IJST (2011) A1: 33-38
Iranian Journal of Science & Technology
http://www.shirazu.ac.ir/en

Application of the multistage homotopy perturbation method to some dynamical systems

M. H. Alnasr[1] and G. H. Erjaee[2]*

[1]*Mathematics & Physics Department, Qatar University, Doha, Qatar*
[2]*Mathematics Department, Shiraz University, Shiraz, Iran*
Email: modialnasr@qu.edu.qa, erjaee@shirazu.ac.ir

Abstract

In this article, we demonstrate an analytic-numeric solution for some dynamical systems using the multistage homotopy perturbation method. The method yields results that are in complete agreement with their numerical counterparts.

Keywords: Homotopy perturbation method; dynamical systems

1. Introduction

In studying non-linear dynamical systems, researchers usually use software packages to find sensitive bifurcation points since the corresponding analytical solutions either do not exist or are extremely difficult to find. The Homotopy Perturbation Method (HPM) was first introduced by J. -H. He [1-4] in 1998. This method has been applied to a number of mathematical and engineering problems; e.g., cf. [2-6] and the references therein. The goal of this paper is to apply the HPM to dynamical systems. Specifically, we find approximate analytic-numeric solutions for some dynamical systems by using the Multistage Homotopy Perturbation Method (MHPM). Some researchers have attempted to use HPM to find analytic-numeric solutions for systems of ODEs in general, or to find chaotic solutions in specific cases [5-14]. Other analytic-numeric solution methods such as Adomian decomposition [15], have also been applied to some chaotic systems [16-17]. However, our attempt is more general in applying MHPM to dynamical systems and determining their various bifurcation solutions.

We begin this article by briefly describing the MHPM. We then apply the method to the Helmholtz and Sprott-S dynamical systems to determine their steady state, unsteady state, and periodic and chaotic solutions for differing system parameters.

We then demonstrate the complete agreement between our MHPM analytic-numeric solutions and fourth-order Runge-Kutta solutions. We conclude with some final remarks.

2. Multistage homotopy perturbation method

J.-H. He established the general HPM [1] by introducing a homotopy parameter $p \in [0,1]$ into an ordinary differential equation such that, if $p = 0$, then the equation takes its simplest possible form. As p varies from 0 to 1, we obtain a sequence of equations in which the solution at any stage is close to solutions at nearby stages. When $p = 1$, we have the original form of the equation, so if we can solve this sequence of deformation equations from the first stage when p=0, then at the end, when the parameter $p = 1$, the procedure provides us the desired solution. Although the HPM yields a solution series which converges very rapidly in most linear and nonlinear equations, in the case of a large time interval t it may produce a large error. However, in this case we can subdivide the interval and apply the HPM algorithm on each subinterval to obtain a more accurate solution. Note that each subinterval will require new initial values determined by the solutions at the end point of the previous subinterval. This technique is called MHPM [6].

To be precise, consider a system of dynamical systems,

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots, x_n, \alpha) \tag{1}$$

* Corresponding author
Received: 11 October 2010 / Accepted: 5 January 2011

where α is a bifurcation parameter, $i = 1, 2, \cdots, n$, and the corresponding initial values are

$$x_i(t_0) = c_i \text{ for } i = 1, 2, \cdots, n. \quad (2)$$

Rewriting system (1) in operator form we get the system

$$L(x_i) + N_i(t, x_i, x_2, \cdots, x_n, \alpha) = 0, \quad (3)$$

with the corresponding initial condition (2), where $L = d/dt$ is a linear operator and N_i is a nonlinear operator for each $i = 1, 2, \cdots, n$. To apply the MHPM we first construct a homotopy for system (3) as follows:

$$\begin{aligned} &L(x_i) - L(v_i) + pL(v_i) + \\ &PN_i(t^*, x_i, x_2, \cdots, x_n, \alpha) = 0 \end{aligned} \quad (4)$$

where $p \in [0,1]$ is the homotopy parameter, v_i is an initial approximation satisfying the given initial values for $i = 1, 2, \cdots, n$ and $t^* \in [t_{j-1}, t_j)$, the j^{th} subinterval of $[t_0, t]$, for some $j = 1, 2, \cdots, m$. As noted above, if the parameter p is zero in (4), then the system will be a simple linear system, and if $p = 1$ the system will be equivalent to the original system (1) with $t^* = t$.

