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Solutions of the perturbed Klein-Gordon equations

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Abstract

This paper studies the perturbed Klein-Gordon equation by the aid of several methods of integrability. There are six forms of nonlinearity that are considered in this paper. The parameter domains are thus identified.

Keywords*:* The perturbed Klein-Gordon equation; integrability; nonlinearity

1. Introduction

The nonlinear Klein-Gordon equation (KGE) appears in Theoretical Physics in the context of relativistic quantum mechanics. There have been several studies conducted with this equation by Physicists and Applied Mathematicians across the globe [1-10]. One of the most important tasks is to carry out the integration of the perturbed KGE. This paper will focus on obtaining the solution of the perturbed KGE by the aid of G'/G method, exp-function method and, finally, the traveling wave solution will be obtained. There are six types of nonlinearities of the KGE that will be considered in this paper.

2. Mathematical Analysis

The dimensionless form of the nonlinear KGE that studied in this paper is given by [4]

$$q_{tt} - k^2 q_{xx} + F(q) = 0 . \tag{1}$$

Here, in (1), the dependent variable q represents the wave profile, while x and t are the independent variables that represent the spatial and temporal variables respectively. Also, k is a real valued constant.The function $F(q)$ represents the nonlinear function. The mathematical analysis of the nonlinear function will now be related to the nonlinear function of the sine-Gordon equation (SGE).This will allow us to justify the study of perturbation terms that arise in the theory of long Josephson junctions that are modeled by the SGE.

2.1. Six Forms of Nonlinearity

There are six types of nonlinearity to be considered. They are labeled as Forms I through VI. For each of these forms, the connection to SGE will be illustrated in the next six subsections. In each of the following forms of nonlinearity, a, b and c are real valued constants. The exponent n is a positive integer.

2.1.1. Form-I

In this case, the nonlinear function $F(q)$ is given by

$$F(q) = aq - bq^2 \tag{2}$$

Equation (2) can be approximated by

$$q_{tt} - k^2 q_{xx} + a\sin q + b_1 \cos q + b_2 \cos 2q = 0 \tag{3}$$

for small values of q, where b_1 and b_2 must be chosen such that there is no constant term and the quadratic term has coefficient $-b$. For better approximations, higher terms such as $\sin 2q$ must be added.

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2.1.2. Form-II

In this case,

$$F(q) = aq - bq^3 \tag{4}$$

and KGE with this form is sometimes known as the Φ^4 equation . This form can be approximated by

$$q_{tt} - k^2 q_{xx} + a_1 \sin q + a_2 \sin 2q = 0 \tag{5}$$

for small values of q, where a_1 and a_2 must be chosen such that the linear coefficient is a and the cubic coefficient is $-b$. For better approximations, higher terms such as $\sin 3q$ must be added.

2.1.3. Form-III

Here,

$$F(q) = aq - bq^n \tag{6}$$

This encompasses the previous two forms in a generalized format. The special case is when $n = 1$ and $n = 2$ collapses to the previous two forms. If n is odd, then this can be approximated by

$$q_{tt} - k^2 q_{xx} + a_1 \sin q + a_2 \sin 2q + \ldots + a_{\frac{n+1}{2}} \sin\left(\left[\frac{n+1}{2}\right] q\right) = 0 \tag{7}$$

for small values of q, where a_1 through $a_{(n+1)/2}$ must be chosen such that only the linear term and q^n survive and have coefficients a and $-b$ respectively. For better approximations, higher terms must be added. If n is even, then this can be approximated by

$$q_{tt} - k^2 q_{xx} + a_1 \sin q + a_2 \sin 2q + \ldots + a_{\frac{n}{2}-1} \sin\left(\left[\frac{n}{2}+1\right] q\right) + b_1 \cos q + b_2 \cos 2q + \ldots + b_{\frac{n}{2}+1} \cos\left(\left[\frac{n}{2}+1\right] q\right) = 0 \tag{8}$$

for small values of q, where a_1 through $a_{n/2-1}$ must be chosen such that only the linear term survives with coefficient a, and b_1 through $b_{n/2+1}$ must be chosen such that there are no even terms until the q^n term and it has coefficient $-b$. For better approximations, higher terms must be added.

