \Gamma b, x \in x \Gamma c
\]

\[
\Rightarrow a \in (x \Gamma c) \Gamma b = x \Gamma (c \Gamma b)
\]

\[
\Rightarrow \exists z \in c \Gamma b, a \in x \Gamma z
\]

\[
\Rightarrow x \in a/z \subseteq a/(c \Gamma b) \subseteq A/(A \Gamma B).
\]

Thus, \((A/B)/C \subseteq A/(A \Gamma B)\).

Conversely, suppose that \( x \in A/(A \Gamma B) \). Then there exist \( a \in A, b \in B \) and \( c \in C \) such that \( x \in a/(c \Gamma b) \). So there exists \( y \in c \Gamma b \) such that \( x \in a/y \). So \( a \in x \Gamma y \subseteq x \Gamma (c \Gamma b) = (x \Gamma c) \Gamma b \). Thus there exists \( z \in x \Gamma c \) such that \( a \in z \Gamma b \) and so \( x \in z/c, z \in a/b \). Therefore, \( x \in (A/B)/C \).

(3) It is similar to (2).

(4) Let \( a \in x/y \neq \emptyset \). Then \( x \in a \Gamma y \), so \( y \in x \setminus a \subseteq x \setminus (x/y) \).

(5) it is similar to (4).

(6) If \( x \in A/A \), then \( x \in a_1/a_2 \). So \( a_1 \in x \Gamma a_2 \subseteq x \Gamma A \cap A \). Since \( A \) is a closed subset of \( S \), it follows that \( x \in A \). Therefore, \( A / A \subseteq A \).

(7) Suppose that \( x \in A \) and \( y \in x \Gamma B \). Then \( x \in y/B \subseteq (A \Gamma B)/B \).

(8) Suppose that \( H \) is a \( \Gamma \)-subhypergroup and \( h_1 \in H \). Then there exists \( h_2 \in H \) such that \( h_1 \in h_1 \Gamma h_2 \) thus \( h_1 h_1 / h_2 \subseteq H/H \), so \( H \subseteq H/H \).

**Theorem 4.8.** Let \( S \) be a \( \Gamma \)-hypergroup and \( H \) be a \( \Gamma \)-subhypergroup of \( S \). Then \( H \) is a closed \( \Gamma \)-subhypergroup if and only if \( H \subseteq H/H \).

**Proof:** Suppose that \( H \) is a closed \( \Gamma \)-subhypergroup. Then, by the previous lemma, \( H \subseteq H/H \subseteq H \). Thus \( H = H/H \).

Conversely, suppose that \( H/H = H \). If \( y \in x \alpha h \cap H \), for \( h \in H \), then \( x \in y/h \subseteq H/H = H \). Therefore, \( H \) is a closed \( \Gamma \)-subhypergroup of \( S \).

**Example 16.** Let \( G \) be a group with a non trivial center. Let \( P, Q \subseteq Z(G) \) and put \( \alpha, \beta \). For every \( x, y \in G \) we define \( x \alpha y = xyP \) and \( x \beta y = xyQ \). Then \( G \) is a \( \Gamma \)-hypergroup.

Let \( a, b \in G \). Then

\[
a/b = \{x \in G \mid a \in x \Gamma b\} = \{x \in G \mid a \in x \alpha b \cup x \beta b\} = \{x \in G \mid a \in xbP \cup xQb\} = ab^{-1}P^{-1} \cup ab^{-1}Q^{-1}.
\]

If \( H \) is a \( \Gamma \)-subhypergroup of \( G \) containing \( P \) and \( Q \), then for every \( a, b \in H \) we have \( a/b = ab^{-1}P^{-1} \cup ab^{-1}Q^{-1} \subseteq H \), so by the above theorem, \( H \) is a closed \( \Gamma \)-subhypergroup of \( G \).

**Lemma 4.9.** Let \( S \) be a \( \Gamma \)-semihypergroup and \( H \) and \( K \) be two closed \( \Gamma \)-subhypergroups of \( S \). Then \( H \cap K = H \cap K \).