To proceed to the next step in MHPM, assume

$$\begin{aligned} x_i(t^*) &= u_{i,0}(t^*) + pu_{i,1}(t^*) + p^2 u_{i,2}(t^*) \\ &+ p^3 u_{i,3}(t^*) + \cdots \end{aligned}, \quad (5)$$

for $i = 1, 2, \cdots, n$ and initial values

$$u_{i,0}(t_0) = v_i(t_0) = x_i(t_0) = c_i. \quad (6)$$

To find the unknown functions $u_{i,k}(t^*)$ for $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$ and $t^* \in [t_{j-1}, t_j)$, we substitute (5) into (4) and rearrange the coefficients of the powers of p to get

$$L(u_{i,1}) + L(v_i) + N_i(u_{1,0}, u_{2,0}, \cdots, u_{n,0}, \alpha) = 0, \quad (7)$$

for the zero$^{\text{th}}$ power of p with the initial values as in (6) for $i = 1, 2, \cdots, n$. For other powers of p we get,

$$L(u_{i,k}) + N_i(u_{1,k-1}, u_{2,k-1}, \cdots, u_{n,k-1}, \alpha) = 0, \quad (8)$$

with the initial values $u_{i,k} = 0$ for $i = 1, 2, \cdots, n$ and any $k \geq 2$. Here, we again note that in the first subinterval $[t_0, t_1)$, when $k = 1$, the initial value is given by (6), and for the rest of the systems, when $k \geq 2$, the initial values are all zero. Then, in the next subinterval, say $[t_1, t_2)$, when $k = 1$ the initial values are the approximation values of $x_i(t_1)$ for $i = 1, 2, \cdots, n$, and for $k \geq 2$, the initial values are zero. Finally, the MHPM approximate solution for K solution terms of the systems (6-8) is given as follows

$$x_i(t) = \sum_{k=0}^{K-1} u_{i,k}(t), \quad (9)$$

for $i = 1, 2, \cdots, n$.

3. Examples involving dynamical systems

In this section we apply the MHPM to two different dynamical systems and find their solutions for varying values of the system parameters. Then we compare our results with those found by Runge-Kutta order 4.

First, we consider the second order Helmholtz equation in ODE form. This equation often arises in the study of physical problems involving partial differential equations. This equation, in the form of a second order ODE, results from separating the variables in the original equation [18]. This equation, as a dynamical system with the bifurcation parameter α, can be written in the form

$$z''(t) + z(t) + \alpha z^2(t) = 0, \quad (10)$$

subject to initial values $z(0) = c_1$ and $z'(0) = c_2$.

Converting this equation to a system of first order ODEs yields

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -x_1(t) - \alpha x_1^2(t), \end{cases} \quad (11)$$

with initial conditions $x_1(0) = c_1$ and $x_2(0) = c_2$.

We apply the first step of MHPM to express system (11) as

$$\begin{cases} x_1'(t) - v_1'(t) + pv_1'(t) + p[-x_2(t)] = 0 \\ x_2'(t) - v_2'(t) + pv_2'(t) + p[x_1(t) + \alpha x_2^2(t)] = 0, \end{cases} \quad (12)$$

with

$$\begin{cases} v_1(t) = x_1(0) = c_1 \\ v_2(t) = x_2(0) = c_2. \end{cases} \quad (13)$$

For the second step of MHPM, we let

$$\begin{cases} x_1(t) = u_{1,0}(t) + pu_{1,1}(t) + p^2u_{1,2}(t) + p^3u_{1,3}(t) + \cdots \\ x_2(t) = u_{2,0}(t) + pu_{2,1}(t) + p^2u_{2,2}(t) + p^3u_{2,3}(t) + \cdots . \end{cases} \quad (14)$$

Note that for simplicity we use t instead of t^*. Then substituting (14) into (12) and rearranging the coefficients of the same power of p, we get

$$\begin{cases} u'_{1,1}(t) + v'_1(t) - u_{2,0}(t) = 0 \\ u'_{2,1}(t) + v'_2(t) + u_{1,0}(t) + \alpha\, u_{1,1}^2 = 0 \end{cases} \quad (15)$$

with initial values

$$\begin{cases} u_{1,0}(0) = v_1(t) = x_1(0) = a \\ u_{2,0}(0) = v_2(t) = x_2(0) = b. \end{cases} \quad (16)$$

We again note that the starting initial values in the first interval are the same as the original values, namely $a = c_1$, $b = c_2$ and the initial values in the j^{th} interval $[t_{j-1}, t_j)$ are the approximate solutions at the endpoint of the $(j-1)^{\text{th}}$ interval; viz. $a = x_1(t_{j-1})$ and $b = x_2(t_{j-1})$.