2.1.4. Form-IV

Here,

$$F(q) = aq - bq^n + cq^{2n-1} \tag{9}$$

In this case, if n is odd, then this can be approximated by

$$q_{tt} - k^2 q_{xx} + a_1 \sin q + a_2 \sin 2q + \ldots + a_n \sin nq = 0 \tag{10}$$

for small values of q, where a_1 through a_n must be chosen to exactly satisfy the coefficients of q, q^n, and q^{2n-1}. For better approximations, higher terms must be added. If n is even, then this can be approximated by

$$q_{tt} - k^2 q_{xx} + a_1 \sin q + a_2 \sin 2q + \ldots + a_n \sin nq + b_1 \cos q + b_2 \cos 2q + \ldots + b_{\frac{n}{2}+1} \cos\left(\left[\frac{n}{2}+1\right] q\right) = 0 \tag{11}$$

for small values of q, where a_1 through a_n must be chosen to exactly satisfy the coefficients of q and q^{2n-1}, and b_1 through $b_{n/2+1}$ must be chosen such that there are no even terms until the q^n term and it has coefficient $-b$. For better approximations, higher terms must be added.

2.1.5. Form-V

For this form,

$$F(q) = aq - bq^{1-n} + cq^{n+1}. \tag{12}$$

This is similar to the other forms, and can be approximated by a series of sine and cosine terms.

2.1.6. Form-VI

This is the logarithmic form of nonlinearity. In this case [6],

$$F(q) = aq + bq \ln q \tag{13}$$

Define

$$f(x) = x \log x^2 \tag{14}$$

It is clear that f is an odd function, thus the Fourier series collapses to just the sine terms

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \tag{15}$$

where the coefficients are defined by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \tag{16}$$

Solving these coefficients leads to the sine expansion of $x\log x^2$

$$x\log x^2 = 4\sum_{n=1}^{\infty}[\frac{(-1)^{n+1}}{n}\log(\pi)$$
$$-\frac{1}{\pi n^2}Si(n\pi)]\sin(nx) \tag{17}$$

where Si is the sine integral defined by

$$Si(x) = \int_0^x \frac{\sin t}{t}dt \tag{18}$$

2.2. Perturbation Terms

The perturbed KGE that will be studied in this paper is given by [4]

$$q_{tt} - k^2 q_{xx} + F(q) = \alpha q + \beta q_t + \gamma q_x$$
$$+\delta q_{xt} + \lambda q_{tt} + \sigma q_{xxt} + \nu q_{xxxx} \tag{19}$$

These perturbation terms typically arise in the study of long Josephson junction in the context of sine-Gordon equation (SGE). Since SGE can be approximated by KGE as seen in the previous sub-section, an exact solution of the perturbed KGE will make sense in this context [9].

For the perturbation terms, α represents losses across the junction, β accounts for dissipative losses in Josephson junction theory due to tunneling of normal electrons across the dielectric barrier, γ is generated by a small inhomogeneous part of the local inductance, δ represents diffusion and λ is the capacity inhomogeneity. Finally, σ accounts for losses due to a current along the barrier [9].

In this paper, therefore the perturbed KGE, given by (19) for the six forms of nonlinearity as defined in (2)-(13), will be integrated in the subsequent three sections.

3. Traveling Wave Solutions

In this section, the traveling wave solutions will be obtained. This is the most fundamental approach to the solution of the perturbed KGE. The saerach will be for soliton solutions only, in this section. After a general description of the method, the six forms of nonlinearity will be considered in separate subsections.