**Proof:** Since \( H \cap K \subseteq H \cap K \), it follows that \( H \cap K \subseteq H \cap K \). Now, we prove the converse of inclusion. Since \( H \) and \( K \) are closed \( \Gamma \)-subhypergroups of \( S \), it follows that \( H \cap K \) is a closed subset of \( S \). Now, by the previous theorem and Lemma 4.7, we have

\[
H = H/H \subseteq (H \cap K)/H/H = (H \cap K)/(H \cap K) \subseteq H \cap K.
\]

Similarly, \( K \subseteq H \cap K \). Therefore, \( H \cap K = H \cap K \).

**5.** \( \Gamma \)-semihypergroups associated to binary relations

The connections between hyperstructures and binary relations have been analyzed by many
researchers, such as Rosenberg [13], Corsini [14], Cristea and Stefanescu [15] and others [16, 17, 18].

In this section we associate to a set of binary relations on a non-empty set $S$, say $\Gamma$, a partial $\Gamma$-hypergroupoid and get necessary and sufficient conditions such that it is a $\Gamma$-semihypergroup or a $\Gamma$-hypergroup.

Rosenberg [13] has associated a partial hypergroupoid $RH$, with a binary relation $R$ defined on a non-empty set $H$, where, for any $Hyx, xxx = L = \{ z H | (x, z) R \}$ and $x y = x x y y$.

An element $Hx$ is called an outer element for $R$ if there exists $Hh$ such that $.),( 2 Rxh .)$

Rosenberg proved the next theorem.

**Theorem 5.1.** [13] $RH$ is a hypergroup if and only if

1. $R$ has full domain;
2. $R$ has full range;
3. $R \subseteq R^2$;
4. If $(a, x) \in R^2$, then $(a, x) \in R$, whenever $x$ is an outer element.

Let $R$ be a binary relation on a non-empty set $S$. Then an element $x \in S$ is called a semiouter element for the relation $R$ if there exists $h \in S$ such that $(h, x) \not\in R$.

Let $R$ be a binary relation on a non-empty set $S$, $A \subseteq S$ and $x, y \in S$. Then we use the following notations:

$$L_*^R = R(x) = \{ z \in S | (x, z) \in R \};$$

$$R(x, y) = \{ z \in S | (x, z) \in R \lor (y, z) \in R \};$$

$$R(A) = \{ z \in S | (a, z) \in R, \exists a \in A \};$$

$$R^{-1}(A) = \{ z \in S | (z, a) \in R, \exists a \in A \}. $$

**Definition 5.2.** Let $S$ be a non-empty set and $\mathcal{R}$ be a set of binary relations on $S$. Then for every $\alpha \in \mathcal{R}$ we can associate a hyperoperation $\circ_{\alpha}$ on $S$ as follows:

$$x \circ_{\alpha} y = \alpha(x, y) = L^\alpha_x \cup L^\alpha_y, \forall x, y \in S.$$

So $(S, \circ_{\alpha})$ is a partial hypergroupoid. Now, let $\Gamma = \{ \circ_{\alpha} | \alpha \in \mathcal{R} \}$. Then $S$ is a partial $\Gamma$-hypergroupoid and is denoted by $S_\Gamma$.

To simplify, we write $\circ_{\alpha}$ by $\alpha$ and consider $\Gamma = \mathcal{R}$, in this way for every $\alpha \in \Gamma$ and $x, y \in S$ we have $x \alpha y = x \circ_{\alpha} y = \alpha(x, y) = L^\alpha_x \cup L^\alpha_y$.

It is easy to see that if for every $\alpha \in \Gamma$ we have $\alpha^{-1}(S) = S$, then $S_\Gamma$ is a $\Gamma$-hypergroupoid.

**Example 17.** Let $S = \{1,2,3,4,5\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ such that

$$\alpha = \{(1,1),(1,2),(2,4),(3,4),(4,5),(4,4),(5,2)\},$$

$$\beta = \{(1,1),(1,3),(1,4),(2,5),(3,3),(4,1),(5,4),(5,3)\},$$

$$\gamma = \{(1,3),(2,3),(3,4),(4,5),(5,1),(5,5)\}.$$

Then $S_\Gamma$ is a $\Gamma$-hypergroupoid.

