IJST (2011) A2: 69-80
Iranian Journal of Science & Technology

http://www.shirazu.ac.ir/en

Γ-hypergroups and Γ-semihypergroups associated to binary relations

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Abstract

The concept of Γ-semihypergroups is a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups. In this paper, we study the concept of semiprime ideals in a Γ-semihypergroup and prove some results. Also, we introduce the notion of Γ-hypergroups and closed Γ-subhypergroups. Finally, we study the concept of Γ-semihypergroups associated to binary relations and give necessary and sufficient conditions on a set of binary relations Γ on a non-empty set S such that S becomes a Γ-semihypergroup or a Γ-hypergroup.

Keywords: Hypergroup; semihypergroup; Γ-semigroup; Γ-semihypergroup; binary relation

1. Introduction

The *hyperstructure* theory was born in 1934, when Marty introduced the notion of a *hypergroup* [1]. Since then, hundreds of papers and several books have been written on this topic, see [2-5]. A recent book on hyperstructures [6] points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a non-empty set. More exactly, let H be a non-empty set. Then the map $\circ : H\times H \to P^*(H)$ is called a *hyperoperation* where $P^*(H)$ is the family of non-empty subsets of H. The couple $(H,\circ)$ is called a *hypergroupoid*.

In the above definition, if A and B are two non-empty subsets of H and $x\in H$, then we define

$$A\circ B = \bigcup_{a\in A, b\in B} a\circ b;\;\; x\circ A = \{x\}\circ A \;\; and \;\; A\circ x = A\circ \{x\}.$$

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Received: 12 October 2010 / Accepted: 20 February 2011

A hypergroupoid $(H,\circ)$ is called a *semihypergroup* if for every $x,y,z\in H$, we have $x\circ(y\circ z)=(x\circ y)\circ z$, and is called a *quasihypergroup* if for every $x\in H$, $x\circ H = H = H\circ x$. This condition is called the *reproduction axiom*. The couple $(H,\circ)$ is called a hypergroup if it is a semihypergroup and a quasihypergroup.

The notion of Γ-*semigroups* was introduced by Sen in [7, 8]. Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S\times\Gamma\times S \to S$, written (a,γ,b) by $a\gamma b$, such that it satisfies the identities $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c\in S$ and $\alpha,\beta\in\Gamma$. Let S be an arbitrary semigroup and Γ a non-empty set. Define a mapping $S\times\Gamma\times S\to S$ by $a\alpha b = ab$ for all $a,b\in S$ and $\alpha\in\Gamma$. It is easy to see that S is a Γ-semigroup. Thus a semigroup can be considered to be a Γ-semigroup. Many classical notions of semigroups have been extended to Γ-semigroups, see ([9, 10]).

Let S be a Γ-semigroup and α be a fixed element in Γ. We define $a\cdot b = a\alpha b$ for all $a,b\in S$. Then $(S,\cdot)$ is a semigroup and is denoted by S_α.

2. Preliminaries and basic definitions

The concept of Γ*-semihypergroups* was introduced by Davvaz et al. [11, 12]. In this section we introduce some preliminaries and basic definitions of Γ-semihypergroups and give some examples.

Definition 2.1. Let S and Γ be two non-empty sets. Then S is called a Γ-semihypergroup if each $\gamma \in \Gamma$ be a hyperoperation on S, i.e., $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property that is $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Let A and B be two non-empty subsets of S and $\gamma \in \Gamma$. Then we define:

$$A\gamma B = \cup\{a\gamma b \mid a \in A, b \in B\},$$

and

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B = \cup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

A Γ-semihypergroup S is called *commutative* if for every $x, y \in S$ and for every $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$. A non-empty subset A of S is called a Γ*-subsemihypergroup* of S if $A\Gamma A \subseteq A$.

Let $(S,\circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then S is a Γ-semihypergroup. So every semihypergroup is a Γ-semihypergroup.

Let S be a Γ-semihypergroup and $\alpha \in \Gamma$, if we define $a \circ b = a\alpha b$ for every $a, b \in S$ then $(S,\circ)$ becomes a semihypergroup, we denote it by S_α.

Now, we give some other examples of Γ-semihypergroups.

Example 1. Let G be a group and $\Gamma = \{\alpha, \beta\}$. Then for every $x, y \in G$, we define $x\alpha y = xy$ and $x\beta y = G$. Then G is a Γ-semihypergroup.

Example 2. Let $(S,\leq)$ be a totally ordered set and Γ be a non-empty subset of S. We define

$$x\gamma y = \{z \in S \mid z \geq \max\{x, \gamma, y\}\},$$

for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semihypergroup.

Example 3. Let S be a Γ-semigroup and P be a non-empty subset of S. Let $\Gamma_P = \{\alpha_P : \alpha \in \Gamma\}$. If we define $x\alpha_P y = x\alpha P\alpha y$, for every $x, y \in S$ and $\alpha \in \Gamma$, then S is a Γ_P-semihypergroup.

Let S be a Γ-semihypergroup. We define a relation ρ on $S \times \Gamma$ as follows:

$$(x,\alpha)\rho(y,\beta) \Leftrightarrow x\alpha s = y\beta s, \forall s \in S.$$

Obviously ρ is an equivalence relation. Let $[x,\alpha]$ denote the equivalence class containing (x,α). Let $M = \{[x,\alpha] : x \in S, \alpha \in \Gamma\}$. We define the hyperoperation $\circ$ on M as follows:

$[x,\alpha] \circ [y,\beta] = \{[z,\beta] : z \in x\alpha y\}$,
for all $[x,\alpha],[y,\beta] \in M$.

Since $(x\alpha y)\beta z = x\alpha(y\beta z)$, for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, then

$[x,\alpha] \circ ([y,\beta] \circ [z,\gamma]) = ([x,\alpha] \circ [y,\beta]) \circ [z,\gamma]$,
for all $[x,\alpha],[y,\beta],[z,\gamma] \in M$.

Thus the hyperoperation $\circ$ is associative, so $(M,\circ)$ is a semihypergroup. This semihypergroup is called the left operator semihypergroup of S.

