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Existence of differentiable connections on top spaces

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Abstract

In this paper, differentiable connections on top spaces are studied and some conditions on which there is no differentiable connection passing from a given point in the top space are found. In a special case, the Euclidean space $\mathbb{R}^2$ is considered as a top space and the existence of differentiable connections is studied. Finally, we prove that the smoothness condition of the inverse map in the definition of a top space is redundant.

***Keywords**:* Lie group; generalized topological group; top space; differentiable connection

1. Introduction

A top space is a generalization of the concept of Lie groups [1, 2]. According to what has been proven already, each top space is a union of disjoint diffeomorphic Lie groups, and these diffeomorphic Lie groups can be considered as vertical lines [2-4].

A differentiable connection in a top space T is a one to one, C^∞ map $\xi : [0,1] \to T$ that intersects each of the vertical lines of the top space in at most one point, and it can be considered as a horizontal line [1]. Note that, we can extend these structures on generalized local groups [5].

In sections 2 and 3, the existence of differentiable connections in some special cases are studied, and in section 4 we prove in proposition 14 that, under a poor condition, the smoothness condition of the inverse map in the definition of a top space is redundant.

Now, let us recall the definition of a top space:

Definition 1. A top space T is a smooth manifold with a generalized group structure such that the multiplication operation and the inverse map are smooth and for every $s, t \in T$, we have: $e(s.t) = e(s).e(t)$, where $e(t)$ is the identity element of t [1, 2].

The following lemma is a corollary in [3].

Lemma 2. Let T be a top space. The map $e : T \to T$ defined by $t \longmapsto e(t)$, is a continuous map.

Example 3. The Euclidean space $\mathbb{R}^2$ with the multiplication:

$$(a,b).(c,d) = (a, b+d), \text{ for any } (a,b),(c,d) \in \mathbb{R}^2$$

is a top space. In this example, the identity element of (a,b) is $(a,0)$ and its inverse is $(a,-b)$.

Theorem 4. Let T be a top space, $e(T)$ be the set of all identity elements of T and $G_{e(t)} = e^{-1}(e(t))$, then $G_{e(t)}$ is a Lie group with the identity element $e(t)$ and for all $e(t_1), e(t_2) \in e(T)$; $G_{e(t_1)}$ is diffeomorphic to $G_{e(t_2)}$, and we have:

$$T = \mathop{\mathring{\bigcup}}_{e(t)\in e(T)} G_{e(t)} \cong \prod_{e(t)\in e(T)} G_{e(t)}$$

(Note that, the first union and $\prod$ denote the disjoint union and the direct sum of Lie groups, respectively) [3].

Example 5. In example 3, we have $e^{-1}\big((a,0)\big) = \{a\} \times \mathbb{R}$ and

$$\mathbb{R}^2 = \mathop{\mathring{\bigcup}}_{a\in\mathbb{R}} (\{a\} \times \mathbb{R}).$$

Now, we define a differentiable connection:

Definition 6. A differentiable connection in a top space T is a one to one, C^∞ map $\xi : [0,1] \to T$ such that $card(\xi[0,1] \cap e^{-1}(e(t))) \leq 1$, for any $e(t) \in e(T)$ [6].

Example 7. In example 3, the map $\xi : [0,1] \to \mathbb{R}^2$ defined by $\xi(t) = (t,t)$, is a differentiable connection.

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2. Cases in which there is no differentiable connection

Let us begin this section with the following proposition, which has been stated as a corollary in [6].

Proposition 8. Let T be a top space such that $e(T)$, the set of all identity elements of T, be finite or countable, then there is no differentiable connection.

Before bringing the theorem, we need the following lemma:

Lemma 9. Let T be a top space and

$$T = \overset{\circ}{\bigcup_{e(t)\in e(T)}} G_{e(t)}.$$

If the dimension of $G_{e(t)}$ is equal to the dimension of T, then $G_{e(t)}$ has at least one interior point in T.

Proof: Let $G_{e(t)}$ have no interior point in T. The map e is continuous, so $G_{e(t)}$ is closed, and hence

$$G_{e(t)} = \overline{G_{e(t)}} = \partial G_{e(t)}$$

where $\overline{G_{e(t)}}$ and $\partial G_{e(t)}$ denote the closure and the set of boundary points of $G_{e(t)}$, respectively. Therefore, $G_{e(t)}$ is equal to its boundary, so its dimension is less than the dimension of T and it is a contradiction.