For powers of p greater than 1 we get

$$\begin{cases} u'_{1,k}(t) - u_{2,k-1}(t) = 0 \\ u'_{2,k}(t) + u_{1,k-1}(t) + \alpha\, u_{1,k-1}^2(t) = 0, \end{cases} \quad (17)$$

for $k \geq 2$ and zero initial values for both sets of components $u_{1,k}$ and $u_{2,k}$. Now integrating systems (15) and (17) from t^* to t with their corresponding initial values we obtain

$$\begin{cases} u_{1,1}(t) = b(t - t^*) \\ u_{2,1}(t) = -(a + \alpha\, a^2)(t - t^*) \end{cases} \quad (18)$$

and

$$\begin{cases} u_{1,k}(t) = \int_{t^*}^{t} u_{2,k-1}(\tau)d\tau \\ u_{2,k}(t) = -\int_{t^*}^{t} \left[u_{1,k-1}(\tau) + \alpha\, u_{1,k-1}^2(\tau) \right] d\tau \end{cases} \quad (19)$$

for $k \geq 2$. It is clear that in integrating (19) for each k, the functions $u_{1,k-1}(t)$ and $u_{2,k-1}(t)$ are known from the previous stage. In this example we calculate these functions for $k = 0, 1, 2, 3, 4$, yielding the approximate analytical solutions for the Helmholtz equation (11) as

$$\begin{aligned} x_1(t) \cong \sum_{k=0}^{4} u_{1,k}(t) &= a + b(t - t^*) \\ &- 0.5(a + \alpha a^2)(t - t^*)^2 - \frac{1}{6} b(t - t^*)^3 \\ &+ \left[-\frac{1}{12}\alpha b^2 + \frac{1}{24}(a + \alpha b^2) \right](t - t^*)^4 \\ &- \frac{\alpha}{120}(a + \alpha b^2)^2 (t - t^*)^6 \end{aligned} \quad (20)$$

and

$$\begin{aligned} x_2(t) \cong \sum_{k=0}^{4} u_{2,k}(t) &= b - (a + \alpha b^2)(t - t^*) - 0.5b(t - t^*)^2 \\ &+ \left[-\frac{1}{3}\alpha b^2 + \frac{1}{6}(a + \alpha b^2) \right](t - t^*)^3 + \frac{1}{24} b(t - t^*)^4 \\ &+ \left[-\frac{\alpha}{20}(a + \alpha b^2)^2 + \frac{1}{60}\alpha b^2 \right](t - t^*)^5 - \frac{\alpha}{252} b^2 (t - t^*)^7 \\ &- \frac{\alpha^2}{288} b^3 (t - t^*)^8 - \frac{\alpha^3}{1296} b^4 (t - t^*)^9. \end{aligned} \quad (21)$$

Our next step is to determine the solutions $x_1(t)$ and $x_2(t)$ for different values of the bifurcation parameter α and compare those with their numerical counterparts found by Runge-Kutta of order 4. Here, in our MHPM solutions (20-21) we choose $t \in [0, 10]$ and divide this interval to 100 subintervals with $\Delta t = 0.001$ for all subintervals. The solutions of (20) and (21) for differing values of α are illustrated in Figures 1(a) to 1(e). Figures 1(a) and 1(b) show the periodic solutions for the small value $\alpha = 0.01$ and the larger value $\alpha = 0.25$, respectively. These solutions are in complete agreement with those found by using the numerical Runge-Kutta method of order 4. Indeed, the maximum difference between these two sets of solutions through the entire interval [0, 10] is less than 10^{-3}. The same situation exists for the steady state solutions if, for example, the bifurcation parameter is $\alpha = -1$, (see Fig. 1(e)).

For some other positive or negative values of α for which unsteady state solutions exist, the MHPM and Runge-Kutta methods continue to show very similar behavior and the difference between the two solutions in some reasonable interval, say [0, 4], is again less than 10^{-3}.