Let S be a Γ-semihypergroup. If there exist elements $e \in S$ and $\delta \in \Gamma$ such that $e\delta x = x$ for every $x \in S$, then S is said to have a left partial unity which is denoted by e_δ. It is easy to check whether e_δ is a left partial unity of S, then $[e,\delta]$ is a left unity of the left operator semihypergroup M.

Example 4. Consider Example 1 and let e be the identity element of G. Then $e_\alpha = e$ is a left partial unity of the Γ-semihypergroup G.

The concept of Γ*-hyperideals* of a Γ-semihypergroup was defined and studied in [12].

Definition 2.2. A non-empty subset I of a Γ-semihypergroup S is called a left (right) Γ-hyperideal, "ideal, for short" of S, if $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). S is called a left (right) simple Γ-semihypergroup if it has no proper left (right) ideal. S is *simple* if S has no proper left and right ideals.

Let A be a non-empty subset of a Γ-semihypergroup S. Then the intersection of all ideals of S containing A is an ideal of S generated by A, and denoted by $< A >$.

Example 5. Consider Example 4. Put $S = N$ with natural order. Then the subset $I_n = \{n, n+1, n+2, \cdots\}$ is an ideal of S, for every $n \in \mathbb{N}$

The following lemmas and theorem were proved in [12].

Lemma 2.3. Let S be a Γ-semihypergroup. If A is a non-empty subset of S, then

$$< A > = A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S.$$

One can see that, if S is a commutative Γ-semihypergroup and $\phi \neq A \subseteq S$, then $< A > = A \cup A\Gamma S$. If S is a commutative Γ-semihypergroup with left partial unity, then $< A > = A\Gamma S$.

Lemma 2.4. Let S be a Γ-semihypergroup and Λ be a non-empty set such that for every $\lambda \in \Lambda$, I_λ is an ideal of S. Then the following assertions hold:

(1) $\bigcup_{\lambda\in\Lambda} I_\lambda$ *is an ideal of* S;

(2) $\bigcap_{\lambda\in\Lambda} I_\lambda$ *is an ideal of* S.

Definition 2.5. A proper ideal P of Γ-semihypergroup S is called a *prime* ideal, if for every ideal I and J of S, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If a Γ-semihypergroup S is commutative, then a proper ideal P is prime if and only if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a, b \in S$.

Example 6. Consider Example 2. Put $S = \Gamma = \{1, 2, \cdots, n\}$ for some natural number n. Then, all ideals of S have the form $I_i = \{i, i+1, \cdots, n\}$, for every $i \in S$ and I_2 is a prime ideal of S.

Theorem 2.6. Let S be a Γ-semihypergroup and P be a left ideal of S. Then P is a prime ideal of S if and only if for all $x, y \in S$,

$$x\Gamma S\Gamma y \subseteq P \text{ implies that } x \in P \text{ or } y \in P.$$

Lemma 2.7. Let S be a commutative Γ-semihypergroup with a left partial unity and M be a maximal ideal of S. Then M is a prime ideal of S.

Proof: Suppose that M is a maximal ideal and e_δ is the left partial unity of S. Let $x, y \in S$ such that $x\Gamma y \subseteq M$. Then we prove that $x \in M$ or $y \in M$. If $x \notin M$, then $M \subset < M, x >$, so by maximality of M we have $S = < M, x >$. Since $e_\delta \notin M$, it follows that there exist $s \in S$ and $\gamma \in \Gamma$ such that $e_\delta \in x\gamma s$. Then, we have

$$y = e_\delta \delta y \in (x\gamma s)\delta y \subseteq x\Gamma y\delta s \subseteq M.$$

Similarly, if $y \notin M$, then one proves that $x \notin M$. Therefore, M is a prime ideal of S.

Proposition 2.8. Let S be a Γ-semihypergroup with a left partial unity and I be a proper ideal of S. Then there exists a maximal ideal of S containing I.

Proof: By Lemma 2.4 and Zorn's lemma the proof is obvious.

Let S be a Γ-semihypergroup and M be the left operator semihypergroup of S. Then for $A \subseteq M$, Davvaz et al. in [12] defined A^+ as follows:

$$A^+ = \{x \in S : [x, \alpha] \in A \ \text{ for all } \ \alpha \in \Gamma\}.$$

Similarly, for $I \subseteq S$, they defined $I^{+'}$ as follows:

$$I^{+'} = \{[x, \alpha] \in M : x\alpha s \subseteq I \ \text{ for all } \ s \in S\}.$$

If I is an ideal of S and A is a hyperideal of M, then $I \subseteq (I^{+'})^+$ and $A \subseteq (A^+)^{+'}$.

We recall the following theorems from [12].

Theorem 2.9. [12] Let S be a Γ-semihypergroup and M be its left operator semihypergroup. Then the following assertions hold:

(1) If A is a right hyperideal of M, then A^+ is a right ideal of S.

(2) If I is a right ideal of S then, $I^{+'}$ is a right hyperideal of M.

Theorem 2.10. [12] Let S be a Γ-semihypergroup with a left partial unity and M be its left operator semihypergroup. If I is a right ideal of S, then $I = (I^{+'})^{+}$.

3. Semiprime ideals of Γ-semihypergroups

In this section, we introduce the concept of semiprime ideals of a Γ-semihypergroup and prove some results.

Definition 3.1. Let S be a Γ-semihypergroup. Then a proper left (right) ideal P of S is said to be a left (right) semiprime ideal, if for any left (right) ideal A of S, $A\Gamma A \subseteq P$ implies that $A \subseteq P$. A proper ideal P is called semiprime ideal if P is both left and right semiprime ideal of S.

Example 7. Let $S = \Gamma = \{1,2,3,\cdots,n\}$ for some $n \in \mathbb{N}$. For every $x, y \in S$ and $\alpha \in \Gamma$ we define the following hyperoperation on S

$$x\alpha y = \{s \in S \mid s \geq \max\{x,\alpha,y\}\}.$$

Then S is a Γ-semihypergroup and $I_i = \{i, i+1, \cdots, n\}$ is a semiprime ideal of S for $1 < i \leq n$.