Now, we state our main result.

Theorem 10. Let T be a top space and

$$T = \overset{\circ}{\bigcup_{e(t)\in e(T)}} G_{e(t)},$$

where $G_{e(t)}$ is a Lie group in which its dimension is equal to the dimension of T, and g_o be an interior point of $G_{e(s)}$ for some $e(s) \in e(T)$, then there exists no differentiable connection in T passing from g_o.

Proof: Let $\xi : [0\,,1] \to T$ be a differentiable connection in T passing from g_o, i.e. there exists $r_o \in [0\,,1]$ such that $\xi(r_o) = g_o$. Suppose U be an open neighborhood of g_o such that $U \subseteq G_{e(s)}$. Since ξ is continuous, the set $\xi^{-1}(U)$ is open in the closed interval $[0\,,1]$, and so there is a base V such that $r_o \in V \subseteq \xi^{-1}(U)$. V is an uncountable set, and

$$\xi(V) \subseteq U \subseteq G_{e(s)}.$$

since ξ is one to one, $card\,\xi(V) = card(V) = c$, then

$$card\left(\xi([0\,,1]) \cap G_{e(s)}\right) = c$$

which is in contradiction to the definition of a connection. Therefore, there is no differentiable connection in T passing from g_o

Corollary 11. Let T be a top space and

$$T = \overset{\circ}{\bigcup_{e(t)\in e(T)}} G_{e(t)},$$

where the Lie group $G_{e(t)}$ is open in T, then there exists no differentiable connection passing from each point of $G_{e(t)}$.

Proof: Each point of $G_{e(t)}$ is an interior point, so one gets the result by the same proof of theorem 10.

Example 12. The space $\mathbb{R} - \{0\}$ with the multiplication:

$$a\,.\ b = a|b|, \text{ for every } a, b \in \mathbb{R} - \{0\}$$

is a top space with the identity elements $\{1\,,-1\}$ and $G_1 = \mathbb{R}^+, G_{-1} = \mathbb{R}^-$. In this example, we see that the dimension of $\mathbb{R} - \{0\}$, G_1 and G_{-1} are equal and so according to theorem 10, there is no differentiable connection passing from each point of $\mathbb{R} - \{0\}$.

3. One special case: the euclidean space $\mathbb{R}^2$

In this section, we study the existence of differentiable connections in the Euclidian space $\mathbb{R}^2$ with different top structures and determine the relation between the tangent space at a point t on the top space $\mathbb{R}^2$, with the tangent spaces at this point on a Lie group which contains t (by theorem 4) and on the image of a connection passing from t (if it exists).

At first, we show by the following example that one cannot necessarily write the tangent space of T at t by any horizontal and vertical structures.

Example 13. The Euclidean space $\mathbb{R}^2$ with the multiplication:

$$(a\,,b).(c\,,d) = (a + c\,,b), \text{ for any } (a\,,b),(c\,,d) \in \mathbb{R}^2$$

is a top space, and

$$\mathbb{R}^2 = \bigcup_{a\in\mathbb{R}} (\mathbb{R} \times \{a\}).$$

In this example, $\gamma : [0,1] \to \mathbb{R}^2$ defined by $\gamma(t) = (t - 1/2, (t - 1/2)^3)$, is a differentiable connection with $\gamma(1/2) = (0,0)$ and $\gamma_*(1/2) = (1,0)$. We see that this tangent vector is in the tangent space on the Lie group $\mathbb{R} \times \{0\}$ at $(0,0)$. Therefore, they do not produce the tangent space on $\mathbb{R}^2$ at $(0,0)$.

Note that in the previous example, the map $\xi(t) = (0, t - 1/2)$, for any $t \in [0,1]$ is a connection with $\xi(1/2) = (0,0)$ and $\xi_*(1/2) = (0,1)$, so these vertical and horizontal structures produce the tangent space on $\mathbb{R}^2$ at $(0,0)$.