For our next example we consider the well-known Sprott-S dynamical system, with its rich dynamics and its variety of solutions, from steady state to the strange attractor, for different values of the system parameter [19 & 20].

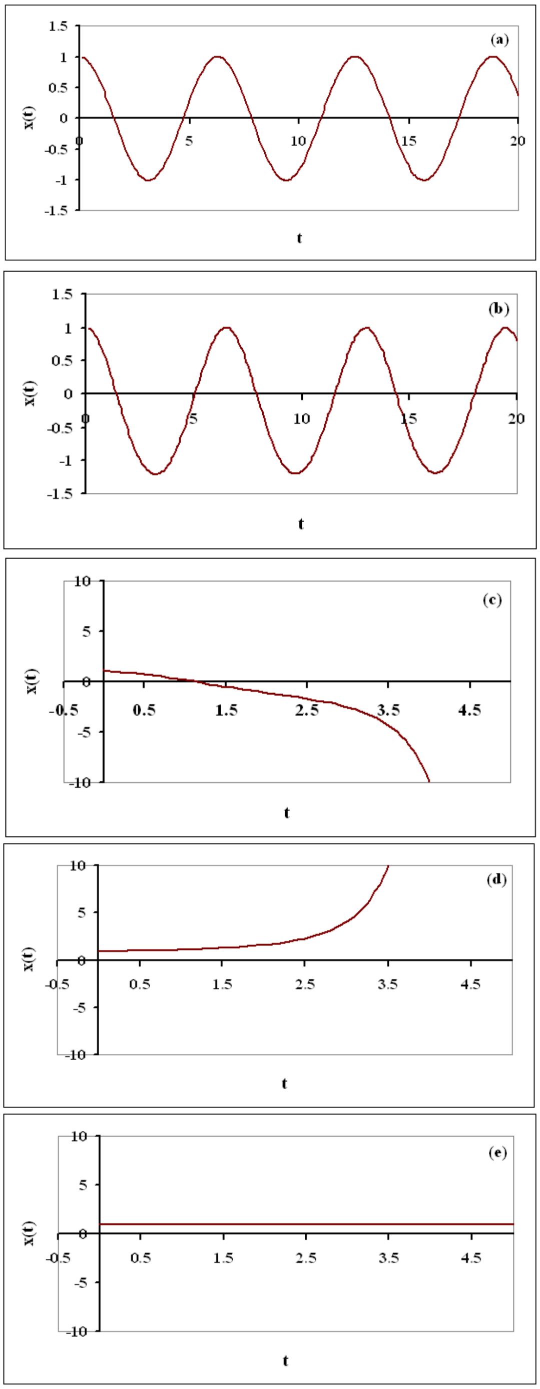


Fig. 1. The solutions of Helmholtz equation using MHPM for different values of bifurcation parameter α. (a) and (b) show the periodic solutions for $\alpha = 0.01$ and $\alpha = 0.25$, respectively. (c) and (d) show the unsteady state solutions for $\alpha = 1$ and $\alpha = -1.2$, respectively. (d) shows the steady state solution for $\alpha = -1$.

In 3-dimensional space, and with α as its bifurcation parameter, this system can be defined by

$$\begin{cases} x_1' = -x_1 - 4x_2 \\ x_2' = x_1 + x_3^2 \\ x_3' = \alpha + x_1, \end{cases} \tag{22}$$

with initial values

$$\begin{cases} x_1(0) = c_1 \\ x_2(0) = c_2 \\ x_3(0) = c_3. \end{cases} \tag{23}$$

Constructing the homotopy for this system yields

$$\begin{cases} u_1' - v_1' + p(v_1' + u_1 + 4u_2) = 0 \\ u_2' - v_2' + p(v_2' - u_1 - u_3^2) = 0 \\ u_3' - v_3' + p(v_3' - \alpha - u_1) = 0, \end{cases} \tag{24}$$

with its initial approximation as

$$\begin{cases} u_1(0) = v_1(t) = x_1(0) = c_1 \\ u_2(0) = v_2(t) = x_2(0) = c_2 \\ u_3(0) = v_3(t) = x_3(0) = c_3. \end{cases} \tag{25}$$