Lemma. 3.2 Let S be a Γ-semihypergroup with a left partial unity and P be a left ideal of S. Then P is a left semiprime ideal of S if and only if for every $x, y \in S$ we have

$$x\Gamma S\Gamma x \subseteq P \Rightarrow x \in P.$$

Proof: Suppose that P is a left semiprime ideal of S and $x\Gamma S\Gamma x \subseteq P$ for $x \in S$. Then $S\Gamma x\Gamma S\Gamma x \subseteq S\Gamma P \subseteq P$. Since P is a left semiprime ideal and $S\Gamma x$ is a left ideal of S, it follows that $x \in S\Gamma x \subseteq P$.

Conversely, let A be an ideal of S such that $A\Gamma A \subseteq P$. If $a \in A$, then $a\Gamma S\Gamma a \subseteq A\Gamma A \subseteq P$. So, by the above implication $a \in P$ thus $A \subseteq P$.

Lemma 3.3. Let S be a Γ-semihypergroup and M be its left operator semihypergroup. Then the following statements hold:

(1) If P is a semiprime ideal of M, then P^{+} is a semiprime ideal of S.

(2) If S has a left partial unity and Q is a semiprime ideal of S, then $Q^{+'}$ is a semiprime ideal of M.

Proof: (1) Suppose that P is a semiprime ideal of M and A is an ideal of S such that $A\Gamma A \subseteq P^{+}$. Then $[A\Gamma A, \Gamma] \subseteq P$ so $[A,\Gamma] \circ [A,\Gamma] \subseteq P$. Since $[A,\Gamma]$ is an ideal of M and P is a semiprime ideal of M, it follows that $[A,\Gamma] \subseteq P$ hence $A \subseteq P^{+}$. Thus P^{+} is a semiprime ideal of S.

(2) Suppose that Q is a semiprime ideal of S and A is an ideal of M such that $A \circ A \subseteq Q^{+'}$. First, we show that $A^{+}\Gamma A^{+} \subseteq (A \circ A)^{+}$. Let $t \in A^{+}\Gamma A^{+}$. Then there exist $x, y \in A^{+}$ and $\gamma \in \Gamma$ such that $t \in x\gamma y$. So $[t,\alpha] \in [x,\gamma] \circ [y,\alpha] \subseteq A \circ A$ for every $\alpha \in \Gamma$. Thus $t \in (A \circ A)^{+}$, so $A^{+}\Gamma A^{+} \subseteq (A \circ A)^{+}$. Now, from $A \circ A \subseteq Q^{+'}$ and Theorem 2.10 we have

$$A^{+}\Gamma A^{+} \subseteq (A \circ A)^{+} \subseteq (Q^{+'})^{+} = Q.$$

Since Q is a semiprime ideal and A^{+} is an ideal of S, it follows that $A^{+} \subseteq Q$. Thus $A \subseteq (A^{+})^{+'} \subseteq Q^{+'}$. Therefore, $Q^{+'}$ is a semiprime ideal of M.

Lemma 3.4. Let P_i be a prime ideal of a Γ-semihypergroup S for every $i \in I$ and let $P = \bigcap_{i \in I} P_i$. Then if $P \neq \varnothing$, then P is a semiprime ideal of S.

Proof: It is immediate.

Lemma 3.5. Let T be a Γ-subsemihypergroup and I be an ideal of the Γ-semihypergroup S such that $I \cap T = \varnothing$. Then T is contained in a Γ-subsemihypergroup that is maximal with respect to the property of not meeting I.

Proof: Since the set $\mathcal{A}=\{K \mid T \leq K \leq S \text{ and } K \cap I=\varnothing\}$ is non-empty, it follows that by Zorn's lemma, $\mathcal{A}$ has a maximal element that satisfies the theorem.

Lemma 3.6. Let T be a commutative Γ-subsemihypergroup and I be an ideal of the Γ-semihypergroup S such that $I \cap T=\varnothing$. Then there exists a prime ideal of S, say P, such that $I \subseteq P$ and $P \cap T=\varnothing$.

Proof: By Zorn's lemma, there exists an ideal P such that P is maximal with respect to properties of $I \subseteq P$ and $P \cap T=\varnothing$. We claim that P is a prime ideal of S. Suppose that $x, y \in S \setminus P$. Then, we show that $x \Gamma S \Gamma y \not\subset P$. Since $x, y \notin P$ and P is maximal, it follows that $<P, x> \cap T \neq \varnothing$ and $<P, y> \cap T \neq \varnothing$. Thus, there exist $s, t \in S$ such that $s \in <P, x> \cap T$ and $t \in <P, y> \cap T$. From the property $P \cap T=\varnothing$, we have only four cases: (i) $s \in s_1 \alpha x$ and $t \in t_1 \beta y$ for some $s_1, t_1 \in S$ and $\alpha, \beta \in \Gamma$, (ii) $s \in s_1 \alpha x$ and $t=y$ for some $s_1 \in S$ and $\alpha \in \Gamma$, (iii) $s=x$ and $t \in t_2 \beta y$ for some $t_2 \in S$ and $\beta \in \Gamma$ and (iv) $s=x$ and $t=y$. If (i) holds, then $s \Gamma t \subseteq (s_1 \alpha x) \Gamma (t_1 \beta y) \subseteq x \Gamma S \Gamma y$.

Now, since T is a Γ-subsemihypergroup, it follows that $s \Gamma t \subseteq T$. Thus $x \Gamma S \Gamma y \not\subset P$. Similarly, in the other cases we conclude that $x \Gamma S \Gamma y \not\subset P$. Therefore, P is a prime ideal of S.

Let S be a Γ-semihypergroup and I be an ideal of S. A prime ideal P of S is called a minimal prime ideal belonging to I, if $I \subseteq P$ and there is no prime ideal containing I and properly contained in P.

Corollary 3.7. If Q is a prime ideal containing an ideal I, then there exists a minimal prime ideal belonging to I which is contained in Q.

Definition 3.8. Let S be a Γ-semihypergroup and I be an ideal of S. Then the prime radical of I is defined as the intersection of all prime ideals of S containing I and is denoted by $\sqrt{I}$.