3.1. Description of the method

The perturbed KGE with general form of nonlinearity is given by Equation (19). For traveling wave solution the hypothesis

$$q(x,t) = g(x - vt) \tag{20}$$

is substituted into the (19) where v represents the velocity of the wave and $g(x,t)$ represents the wave profile. Thus equation (19) reduces to

$$\left(v^2 - k^2 + \delta v - \lambda v^2\right)g'' + F(g) - \alpha g$$
$$+\left(\beta v - \gamma\right)g' + \sigma v g''' - \nu g'''' = 0 \tag{21}$$

where $g' = dg/ds$, $g'' = d^2g/ds^2$ and so on. Here,

$$s = x - vt \tag{22}$$

In order to solve (21) by the traveling wave hypothesis, it is necessary to set

$$\sigma = \beta = \gamma = \nu = 0 \tag{23}$$

Thus equation of study given by (19) reduces to

$$q_{tt} - k^2 q_{xx} + F(q) = \alpha q + \delta q_{xt} + \lambda q_{tt} \tag{24}$$

and hence (21) reduces to

$$\left(v^2 - k^2 + \delta v - \lambda v^2\right)g'' + F(g) - \alpha g = 0. \tag{25}$$

Multiplying both sides by g' and choosing the integration constant to be zero, since the search is for soliton solution, gives

$$\left(v^2 - k^2 + \delta v - \lambda v^2\right)\left(g'\right)^2$$
$$+2\int g' F(g)ds - \alpha g^2 = 0. \tag{26}$$

Separating variables and integrating leads to

$$\frac{x - vt}{\sqrt{v^2 - k^2 + \delta v - \lambda v^2}}$$
$$= \int \frac{dg}{\left[\alpha g^2 - 2\int g' F(g)ds\right]^{\frac{1}{2}}}. \tag{27}$$

This expression will be evaluated for the six different functions F that will be studied in the following subsections.

3.1.1. Form-I

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^2 = \alpha q + \delta q_{xt} + \lambda q_{tt} \quad (28)$$

for the nonlinearity that is given by (2). This is the quadratic form of nonlinearity and hence (28) is the perturbed Φ^4 equation. In this case (27) reduces to

$$\frac{x-vt}{\sqrt{v^2-k^2+\delta v-\lambda v^2}} = \sqrt{3}\int \frac{dg}{g\left\{3(\alpha-a)+2bg\right\}^{\frac{1}{2}}} \quad (29)$$

which gives, upon integration

$$\frac{x-vt}{\sqrt{v^2-k^2+\delta v-\lambda v^2}} = -\frac{2}{\sqrt{\alpha-a}}\tanh^{-1}\left\{\sqrt{1+\frac{2bg}{3(\alpha-a)}}\right\} \quad (30)$$

which leads to the soliton solution

$$g(x-vt) = q(x,t) = A\,sech^2[B(x-vt)], \quad (31)$$

where the amplitude A and the inverse width B of the soliton are given by

$$A = \frac{3(a-\alpha)}{2b} \quad (32)$$

and

$$B = \frac{1}{2}\sqrt{\frac{\alpha-a}{v^2-k^2+\delta v-\lambda v^2}}, \quad (33)$$

respectively. The width of the soliton forces the constraint

$$(\alpha-a)\left(v^2-k^2+\delta v-\lambda v^2\right) > 0 \quad (34)$$

3.1.2. Form-II

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^3 = \alpha q + \delta q_{xt} + \lambda q_{tt} \quad (35)$$

which is from the nonlinearity that is given by (4). Thus, in this case, using (27) we obtain

$$\frac{x-vt}{\sqrt{v^2-k^2+\delta v-\lambda v^2}} = -\frac{1}{\sqrt{\alpha-a}}\tanh^{-1}\left\{\sqrt{1+\frac{bg^2}{2(\alpha-a)}}\right\} \quad (36)$$

which leads to the soliton solution

$$g(x-vt) = q(x,t) = A\,sech[B(x-vt)], \quad (37)$$

where the amplitude of the soliton is given by

$$A = \sqrt{\frac{2(a-\alpha)}{b}} \quad (38)$$

and

$$B = \frac{1}{2}\sqrt{\frac{\alpha-a}{v^2-k^2+\delta v-\lambda v^2}} \quad (39)$$

is the width of the soliton. Hence the constraint (34) is still valid and it is additionally necessary that

$$b(\alpha-a) < 0 \quad (40)$$

must also hold.