**Lemma 5.3.** Let $S$ be a non-empty set and $\Gamma$ be a set of binary relations on $S$ such that $S_\Gamma$ is a $\Gamma$-hypergroupoid. Then the following assertions hold:

1. $S_\Gamma$ is a commutative $\Gamma$-hypergroupoid;
2. For every $x \in S$ and $\alpha \in \Gamma$, $x \alpha x = \alpha(x)$;
3. For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, $x \alpha(y \beta z) = \alpha(x) \cup \beta \alpha(y, z)$;
4. For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, $(x \alpha y) \beta z = \alpha \beta(x, y) \cup \beta(z)$.

**Proof:** The proof is straightforward.

In the following we provide some conditions on $\Gamma$ such that $S_\Gamma$ be a $\Gamma$-semihypergroup.

**Theorem 5.4.** Let $S$ be a non-empty set and $\Gamma$ be a set of binary relations on $S$ such that $S_\Gamma$ be a $\Gamma$-hypergroupoid. Then $S_\Gamma$ is a $\Gamma$-semihypergroup if and only if the following conditions hold:

1. (GSH1) For every $\alpha, \beta \in \Gamma$, $\alpha \subseteq \alpha \beta$;
2. (GSH2) If $x$ is a semiouter element for the relation $\alpha \beta$ and $(a, x) \in \beta \alpha$, then $(a, x) \in \beta$ for every $a \in S$ and $\alpha, \beta \in \Gamma$;
If \( x \) is a semiouter element for the relations \( \alpha \beta \) and \( \beta \) and \( (a,x) \in \beta \alpha \), then \( (a,x) \in \alpha \beta \), for every \( a \in S \) and \( \alpha, \beta \in \Gamma \).

**Proof:** Suppose that \( S_\Gamma \) is a \( \Gamma \)-semihypergroup. We prove the conditions \((\Gamma \text{ SH}1), (\Gamma \text{ SH}2)\) and \((\Gamma \text{ SH}3)\) of the theorem.

\( (\Gamma \text{ SH}1) \) Let \( x, y \in S \) and \( \alpha, \beta \in \Gamma \) such that \( y \in \alpha (x) \). Then we consider two cases:

Case (i) \( y \in \beta (y) \). Then \( y \in \alpha \beta (x) \).

Case (ii) \( y \notin \beta (y) \). Then we have \((x \alpha \alpha) \beta y = x \alpha (x \beta y)\) so the associativity axiom and the previous lemma conclude that \( \alpha \beta (x) \cup \beta (y) = \alpha (x) \cup \beta \alpha (x) \cup \beta \alpha (y) \).

Now, since \( y \in \alpha (x) \) and \( y \notin \beta (y) \), it follows that \( y \notin \alpha \beta (x) \). Therefore, \( \alpha \subseteq \alpha \beta \).

\( (\Gamma \text{ SH}2) \) Suppose that \( x \) is a semiouter element for the relation \( \alpha \beta \) and \( x \in \beta \alpha (a) \). So there exists \( h \in S \) such that \( x \notin \alpha \beta (h) \). Thus the associativity axiom and the previous lemma conclude that \( (h \alpha h) \beta h = h \alpha (h \beta a) \), thus \( \alpha \beta (h) \cup \beta (a) = \alpha (h) \cup \beta \alpha (h) \cup \beta \alpha (a) \).

Since \( x \in \beta \alpha (a) \) and \( x \notin \alpha \beta (h) \), it follows that \( x \notin \beta (a) \).

\( (\Gamma \text{ SH}3) \) Suppose that \( x \) is a semiouter element for the relations \( \alpha \beta \) and \( \beta \) and let \( x \in \beta \alpha (a) \). So there exist \( h, t \in S \) such that \( (h, x) \notin \alpha \beta \) and \( (t, x) \notin \beta \). Now, we have \( h \alpha (\alpha \beta) = (h \alpha a) \beta t \) thus \( \alpha (h) \cup \beta \alpha (a, t) = \alpha \beta (a, h) \cup \beta \alpha (t) \).