Proposition 3.9. Let S be a Γ-semihypergroup and I be an ideal of S. Then the following statements hold:

(1) $\sqrt{I}$ is a semiprime ideal of S;

(2) $\sqrt{I}=\bigcap\{P \mid P \text{ is a minimal prime ideal belonging to } I\}$.

Proof: (1) It is straightforward.

(2) It is taken from Corollary 3.7.

4. Γ-hypergroups

In this section we study the concept of Γ-hypergroups and give some examples. Also, we introduce the concept of closed Γ-subhypergroups of a Γ-hypergroup.

Definition 4.1. A Γ-semihypergroup S is called a Γ-hypergroup if (S_α, α) is a hypergroup for every $\alpha \in \Gamma$.

Example 8. Let $S=\{a, b, c, d\}$ and $\Gamma=\{\alpha, \beta\}$. We define the hyperoperations α and β as follows:

α	a	b	c	d
a	$\{a,b\}$	$\{b,c\}$	$\{c,d\}$	$\{a,d\}$
b	$\{b,c\}$	$\{c,d\}$	$\{a,d\}$	$\{a,b\}$
c	$\{c,d\}$	$\{a,d\}$	$\{a,b\}$	$\{b,c\}$
d	$\{a,d\}$	$\{a,b\}$	$\{b,c\}$	$\{c,d\}$

β	a	b	c	d
a	$\{b,c\}$	$\{c,d\}$	$\{a,d\}$	$\{a,b\}$
b	$\{c,d\}$	$\{a,d\}$	$\{a,b\}$	$\{b,c\}$
c	$\{a,d\}$	$\{a,b\}$	$\{b,c\}$	$\{c,d\}$
d	$\{a,b\}$	$\{b,c\}$	$\{c,d\}$	$\{a,d\}$

Then S is a Γ-hypergroup.

Example 9. Let S be a non-empty set and $\Gamma=\{\alpha, \beta\}$. Then for every $x, y \in S$ and $\alpha, \beta \in \Gamma$ we define $x \alpha y=S$ and $x \beta y=\{x, y\}$. Then S is a Γ-hypergroup.

Example 10. Let $(S, \cdot)$ be a group. Let $\Gamma \subseteq \mathrm{P}^*(S)$. We define $x \alpha y=x \cdot \alpha \cdot y$ for every

$x, y \in S$ and $\alpha \in \Gamma$. Then S is a Γ-hypergroup.

Example 11. Let $(S, \circ)$ be a hypergroup and $\varnothing \neq \Gamma \subseteq S$. We define $x\alpha y = x \circ \alpha \circ y$ for every $x, y \in S$ and $\alpha \in \Gamma$. Then S is a Γ-hypergroup.

Example 12. Let $(G, \cdot)$ be a group and $\{A_g\}_{g\in G}$ be a collection of disjoint sets. Consider $S = \bigcup_{g\in G} A_g$ and $\Gamma = G$. For $x, y \in S$ there exist $g_x, g_y \in G$ such that $x \in A_{g_x}$ and $y \in A_{g_y}$. We define $x\alpha y = A_{g_x \alpha g_y}$. Then S is a Γ-hypergroup.

Example 13. Let S be a Γ-group and P be a Γ-subgroup of S. Let $\Gamma' = \{\gamma' \mid \gamma \in \Gamma\}$. Now, for every $x, y \in S$ and $\alpha' \in \Gamma$ we define $x\alpha' y = x\alpha y \cup P$. Then, S is a Γ'-hypergroup.

Theorem 4.2. [12] Let S be a Γ-semihypergroup. Then S is a simple Γ-semihypergroup if and only if S_α is a hypergroup for every $\alpha \in \Gamma$.

Theorem 4.3. Let S be a Γ-semihypergroup. Then for every $\alpha \in \Gamma$, S_α is a hypergroup if and only if S is left and right simple.

Proof: Suppose that S_α is a hypergroup and I is a left (right) ideal of S. If $x \in I$, then the reproduction axiom implies that $x\alpha S = S = S\alpha x$. On the other hand, we have $S\alpha x \subseteq I$ ($x\alpha S \subseteq I$). Therefore, $I = S$.

Conversely, suppose that S is left and right simple. Then for every $x \in S$ and $\alpha \in \Gamma$, put $I = x\alpha S$. Thus, I is a right ideal of S, for

$$I\Gamma S = (x\alpha S)\Gamma S = x\alpha(S\Gamma S) \subseteq x\alpha S = I$$

so $x\alpha S = S$. Similarly, we have $S = S\alpha x$. Therefore, S is a Γ-hypergroup.

Corollary 4.4. If S_α is a hypergroup for some $\alpha \in \Gamma$, then for every $\alpha \in \Gamma$, S_α is a hypergroup.

Definition 4.5. A subset H of a Γ-hypergroup is called a Γ-subhypergroup if for every $h, k \in H$ and $\alpha \in \Gamma$ we have $h\alpha k \subseteq H$ and $h\alpha H = H = H\alpha h$.

Definition 4.6. Let S be a Γ-hypergroup. Then a subset H of S is called closed if for every $h, k \in H$, $x \in S$ and $\alpha \in \Gamma$ we have the following implication

$$h \in x\,\alpha H \Rightarrow x \in H.$$

Example 14. Consider $(\mathbb{Z}, +)$ and let $\Gamma = \{\alpha, \beta\}$ where $\alpha = \{-1, 1\}$ and $\beta = \{-2, +2\}$. If for every $x, y \in \mathbb{Z}$ we define:

$$x\alpha y = \{x + y - 1, x + y + 1\}, x\beta y$$
$$= \{x + y - 2, x + y + 2\}.$$

Then, $\mathbb{Z}$ is a Γ-hypergroup and $H = 2\mathbb{Z}$ is a closed subset of $\mathbb{Z}$.

Example 15. Consider $(\mathbb{Z}, +)$ and let $\Gamma = \{\alpha, \beta\}$ where $\alpha = \{-2, 2\}$ and $\beta = \{-4, 4\}$. If for every $x, y \in \mathbb{Z}$ we define:

$$x\alpha y = \{x + y - 2, x + y + 2\}, x\beta y$$
$$= \{x + y - 4, x + y + 4\}.$$

Then $\mathbb{Z}$ is a Γ-hypergroup and $H = 2\mathbb{Z}$ is a closed Γ-subhypergroup of $\mathbb{Z}$.