Substituting the same power series of p as in (14) for the three functions $x_1(t), x_2(t)$ and $x_3(t)$ into (24) and collecting like terms, we get

$$\begin{cases} u'_{1,0}(t) - v'_1(t) = 0 \\ u'_{2,0}(t) - v'_2(t) = 0 \\ u'_{3,0}(t) - v'_3 = 0, \end{cases}$$

$$\begin{cases} u'_{1,1}(t) + v'_1(t) + u_{1,0}(t) + 4u_{2,0}(t) = 0 \\ u'_{2,1}(t) + v'_2(t) - u_{1,0}(t) - u^2_{3,0}(t) = 0 \\ u'_{3,1} - v'_3(t) - \alpha - u_{1,0}(t) = 0, \end{cases} \tag{26}$$

with initial values

$$\begin{cases} u_{1,0}(t^*) = a \\ u_{2,0}(t^*) = b \\ u_{3,0}(t^*) = c \end{cases} \quad \text{and} \quad \begin{cases} u_{1,1}(t^*) = 0 \\ u_{2,1}(t^*) = 0 \\ u_{2,1}(t^*) = 0, \end{cases} \tag{27}$$

corresponding to the powers zero and one of p, respectively. Note that here t^* is zero in the first interval $[0, t_1)$, and for the j^{th} interval, $t^* = t_{j-1}$. So in the first interval $[0, t_1)$, $a = c_1, b = c_2$ and $c = c_3$. These initial values in the j^{th} interval are equal to the approximate solutions of x_1, x_2 and

x_3 which have to be evaluated at the point $t^* = t_{j-1}$ through the procedure. The terms corresponding to the powers $k \geq 2$ of p are determined by

$$\begin{cases} u'_{1,k}(t) + u_{1,k-1}(t) + 4u_{2,k-1}(t) = 0 \\ u'_{2,k}(t) - u_{1,k-1}(t) - u^2_{3,k-1}(t) = 0 \\ u'_{3,k-1}(t) - u_{1,k-1}(t) = 0, \end{cases} \quad (28)$$

with all initial values zero. Now integrating the systems (26) for $k = 0, 1$ and (28) for $k = 2, 3, 4$ over the interval (t^*, t), and adding all five terms together yields the approximate analytical solutions for x_1, x_2 and x_3 as

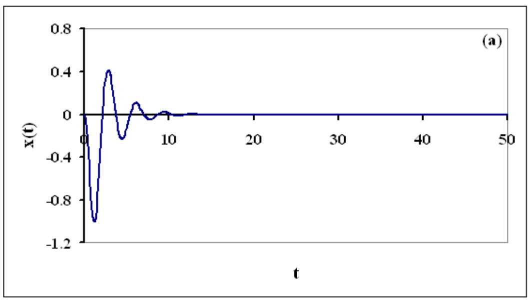
$$\begin{aligned} x_1(t) \cong \sum_{k=0}^{4} u_{1,k}(t) &= a - (a+4b)(t-t^*) + (-\frac{2}{3}a + 2b - 2c^2)(t-t^*)^2 \\ &+ \frac{1}{3}(\frac{7}{2}a + 10b + 2c^2)(t-t^*)^3 - \frac{1}{12}(4\alpha + \frac{3}{2}a + 14b + 6c^2) \\ &.(t-t^*)^4 + \frac{1}{15}(\alpha+a)^2(t-t^*)^5 - \frac{1}{30}(a+4c^2)(t-t^*)^6, \end{aligned} \quad (29)$$

$$\begin{aligned} x_2(t) \cong \sum_{k=0}^{4} u_{2,k}(t) &= b + (a+c^2)(t-t^*) - 0.5(a+4b)(t-t^*)^2 \\ &+ \frac{1}{3}\left[(\alpha+a)^2 - \frac{3}{2}a + 2b - 2c^2\right](t-t^*)^3 + \frac{1}{12}(2a+8b) \\ &.(t-t^*)^4 + \left[\frac{1}{25}(a+4b)^2 - \frac{1}{15}(\alpha+a)\right](t-t^*)^5 \end{aligned} \quad (30)$$

and

$$\begin{aligned} x_3(t) \cong \sum_{k=0}^{4} u_{3,k}(t) &= c + (\alpha+a)(t-t^*) - 0.5(a+4b)(t-t^*)^2 \\ &+ \frac{1}{3}(-\frac{3}{2}a + 2b - 2c^2)(t-t^*)^3 + \frac{1}{4}(\frac{7}{6}a + 2b + \frac{2}{3}c^2) \\ &.(t-t^*)^4 - \frac{1}{12}(\alpha+a)^2(t-t^*)^5. \end{aligned} \quad (31)$$

In order to compare our solutions with the numerical solution found by Runge-Kutta of order 4, we choose the interval $[0, 50]$ with $\Delta t = 0.001$ in each subinterval with length 0.1. The results are illustrated in Figures 2(a) to 2(e) for various values of α.