Since \( x \in \beta \alpha (a) \), \( x \notin \alpha \beta (h) \) and \( x \notin \beta (t) \), it follows that \( x \in \alpha \beta (a) \).

Conversely, suppose that \( S \) is a non-empty set and \( \Gamma \) be a set of binary relations on \( S \) such that \( S_\Gamma \) is a \( \Gamma \)-hypergroupoid and the conditions \((\Gamma \text{ SH}1), (\Gamma \text{ SH}2)\) and \((\Gamma \text{ SH}3)\) of the theorem are satisfied. We prove the associativity axiom for \( S_\Gamma \).

Let \( x, y, z, t \in S \) and \( \alpha, \beta \in \Gamma \) such that \( t \in x \alpha (y \beta z) = \alpha (x) \cup \beta \alpha (y, z) \). Then we have three cases:

Case (i) \( t \in \alpha (x) \). Then by the condition \((\Gamma \text{ SH}1) \) \( t \in \alpha \beta (x) \).

Case (ii) \( t \in \beta \alpha (x) \). Then if \( t \notin \alpha \beta (x) \cup \beta (z) \), then \( t \) is a semiouter element for the relations \( \alpha \beta \) and \( \beta \). So by the condition \((\Gamma \text{ SH}3) \) \( t \in \alpha \beta (y) \).

Case (iii) \( t \in \beta \alpha (z) \). Then if \( t \notin \alpha \beta (x) \), then \( t \) is a semiouter element for the relation \( \alpha \beta \) so by the condition \((\Gamma \text{ SH}2) \), \( t \in \beta (z) \). Thus \( x \alpha (y \beta z) \subseteq (x \alpha y) \beta z \). In the same way, we can prove the converse inclusion. Therefore, \( S_\Gamma \) is a \( \Gamma \)-semihypergroup.

**Example 18.** Let \( S = \{1,2,3\} \) and \( \Gamma = \{\alpha, \beta\} \) such that \( \alpha = \{ (1,2),(2,2),(2,3)\}, (3,2) \} \) and \( \beta = \{ (1,3),(2,2),(3,2)\}, (3,3) \} \). Then we have the table of hyperoperations \( \alpha \) and \( \beta \) as follows:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{2}</td>
<td>{2,3}</td>
<td>{2,3}</td>
</tr>
<tr>
<td>2</td>
<td>{2,3}</td>
<td>{2,3}</td>
<td>{2,3}</td>
</tr>
<tr>
<td>3</td>
<td>{2,3}</td>
<td>{2,3}</td>
<td>{3}</td>
</tr>
</tbody>
</table>

Then \( S_\Gamma \) is a \( \Gamma \)-semihypergroup.

**Theorem 5.5.** Let \( S \) be a non-empty set and \( \Gamma \) be a set of binary relations on \( S \) such that \( S_\Gamma \) is a \( \Gamma \)-semihypergroup. Then \( S_\Gamma \) is a \( \Gamma \)-hypergroup if and only if \( \alpha (S) = S \) for every \( \alpha \in \Gamma \).

**Proof:** Suppose that \( S_\Gamma \) is a \( \Gamma \)-hypergroup. Then \( S_\alpha \) is a hypergroup for every \( \alpha \in \Gamma \). So by Theorem 5.1, \( \alpha \) has full range, thus \( \alpha (S) = S \).

Conversely, suppose that \( \alpha (S) = S \) for every \( \alpha \in \Gamma \) so \( S_\alpha \) is a hypergroup. Therefore, \( S_\Gamma \) is a \( \Gamma \)-hypergroup.

**Example 19.** Let \( S = \{1,2,3\} \) and \( \Gamma = \{\alpha, \beta\} \) such that \( \alpha = \Delta_S \cup \{ (2,1),(3,2) \} \) and \( \beta = \Delta_S \cup \{ (3,1) \} \), where \( \Delta_S \) is the diagonal
relation on $S$. Then we have the table of hyperoperations $\alpha$ and $\beta$ as follows:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${1,2}$</td>
<td>$S$</td>
</tr>
<tr>
<td>2</td>
<td>${1,2}$</td>
<td>${1,2}$</td>
<td>$S$</td>
</tr>
<tr>
<td>3</td>
<td>$S$</td>
<td>$S$</td>
<td>${2,3}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${1,2}$</td>
<td>${1,3}$</td>
</tr>
<tr>
<td>2</td>
<td>${1,2}$</td>
<td>${2}$</td>
<td>$S$</td>
</tr>
<tr>
<td>3</td>
<td>$S$</td>
<td>${1,3}$</td>
<td>$S$</td>
</tr>
</tbody>
</table>

Then $S$ is a $\Gamma$-hypergroup.