Let S be a Γ-hypergroup. Then two new hyperoperations may be defined on S as follows:

$$a / b = \{x \in S \mid a \in x\alpha b, \alpha \in \Gamma\}$$
$$\text{and} \quad a \setminus b = \{x \in S \mid a \in b\alpha x, \alpha \in \Gamma\}.$$

If A and B are non-empty subsets of S, then

$$A/B = \bigcup_{a\in A, b\in B} a/b \quad and \quad A \setminus B = \bigcup_{a\in A, b\in B} a \setminus b.$$

Lemma 4.7. Let S be a Γ-hypergroup, A, B, C and D be non-empty subsets of S and $x, y \in S$. Then the following assertions hold:

(1) If $A \subseteq B$ and $C \subseteq D$, then $A/C \subseteq B/D$;

(2) $(A/B)/C = A/(C\Gamma B)$;

(3) $(A \setminus B) \setminus C = A \setminus (B\Gamma C)$;

(4) $y \in x \setminus (x/y)$;

(5) $y \in x/(x \setminus y)$;

(6) If A is a closed subset of S, then $A/A \subseteq A$;

(7) $A \subseteq (A\Gamma B)/B$;

(8) If H is a Γ-subhypergroup, then $H \subseteq H/H$.

Proof: (1) It is immediate.

(2) Suppose that $x \in (A/B)/C$. Then, there exist $a \in A, b \in B$ and $c \in C$ such that $x \in (a/b)/c$. So, we have

$$\begin{aligned} x \in (a/b)/c \quad &\Rightarrow \exists y \in a/b, x \in y/c \\ &\Rightarrow a \in y\Gamma b, y \in x\Gamma c \\ &\Rightarrow a \in (x\Gamma c)\Gamma b = x\Gamma(c\Gamma b) \\ &\Rightarrow \exists z \in c\Gamma b, a \in x\Gamma z \\ &\Rightarrow x \in a/z \subseteq a/(c\Gamma b) \subseteq A/(C\Gamma B). \end{aligned}$$

Thus, $(A/B)/C \subseteq A/(C\Gamma B)$.

Conversely, suppose that $x \in A/(C\Gamma B)$. Then there exist $a \in A, b \in B$ and $c \in C$ such that $x \in a/(c\Gamma b)$. So there exists $y \in c\Gamma b$ such that $x \in a/y$. So $a \in x\Gamma y \subseteq x\Gamma(c\Gamma b) = (x\Gamma c)\Gamma b$. Thus there exists $z \in x\Gamma c$ such that $a \in z\Gamma b$ and so $x \in z/c, z \in a/b$. Therefore, $x \in (A/B)/C$.

(3) It is similar to (2).

(4) Let $a \in x/y \neq \varnothing$. Then $x \in a\Gamma y$, so $y \in x \setminus a \subseteq x \setminus (x/y)$.

(5) it is similar to (4).

(6) If $x \in A/A$, then $x \in a_1/a_2$. So $a_1 \in x\Gamma a_2 \subseteq x\Gamma A \cap A$. Since A is a closed subset of S, it follows that $x \in A$. Therefore, $A/A \subseteq A$.

(7) Suppose that $x \in A$ and $y \in x\Gamma B$. Then $x \in y/B \subseteq (A\Gamma B)/B$.

(8) Suppose that H is a Γ-subhypergroup and $h_1 \in H$. Then there exists $h_2 \in H$ such that $h_1 \in h_1\Gamma h_2$ thus $h_1 \in h_1/h_2 \subseteq H/H$, so $H \subseteq H/H$.

Theorem 4.8. Let S be a Γ-hypergroup and H be a Γ-subhypergroup of S. Then H is a closed Γ-subhypergroup if and only if $H = H/H$.

Proof: Suppose that H is a closed Γ-subhypergroup. Then, by the previous lemma, $H \subseteq H/H \subseteq H$. Thus $H = H/H$.

Conversely, suppose that $H/H = H$. If $y \in x\alpha h \cap H$, for $h \in H$, then $x \in y/h \subseteq H/H = H$. Therefore, H is a closed Γ-subhypergroup of S.

Example 16. Let G be a group with a non trivial center. Let $P, Q \subseteq Z(G)$ and put $\Gamma = \{\alpha, \beta\}$. For every $x, y \in G$ we define $x\alpha y = xyP$ and $x\beta y = xyQ$. Then G is a Γ-hypergroup.

Let $a, b \in G$. Then

$$\begin{aligned} a/b &= \{x \in G \mid a \in x\Gamma b\} \\ &= \{x \in G \mid a \in x\alpha b \cup x\beta b\} \\ &= \{x \in G \mid a \in xbP \cup xbQ\} \\ &= ab^{-1}P^{-1} \cup ab^{-1}Q^{-1}. \end{aligned}$$

If H is a Γ-subhypergroup of G containing P and Q, then for every $a, b \in H$ we have $a/b = ab^{-1}P^{-1} \cup ab^{-1}Q^{-1} \subseteq H$, so by the above theorem, H is a closed Γ-subhypergroup of G.

Lemma 4.9. Let S be a Γ-semihypergroup and H and K be two closed Γ-subhypergroups of S. Then $< H \cup K >=< H\Gamma K >$.

Proof: Since $H\Gamma K \subseteq < H \cup K >$, it follows that $< H\Gamma K > \subseteq < H \cup K >$. Now, we prove the converse of inclusion. Since H and K are closed Γ-subhypergroups of S, it follows that $H\Gamma K$ is a closed subset of S. Now, by the previous theorem and Lemma 4.7, we have

$$\begin{aligned} H = H/H \quad &\subseteq ((H\Gamma K)/K)/H \\ &= (H\Gamma K)/(H\Gamma K) \subseteq < H\Gamma K >. \end{aligned}$$

Similarly, $K \subseteq < H\Gamma K >$. Therefore, $< H \cup K >=< H\Gamma H >$.

5. Γ-semihypergroups associated to binary relations

The connections between hyperstructures and binary relations have been analyzed by many researchers, such as Rosenberg [13], Corsini [14], Cristea and Stefănescu [15] and others [16, 17, 18] .