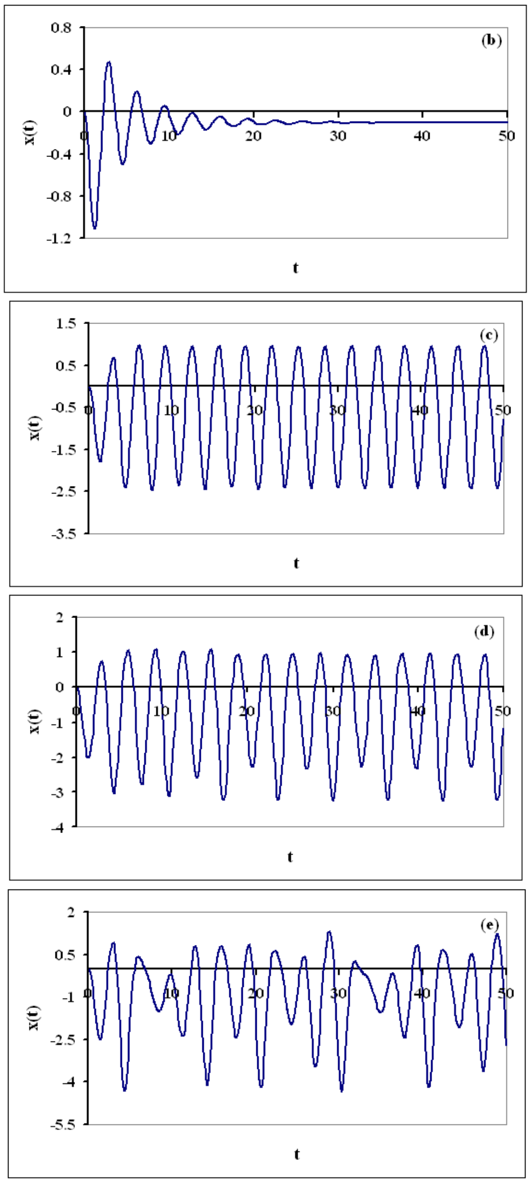


Fig. 2. The solutions of Sprott-S equation using MHPM for different values of system parameter α. (a) and (b) are the study state solutions for $\alpha = 0.0$ and $\alpha = 0.1$, respectively. (c) shows the periodic solution for $\alpha = 0.6$ and (d) shows double periodic solution for $\alpha = 0.75$. The chaotic solution is in (e) for parameter value $\alpha = 1$.

We can see in Figures 2(a) and 2(b) that, for values $\alpha = 0$ and 0.1, the solutions will be steady state after some fluctuation. As we can see in Figures 2(c) and 2(d), different periodic solutions appear for larger values of α in the interval (0.4, 0.8). The double periodic solution in Figure 2(d) is an indication of a chaotic solution which occurs for $\alpha = 1$. This chaotic solution is illustrated in Figure 2(e). All these solutions are in complete agreement with those numerical solutions found by other software packages, such as PHASER [21], and by other numerical schemes. Some of these results can be found in [19-20 and 22].

4. Conclusion

The MHPM can be applied to various dynamical systems to determine their analytical solutions in the form of power series. As we have seen in this article, these solutions are very useful and exhibit sufficient accuracy to analyze the numerical bifurcation of the systems. As discussed elsewhere, the closed form of the solutions can be found by using MHPM for some differential equations [2-4, 23 & 24]. Nevertheless, in the case at hand of highly nonlinear terms in the systems, the MHPM gives very complicated and long series solutions which may create large computational error in evaluating the solution at a specific point in the given interval.

Acknowledgments

This work is partially supported by Qatar National Research Fund under the Grant No. NPRP-08-056-01-014.

References

[1] He, J. H. (1998). Variational iteration method for solving Burger's and coupled Burger's equations. *Comput. Methods Appl. Mech. Engrg. 167,* 57-68.