**Lemma 5.6.** Let $S$ be a non-empty set and $\Gamma$ be a set of binary relations on $S$ such that $S_\Gamma$ is a $\Gamma$-semihypergroup. Then $I = \Gamma(S) = \bigcup_{\alpha \in \Gamma} \alpha(S)$ is a minimal ideal of $S_\Gamma$.

**Proof:** Suppose that $a \in I$, $s \in S$ and $\alpha \in \Gamma$. Then we have $s \alpha a = \alpha(s) \cup \alpha(a) \subseteq \alpha(S) \subseteq I$. So $I$ is an ideal of $S_\Gamma$. Furthermore, if $J$ is an ideal of $S_\Gamma$ and $b \in J$, then for every $s \in S$ and $\alpha \in \Gamma$, $s \alpha b = \alpha(s) \cup \alpha(b) \subseteq J$. So $\alpha(S) \subseteq J$ hence $I \subseteq J$.

**Proposition 5.7.** Let $S$ be a non-empty set and $\Gamma$ be a set of binary relations on $S$ such that $S_\Gamma$ is a $\Gamma$-semihypergroup. Then $S_{\overline{\oplus}}$ is a $\Gamma_{\overline{\oplus}}$-semihypergroup.

**Proof:** We prove that $S_{\overline{\oplus}}$ satisfies the conditions (SH1), (SH2) and (SH3) of Theorem 5.4. Suppose that $\alpha', \beta' \in \Gamma_{\overline{\oplus}}$. Then there exist $\alpha, \beta, \delta, \gamma \in \Gamma$, such that $\alpha' = \alpha \cup \beta$ and $\beta' = \delta \cup \gamma$. Since $S_\Gamma$ is a $\Gamma$-semihypergroup, it follows that $\alpha \subseteq \alpha \delta \cup \alpha \gamma$ and $\beta \subseteq \beta \delta \cup \beta \gamma$. Thus $\alpha' = \alpha \cup \beta \subseteq \alpha \delta \cup \alpha \gamma \cup \beta \delta \cup \beta \gamma = \alpha \cup \beta \delta \cup \beta \gamma = \theta' \phi'$.

So the condition (SH1) holds.

Suppose that $x \in S$ is a semiouter element for the relation $\theta'$ and let $(a, x) \in \phi' \theta'$. Then there exists $h \in S$ such that $(h, x) \notin \theta' \phi'$. Thus $x$ is a semiouter element for the relations $\alpha \delta, \alpha \gamma, \beta \delta$ and $\beta \gamma$. Since $(a, x) \in \phi' \theta'$, it follows that $(a, x) \in \delta \alpha, (a, x) \in \gamma \alpha, (a, x) \in \delta \beta$ or $(a, x) \in \gamma \beta$. From the condition (SH2) for $S_\Gamma$, we conclude that $(a, x) \in \delta \alpha, (a, x) \in \gamma \alpha, (a, x) \in \delta \beta$ or $(a, x) \in \gamma \beta$, then from the condition (SH3) for $S_\Gamma$, we conclude that $(a, x) \in \alpha \delta, (a, x) \in \alpha \gamma, (a, x) \in \delta \beta$ or $(a, x) \in \gamma \beta$, respectively, and the condition (SH3) holds. Therefore, $S_{\overline{\oplus}}$ is a $\Gamma_{\overline{\oplus}}$-semihypergroup.

Let $S_R$ be a hypergroupoid associated to a binary relation $R$. Let $\Gamma_R = \{\alpha_i \mid i \in \mathbb{N}\}$. Now, for every $x, y \in S$ and $\alpha_i \in \Gamma$ we define $x \alpha_i y = \{z \mid (x, z) \in R^i \lor (y, z) \in R^i\} = L_{\alpha_i}^R \cup L_y^{\alpha_i}$.