In this section we associate to a set of binary relations on a non-empty set S, say Γ, a partial Γ-hypergroupoid and get necessary and sufficient conditions such that it is a Γ-semihypergroup or a Γ-hypergroup.

Rosenberg [13] has associated a partial hypergroupoid H_R, with a binary relation R defined on a non-empty set H, where, for any $x, y \in H$

$$\mathrm{x} \circ \mathrm{x} = \mathrm{L_x} = \{\mathrm{z} \in \mathrm{H} \mid (\mathrm{x},\mathrm{z}) \in \mathrm{R}\}$$
and $\mathrm{x} \circ \mathrm{y} = \mathrm{x} \circ \mathrm{x} \cup \mathrm{y} \circ \mathrm{y}.$

An element $x \in H$ is called an outer element for R if there exists $h \in H$ such that $(h, x) \notin R^2$. Rosenberg proved the next theorem.

Theorem 5.1. [13] H_R is a hypergroup if and only if

(1) R has full domain;

(2) R has full range;

(3) $R \subseteq R^2$;

(4) If $(a, x) \in R^2$, then $(a, x) \in R$, whenever x is an outer element.

Let R be a binary relation on a non-empty set S. Then an element $x \in S$ is called a semiouter element for the relation R if there exists $h \in S$ such that $(h, x) \notin R$.

Let R be a binary relation on a non-empty set S, $A \subseteq S$ and $x, y \in S$. Then we use the following notations:

$$L_x^R = R(x) = \{z \in S \mid (x, z) \in R\};$$
$$R(x, y) = \{z \in S \mid (x, z) \in R \vee (y, z) \in R\};$$
$$R(A) = \{z \in S \mid (a, z) \in R, \exists a \in A\};$$
$$R^{-1}(A) = \{z \in S \mid (z, a) \in R, \exists a \in A\}.$$

Definition 5.2. Let S be a non-empty set and $\mathcal{R}$ be a set of binary relations on S. Then for every $\alpha \in \mathcal{R}$ we can associate a hyperoperation $\circ_\alpha$ on S as follows:

$$x \circ_\alpha y = \alpha(x, y) = L_x^\alpha \cup L_y^\alpha, \ \forall\, x, y \in S.$$

So $(S, \circ_\alpha)$ is a partial hypergroupoid. Now, let $\Gamma = \{\circ_\alpha \mid \alpha \in \mathcal{R}\}$. Then S is a partial Γ-hypergroupoid and is denoted by S_Γ.

To simplify, we write $\circ_\alpha$ by α and consider $\Gamma = \mathcal{R}$, in this way for every $\alpha \in \Gamma$ and $x, y \in S$ we have

$$x\alpha y = x \circ_\alpha y = \alpha(x, y) = L_x^\alpha \cup L_y^\alpha.$$

It is easy to see that if for every $\alpha \in \Gamma$ we have $\alpha^{-1}(S) = S$, then S_Γ is a Γ-hypergroupoid.

Example 17. Let $S = \{1,2,3,4,5\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ such that

$$\alpha = \{(1,1),(1,2),(2,4),(3,4),(4,5),(4,4),(5,2)\},$$
$$\beta = \{(1,1),(1,3),(1,4),(2,5),(3,3),(4,1),(5,4),(5,3)\},$$
$$\gamma = \{(1,3),(2,3),(3,4),(4,5),(5,1),(5,5)\}.$$

Then S_Γ is a Γ-hypergroupoid.

Lemma 5.3. Let S be a non-empty set and Γ be a set of binary relations on S such that S_Γ is a Γ-hypergroupoid. Then the following assertions hold:

(1) S_Γ is a commutative Γ-hypergroupoid;

(2) For every $x \in S$ and $\alpha \in \Gamma$, $x\alpha x = \alpha(x)$;

(3) For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, $x\alpha(y\beta z) = \alpha(x) \cup \beta\alpha(y, z)$;

(4) For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, $(x\alpha y)\beta z = \alpha\beta(x, y) \cup \beta(z)$.

Proof: The proof is straightforward.

In the following we provide some conditions on Γ such that S_Γ be a Γ-semihypergroup.

Theorem 5.4. Let S be a non-empty set and Γ be a set of binary relations on S such that S_Γ be a Γ-hypergroupoid. Then S_Γ is a Γ-semihypergroup if and only if the following conditions hold:

($\Gamma SH1$) For every $\alpha, \beta \in \Gamma$, $\alpha \subseteq \alpha\beta$;

($\Gamma SH2$) If x is a semiouter element for the relation $\alpha\beta$ and $(a, x) \in \beta\alpha$, then $(a, x) \in \beta$ for every $a \in S$ and $\alpha, \beta \in \Gamma$;

($\Gamma SH3$) If x is a semiouter element for the relations $\alpha\beta$ and β and $(a,x)\in\beta\alpha$, then $(a,x)\in\alpha\beta$, for every $a\in S$ and $\alpha,\beta\in\Gamma$.

Proof: Suppose that S_Γ is a Γ-semihypergroup. We prove the conditions (Γ SH1), (Γ SH2) and (Γ SH3) of the theorem.

(Γ SH1) Let $x,y\in S$ and $\alpha,\beta\in\Gamma$ such that $y\in\alpha(x)$. Then we consider two cases:

Case (i) $y\in\beta(y)$. Then $y\in\alpha\beta(x)$.

Case (ii) $y\notin\beta(y)$. Then we have $(x\alpha x)\beta y = x\alpha(x\beta y)$ so the associativity axiom and the previous lemma conclude that $\alpha\beta(x)\cup\beta(y)=\alpha(x)\cup\beta\alpha(x)\cup\beta\alpha(y)$. Now, since $y\in\alpha(x)$ and $y\notin\beta(y)$, it follows that $y\in\alpha\beta(x)$. Therefore, $\alpha\subseteq\alpha\beta$.