[2] He, J. H. (1999). Homotopy perturbation technique. *Comput. Meth. Appl. Mech. Eng., 178*, 257-262.

[3] He, J. H. (2000). A coupling method of homotopy technique and perturbation technique for nonlinear problems. *Int. J. Non-Linear Mech., 35*(1), 37-43.

[4] He, J. H. (2003). Homotopy perturbation method: a new nonlinear technique. *Appl. Math. Comput. 135*, 73-79.

[5] He, J. H. (2005). Application of homotopy perturbation method to nonlinear wave equation. *Chaos, Solitons Fractals 26*, 695-700.

[6] Chowdhury, M. S. H., Hashim, I. & Momani, S. (2009). The multistage homotopy perturbation method: A powerful scheme for handling the Lorenz system, *Chaos, Solitons Fractals, 40*(4), 1929-1937.

[7] He, J. H. (2006). Homotopy perturbation method for solving boundary value problems. *Int. J. Phys. Lett. A 20*, 1141.

[8] He, J. H. (2000). A coupling method of homotopy technique and perturbation technique for nonlinear problems. *Int. J. Non-Linear Mech., 35*(1), 37-43.

[9] Cveticanin, L. (2006). Homotopy-perturbation method for pure nonlinear differential equation. *Chaos Solitons & Fractals*, *30*(5), 1221-1230.

[10] He, J. H. (2006). New interpretation of homotopy perturbation method. *International Journal of Modern Physics B, 20*(18), 2561-2568.

[11] Beléndez, A. & Hernández, T. (2007). Application of He's homotopy perturbation method to the Duffing-harmonic oscillator. *International Journal of Non-linear Sciences and Numerical Simulation, 8*(1), 79-88.

[12] Shou, D. H. (2007). Application of parameter-expanding method to strongly nonlinear oscillators. *International Journal of Non-linear Sciences and Numerical Simulation, 8*(1), 121-124.

[13] Öziş, T. & Yıldırım, A. (2007). A comparative study of He’s homotopy perturbation method for determining frequency-amplitude relation of a nonlinear oscillator with discontinuities. *International Journal of Non-linear Sciences and Numerical Simulation, 8*(2), 243-248.

[14] Beléndez, A., Pascual, C., Gallego, S., Ortuño, M. & Neipp, C. (2007). Application of a modified He’s homotopy perturbation method to obtain higher-order approximations of an $x^{1/3}$ force nonlinear oscillator. *Physics Letters A, 371*, 421-426.

[15] Adomian, G. (1988). Nonlinear stochastic systems theory and application to physics. *Dordrecht: Reidel*.

[16] Hashim, I., Noorani, M., Ahmad, R., Baker, S. A., Ismail, E. S. & Zajaria, A. M. (2006). Accuracy of the Adomian decomposition method applied to the Lorenz system. *Chaos Solitons Fractals, 28*, 1149-1158.

[17] Noorani, M., Hashim, I., Ahmad, R., Baker, S. A., Ismail, E. S. & Zajaria, A. M. (2007). Comparing numerical methods for the solutions of the Chen system. *Chaos Solitons Fractals, 32*, 1296-1304.

[18] Riley, K. F., Hobson, M. P. & Bence, S. J. (2002). Mathematical methods for physics and engineering. *Cambridge University Press, ch. 19*, ISBN 0-521-89067-5.

[19] Sprott, J. C. (2000). Some simple chaotic flows. *Phys. Rev. E50,* 647-650.

[20] Zhou, C. & Lai, C. H. (1998). On synchronization with positive conditional Lyapunov exponents. *Phys. Rev. E58,* 5188-5191.

[21] PHASER, (2005). A Universal Simulator for Dynamical Systems, http://www.phaser.com.

[22] Erjaee, G. H., Atabazadeh, M. H. & Saha, L. M. (2004). Interesting synchronization-like behaviour. *Int. J. of Bifurcation and Chaos*, *14*(4), 1447-1453.

[23] Beléndez, A. & Hernández, T. (2007). Application of He's homotopy perturbation method to the Duffing-harmonic oscillator. *Int. J. of Non-linear Sciences and Numerical Simulation, 8*(1), 79-88.

[24] Momani, S. & Ertürk, V. S. (2008). Solutions of non-linear oscillators by the modified differential transform method. *Computers & Math. with Applications*, *55*(4), 833-842.