Now, we study the general state:

Let $(\mathbb{R}^2, .)$ be a top space and with this top structure:

$$\mathbb{R}^2 = \mathop{\mathring{\bigcup}}_{e(t)\in e(T)} G_{e(t)} \cong \prod_{e(t)\in e(T)} G_{e(t)},$$

<u>*Case 1*</u>. $dim\, G_{e(t)} = 0$

In this case, at every point one can find two connections with independent tangent vectors that produce the tangent space on $\mathbb{R}^2$.

<u>*Case 2*</u>. $\dim G_{e(t)} = 1$

Since the Euclidean space $\mathbb{R}^2$ is connected, $G_{e(t)}$ is connected for all $e(t) \in e(T)$. We know that every one dimensional connected Lie group is isomorphic to $\mathbb{R}$ or S^1 [7], and so we have:

$$\mathbb{R}^2 \cong \prod_{e(t)\in e(T)} \mathbb{R}$$

or

$$\mathbb{R}^2 \cong \prod_{e(t)\in e(T)} S^1,$$

since S^1 is compact, $\prod_{e(t)\in e(T)} S^1$ is also compact. So $\mathbb{R}^2 \cong \prod_{e(t)\in e(T)} S^1$ is impossible. Therefore, we just have:

$$\mathbb{R}^2 \cong \prod_{e(t)\in e(T)} \mathbb{R},$$

then there exist some connections at every point similar to example 11.

<u>*Case 3*</u>. $\dim G_{e(t)} = 2$

According to theorem 8, there is no differentiable connection passing from each interior point of $G_{e(t)}$, moreover, the tangent space on $G_{e(t)}$ is equal to the tangent space on $\mathbb{R}^2$ at these points.

4. A redundant condition in definition of top space

In this section, we prove that under a few conditions, checking the differentiability of the inverse map in a top space is not necessary.

Let M be a manifold with a differentiable map $m : M \times M \to M$, which defines an associative multiplication operation on M. Assume that for each $t \in M$ there exists a unique $e(t) \in M$ such that $e(t).t = t.e(t) = t$ and $e(t.s) = e(t).e(s)$, for all $t, s \in M$. Let $e : M \to M$ be the map defined by $t \longmapsto e(t)$ and for all $t \in M$, $e^{-1}(e(t))$ be open. Define $M_{e(t)} = e^{-1}(e(t))$, for all $t \in M$, then $M_{e(t)}$ is an open submanifold of M and the restriction of m to $M_{e(t)}$ gives us a C^∞ associative multiplication operation on the manifold $M_{e(t)}$ denoted by $m_{e(t)}$.

Lemma 14. The differential of the multiplication map on $M_{e(t)}$ at $(e(t), e(t))$ is given by

$$T_{(e(t),e(t))}(m_{e(t)})\,(X, Y) = X + Y,$$

for all $X, Y \in T_{e(t)}(M)$ [7].

Let $G_{e(t)}$ be the set of all invertible elements in $M_{e(t)}$, it is clear that $G_{e(t)}$ is a group and we have:

Lemma 15. The group $G_{e(t)}$ is an open submanifold of $M_{e(t)}$ and with this manifold structure, $G_{e(t)}$ is a Lie group [7].

This lemma implies that the inverse map $\iota_{e(t)}: G_{e(t)} \to G_{e(t)}$ is C^∞.

Let S be the set of all invertible elements in M, then

$$S = \mathop{\mathring{\bigcup}}_{e(t)\in e(M)} G_{e(t)},$$

so S is a generalized group. Moreover, we have:

Proposition 16. Let S be the set of all invertible elements in M, then S is an open submanifold of M and with this manifold structure, S is a top space.

Proof: Since $S = \bigcup_{e(t)\in e(M)} G_{e(t)}$ and $G_{e(t)}$ is open in $M_{e(t)}$ and $M_{e(t)}$ is open in M, S is an open submanifold of M. The inverse map $\iota : S \to S$ is C^∞, because the restriction of ι to the open submanifold $G_{e(t)}$ of S is C^∞, for every $e(t) \in e(M)$.

We conclude this section with an example below.

Example 17. In example 12, G_i are open in $\mathbb{R} - \{0\}$, for $i = 1, 2$. In this example, the inverse map $\iota : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ defined by $x \mapsto 1/x$ is C^∞

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