(Γ SH2) Suppose that x is a semiouter element for the relation $\alpha\beta$ and $x\in\beta\alpha(a)$. So there exists $h\in S$ such that $x\notin\alpha\beta(h)$. Thus the associativity axiom and the previous lemma conclude that $(h\alpha h)\beta h = h\alpha(h\beta a)$, thus $\alpha\beta(h)\cup\beta(a)=\alpha(h)\cup\beta\alpha(h)\cup\beta\alpha(a)$. Since $x\in\beta\alpha(a)$ and $x\notin\alpha\beta(h)$, it follows that $x\in\beta(a)$.

(Γ SH3) Suppose that x is a semiouter element for the relations $\alpha\beta$ and β and let $x\in\beta\alpha(a)$. So there exist $h,t\in S$ such that $(h,x)\notin\alpha\beta$ and $(t,x)\notin\beta$. Now, we have $h\alpha(a\beta t)=(h\alpha a)\beta t$ thus $\alpha(h)\cup\beta\alpha(a,t)=\alpha\beta(a,h)\cup\beta(t)$. Since $x\in\beta\alpha(a)$, $x\notin\alpha\beta(h)$ and $x\notin\beta(t)$, it follows that $x\in\alpha\beta(a)$.

Conversely, suppose that S is a non-empty set and Γ be a set of binary relations on S such that S_Γ is a Γ-hypergroupoid and the conditions (Γ SH1), (Γ SH2) and (Γ SH3) of the theorem are satisfied. We prove the associativity axiom for S_Γ. Let $x,y,z,t\in S$ and $\alpha,\beta\in\Gamma$ such that $t\in x\alpha(y\beta z)=\alpha(x)\cup\beta\alpha(y,z)$. Then we have three cases:

Case (i) $t\in\alpha(x)$. Then by the condition (Γ SH1) $t\in\alpha\beta(x)$.

Case (ii) $t\in\beta\alpha(x)$. Then if $t\notin\alpha\beta(x)\cup\beta(z)$, then t is a semiouter element for the relations $\alpha\beta$ and β. So by the condition (Γ SH3) $t\in\alpha\beta(y)$.

Case (iii) $t\in\beta\alpha(z)$. Then if $t\notin\alpha\beta(x)$, then t is a semiouter element for the relation $\alpha\beta$ so by the condition (Γ SH2), $t\in\beta(z)$. Thus $x\alpha(y\beta z)\subseteq(x\alpha y)\beta z$. In the same way, we can prove the converse inclusion. Therefore, S_Γ is a Γ-semihypergroup.

Example 18. Let $S=\{1,2,3\}$ and $\Gamma=\{\alpha,\beta\}$ such that $\alpha=\{(1,2),(2,2),(2,3),(3,3)\}$ and $\beta=\{(1,3),(2,2),(3,2),(3,3)\}$. Then we have the table of hyperoperations α and β as follows:

α	1	2	3
1	$\{2\}$	$\{2,3\}$	$\{2,3\}$
2	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$
3	$\{2,3\}$	$\{2,3\}$	$\{3\}$

β	1	2	3
1	$\{3\}$	$\{2,3\}$	$\{2,3\}$
2	$\{2,3\}$	$\{2\}$	$\{2,3\}$
3	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$

Then S_Γ is a Γ-semihypergroup.

Theorem 5.5. Let S be a non-empty set and Γ be a set of binary relations on S such that S_Γ is a Γ-semihypergroup. Then S_Γ is a Γ-hypergroup if and only if $\alpha(S)=S$ for every $\alpha\in\Gamma$.

Proof: Suppose that S_Γ is a Γ-hypergroup. Then S_α is a hypergroup for every $\alpha\in\Gamma$. So by Theorem 5.1, α has full range, thus $\alpha(S)=S$.

Conversely, suppose that $\alpha(S)=S$ for every $\alpha\in\Gamma$ so S_α is a hypergroup. Therefore, S_Γ is a Γ-hypergroup.

Example 19. Let $S=\{1,2,3\}$ and $\Gamma=\{\alpha,\beta\}$ such that $\alpha=\Delta_S\cup\{(2,1),(3,2)\}$ and $\beta=\Delta_S\cup\{(3,1)\}$, where Δ_S is the diagonal relation on S. Then we have the table of hyperoperations α and β as follows:

α	1	2	3
1	$\{1\}$	$\{1,2\}$	S
2	$\{1,2\}$	$\{1,2\}$	S
3	S	S	$\{2,3\}$

β	1	2	3
1	$\{1\}$	$\{1,2\}$	$\{1,3\}$
2	$\{1,2\}$	$\{2\}$	S
3	$\{1,3\}$	S	$\{1,3\}$

Then S is a Γ-hypergroup.

Lemma 5.6. Let S be a non-empty set and Γ be a set of binary relations on S such that S_Γ is a Γ-semihypergroup. Then $I=\Gamma(S)=\bigcup_{\alpha\in\Gamma}\alpha(S)$ is a minimal ideal of S_Γ.

Proof: Suppose that $a\in I$, $s\in S$ and $\alpha\in\Gamma$. Then we have $s\alpha a=\alpha(a)\cup\alpha(s)\subseteq\alpha(S)\subseteq I$. So I is an ideal of S_Γ. Furthermore, if J is an ideal of S_Γ and $b\in J$, then for every $s\in S$ and $\alpha\in\Gamma$, $s\alpha b=\alpha(s)\cup\alpha(b)\subseteq J$. So $\alpha(S)\subseteq J$ hence $I\subseteq J$.

Proposition 5.7. Let S be a non-empty set and Γ be a set of binary relations on S such that S_Γ is a Γ-semihypergroup. Let $\Gamma_\cup=\{\alpha\cup\beta\mid\alpha,\beta\in\Gamma\}$. Then $S_{\Gamma_\cup}$ is a $\Gamma_\cup$-semihypergroup.

Proof: We prove that $S_{\Gamma_\cup}$ satisfies the conditions (Γ SH1), (Γ SH2) and (Γ SH3) of Theorem 5.4. Suppose that $\theta',\varphi'\in\Gamma_\cup$. Then there exist $\alpha,\beta,\delta,\gamma\in\Gamma$, such that $\theta'=\alpha\cup\beta$ and $\varphi'=\delta\cup\gamma$. Since S_Γ is a Γ-semihypergroup, it follows that $\alpha\subseteq\alpha\delta\cup\alpha\gamma$ and $\beta\subseteq\beta\delta\cup\beta\gamma$. Thus

$$\begin{aligned}\theta'&=\alpha\cup\beta\subseteq\alpha\delta\cup\alpha\gamma\cup\beta\delta\cup\beta\gamma\\&=(\alpha\cup\beta)(\delta\cup\gamma)=\theta'\phi'.\end{aligned}$$

So the condition (Γ SH1) holds.