Then $S$ is a $\Gamma_R$-hypergroupoid and denoted by $S_{\overline{\oplus}}$. In the following we verify conditions such that $S$ is a $\Gamma_R$-semihypergroup.

**Lemma 5.8.** Let $S_R$ be a semihypergroup associated to a binary relation $R$. Then if $(z, t) \in R^{i+j}$ and $(x, t) \notin R^{i+j}$, then $(z, t) \in R^j$, for every $x, z, t \in S$ and $i, j \in \mathbb{N}$.

**Proof:** We prove by mathematical induction on $i + j$. If $i + j = 2$, $(z, t) \in R^2$ and $(x, t) \notin R^2$, then $t$ is an outer element for $R$ so $(z, t) \in R$. 


Suppose that the result holds for \(i + j - 1\). Now, let \((z, t) \in R^{i+j}\) and \((x, t) \notin R^{i+j}\). Then there exists \(s \in S\) such that \((z, s) \in R^2\) and \((s, t) \in R^{i+j-1}\). Thus \((x, s) \notin R^2\), that is, \(s\) is an outer element for \(R\). Therefore, \((z, t) \in R^{i+j}\). Now, we have \((z, t) \in R^{i+j-1}\) and \((x, t) \notin R^{i+j-1}\). Thus \((z, t) \in R^j\).

**Lemma 5.9.** Let \(S_R\) be a semihypergroup associated to a binary relation \(R\). Then \(S_{T_R}\) is a \(\Gamma_R\)-semihypergroup.

**Proof:** We prove the associativity law. Suppose that \(x, y, z \in S_r\) and \(\alpha_i, \alpha_j \in \Gamma\). Then,

\[
x\alpha_i(y\alpha_jz) = L^{i+j}_x \cup L^{i+j}_y \cup L^{i+j}_z
\]

and \((x\alpha_iy)\alpha_jz = L^{i+j}_x \cup L^{i+j}_y \cup L^{i+j}_z\).

If \(t \in L^{i+j}_x\) and \(t \notin L^{i+j}_x\), then by the previous lemma \(t \in L^{i+j}_x \subseteq (x\alpha_iy)\alpha_jz\). Therefore,

\[
x\alpha_i(y\alpha_jz) \subseteq (x\alpha_iy)\alpha_jz.
\]

In a similar way we have the inverse inclusion.

**Example 20.** Let \(S = \{1, 2, 3\}\) and \(R = \{(1, 2), (1, 3), (2, 2), (3, 2)\}\). Then \(S_R\) is a semihypergroup. Let \(\Gamma_R = \{\alpha_1, \alpha_2\}\). Then we have the following hyperoperations:

<table>
<thead>
<tr>
<th>(\alpha_1)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{1, 3}</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>{2}</td>
<td>{2, 3}</td>
</tr>
<tr>
<td>3</td>
<td>S</td>
<td>{2, 3}</td>
<td>{2}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\alpha_2)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>{2}</td>
<td>{2, 3}</td>
</tr>
<tr>
<td>3</td>
<td>S</td>
<td>{2, 3}</td>
<td>{2}</td>
</tr>
</tbody>
</table>

Then \(S_{T_R}\) is a \(\Gamma_R\)-semihypergroup.

6. Conclusion

In this work, we presented the concept of semiprime ideals in a \(\Gamma\)-semihypergroup and proved some results. Also, we introduced the notion of \(\Gamma\)-hybergroups and closed \(\Gamma\)-subhybergroups. Finally, we defined the concept of \(\Gamma\)-semihypergroups and \(\Gamma\)-hybergroups associated to a set of binary relations. Then we find the necessary and sufficient conditions on a set of binary relations \(\Gamma\) on a non-empty set \(S\) such that \(S\) becomes a \(\Gamma\)-semihypergroup or a \(\Gamma\)-hypergroup.

Our future research will consider \(\Gamma\)-semihyperrings associated to binary relations.

**References**