Suppose that $x\in S$ is a semiouter element for the relation θ' and let $(a,x)\in\varphi'\theta'$. Then there exists $h\in S$ such that $(h,x)\notin\theta'\varphi'$. Thus x is a semiouter element for the relations $\alpha\delta,\alpha\gamma,\beta\delta$ and $\beta\gamma$. Since $(a,x)\in\varphi'\theta'$, it follows that $(a,x)\in\delta\alpha$, $(a,x)\in\gamma\alpha$, $(a,x)\in\delta\beta$ or $(a,x)\in\gamma\beta$. From the condition (Γ SH2) for S_Γ we conclude that $(a,x)\in\delta$, $(a,x)\in\gamma$, $(a,x)\in\delta$ or $(a,x)\in\gamma$. Thus $(a,x)\in\delta\cup\gamma=\varphi'$ and the condition (Γ SH2) holds.

Suppose that $x\in S$ is a semiouter element for the relations $\theta'\varphi'$ and φ' and let $(a,x)\in\varphi'\theta'$. Then there exist $h,t\in S$ such that $(h,x)\notin\theta'\varphi'$ and $(t,x)\notin\varphi'$. So x is a semiouter element for the relations $\alpha\delta,\alpha\gamma,\beta\delta,\beta\gamma,\delta$ and γ. Thus if $(a,x)\in\alpha\delta$, $(a,x)\in\delta\alpha$, $(a,x)\in\gamma\alpha$, $(a,x)\in\delta\beta$ or $(a,x)\in\gamma\beta$, then from the condition (Γ SH3) for S_Γ we conclude that $(a,x)\in\alpha\delta$, $(a,x)\in\alpha\gamma$, $(a,x)\in\delta\beta$ or $(a,x)\in\gamma\beta$, respectively, and the condition (Γ SH3) holds. Therefore, $S_{\Gamma_\cup}$ is a $\Gamma_\cup$-semihypergroup.

Let S_R be a hypergroupoid associated to a binary relation R. Let $\Gamma_R=\{\alpha_i\mid i\in\mathbb{N}\}$. Now, for every $x,y\in S$ and $\alpha_i\in\Gamma$ we define

$$x\alpha_i y=\{z\mid(x,z)\in R^i\vee(y,z)\in R^i\}=L_x^{R^i}\cup L_y^{R^i}.$$

Then S is a Γ_R-hypergroupoid and denoted by S_{Γ_R}. In the following we verify conditions such that S is a Γ_R-semihypergroup.

Lemma 5.8. Let S_R be a semihypergroup associated to a binary relation R. Then if $(z,t)\in R^{i+j}$ and $(x,t)\notin R^{i+j}$, then $(z,t)\in R^j$, for every $x,z,t\in S$ and $i,j\in\mathbb{N}$.

Proof: We prove by mathematical induction on $i+j$. If $i+j=2$, $(z,t)\in R^2$ and $(x,t)\notin R^2$, then t is an outer element for R so $(z,t)\in R$.

Suppose that the result holds for $i+j-1$. Now, let $(z,t)\in R^{i+j}$ and $(x,t)\notin R^{i+j}$. Then there exists $s\in S$ such that $(z,s)\in R^2$ and $(s,t)\in R^{i+j-1}$. Thus $(x,s)\notin R^2$, that is, s is an outer element for R and so $(z,s)\in R$. Therefore, $(z,t)\in R^{i+j}$. Now, we have $(z,t)\in R^{i+j-1}$ and $(x,t)\notin R^{i+j-1}$ thus $(z,t)\in R^j$.

Lemma 5.9. Let S_R be a semihypergroup associated to a binary relation R. Then S_{Γ_R} is a Γ_R-semihypergroup.

Proof: We prove the associativity law. Suppose that $x,y,z\in S_\Gamma$ and $\alpha_i,\alpha_j\in\Gamma$. Then

$$x\alpha_i(y\alpha_j z) = L_x^{R^i}\cup L_y^{R^{i+j}}\cup L_z^{R^{i+j}}$$
and $$(x\alpha_i y)\alpha_j z = L_x^{R^{i+j}}\cup L_y^{R^{i+j}}\cup L_z^{R^j}.$$

If $t\in L_z^{R^{i+j}}$ and $t\notin L_x^{R^{i+j}}$, then by the previous lemma $t\in L_z^{R^j}\subseteq (x\alpha_i y)\alpha_j z$. Therefore, $x\alpha_i(y\alpha_j z)\subseteq (x\alpha_i y)\alpha_j z$. In a similar way we have the inverse inclusion.

Example 20. Let $S=\{1,2,3\}$ and $R=\{(1,2),(1,3),(2,2),(3,2)\}$. Then S_R is a semihypergroup. Let $\Gamma_R=\{\alpha_1,\alpha_2\}$. Then we have the following hyperoperations:

α_1	1	2	3
1	$\{1,3\}$	S	S
2	S	$\{2\}$	$\{2,3\}$
3	S	$\{2,3\}$	$\{2\}$

α_2	1	2	3
1	S	S	S
2	S	$\{2\}$	$\{2,3\}$
3	S	$\{2,3\}$	$\{2\}$

Then S_{Γ_R} is a Γ_R-semihypergroup.

6. Conclusion

In this work, we presented the concept of semiprime ideals in a Γ-semihypergroup and proved some results. Also, we introduced the notion of Γ-hypergroups and closed Γ-subhypergroups. Finally, we defined the concept of Γ-semihypergroups and Γ-hypergroups associated to a set of binary relations. Then we find the necessary and sufficient conditions on a set of binary relations Γ on a non-empty set S such that S becomes a Γ-semihypergroup or a Γ-hypergroup.

Our future research will consider Γ-semihyperrings associated to binary relations.

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