3.1.3. Form-III

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^n = \alpha q + \delta q_{xt} + \lambda q_{tt}. \quad (41)$$

Here equation (27) yields

$$\frac{x-vt}{\sqrt{v^2-k^2+\delta v-\lambda v^2}} = -\frac{2}{(n-1)\sqrt{\alpha-a}}\times \tanh^{-1}\left\{\sqrt{1+\frac{2bg^{n-1}}{(n+1)(\alpha-a)}}\right\}. \quad (42)$$

After simplification, this implies that the 1-soliton solution to (41) is given by

$$q(x,t) = g(x-vt) = A\,sech^{\frac{2}{n-1}}[B(x-vt)], \quad (43)$$

where the amplitude of the soliton is given by

$$A = \left[\frac{(n+1)(a-\alpha)}{2b}\right]^{\frac{1}{n-1}} \quad (44)$$

and the width B is

$$B = \frac{(n-1)}{2}\sqrt{\frac{\alpha - a}{v^2 - k^2 + \delta v - \lambda v^2}} \quad (45)$$

which again requires the same constraint in (34).

3.1.4. Form-IV

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^n + cq^{2n-1} = \alpha q + \delta q_{xt} + \lambda q_{tt}. \quad (46)$$

In this case, equation (27) gives

$$\frac{x - vt}{\sqrt{v^2 - k^2 + \delta v - \lambda v^2}} = \frac{1}{(n-1)\sqrt{\alpha - a}} \ln\left(\frac{g^{n-1}}{G}\right), \quad (47)$$

where

$$G = \frac{2b}{(n+1)(\alpha - a)} g^{n-1} + 2 + 2\sqrt{1 - \frac{2b}{(n+1)(\alpha - a)} g^{n-1} + \frac{c}{n(\alpha - a)} g^{2n-2}} \quad (48)$$

which finally leads to

$$q(x,t) = g(x - vt) = \frac{A}{\{1 + C\cosh[B(x - vt)]\}^{\frac{1}{n-1}}}, \quad (49)$$

where

$$A = \left[-\frac{(n+1)(\alpha - a)}{b}\right]^{\frac{1}{n-1}} \quad (50)$$

$$B = (n-1)\sqrt{\frac{\alpha - a}{v^2 - k^2 + \delta v - \lambda v^2}} \quad (51)$$

$$C = -\frac{2(n+1)(\alpha - a)}{b} \quad (52)$$

with the constraints (34) and

$$b(\alpha - a) < 0 \quad (53)$$

$$\frac{4b^2}{(n+1)^2(\alpha - a)^2} + \frac{4c}{n(\alpha - a)} = 1. \quad (54)$$

3.1.5. Form-V

For this form, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^{1-n} + cq^{n+1} = \alpha q + \delta q_{xt} + \lambda q_{tt} \quad (55)$$

Here, equation (27) reduces to give

$$\frac{x - vt}{\sqrt{v^2 - k^2 + \delta v - \lambda v^2}} = \frac{A}{B}\Pi\left(i\sinh^{-1} C \middle| D\right), \quad (56)$$

where

$$A = 2i\sqrt{g^n + \frac{2J}{1 - \sqrt{1 + 4JK}}} \times \sqrt{g^n + \frac{2J}{1 + \sqrt{1 + 4JK}}} \quad (57)$$

$$B = n\sqrt{1 + Jg^{-n} - Kg^n} \times \sqrt{\frac{2Jg^n}{1 + \sqrt{1 + 4JK}}} \quad (58)$$

$$C = g^{-\frac{n}{2}}\sqrt{\frac{2J}{1 + \sqrt{1 + 4JK}}} \quad (59)$$

$$D = \frac{1 + \sqrt{1 + 4JK}}{1 - \sqrt{1 + 4JK}} \quad (60)$$

$$J = \frac{2b}{(2-n)(\alpha - a)} \quad (61)$$

$$K = \frac{2c}{(2+n)(\alpha - a)} \quad (62)$$

and Π is the elliptic integral of the third kind that is defined as

$$\Pi(n;\phi \mid k) = \int_0^{\sin\phi} \frac{dt}{\left(1 - nt^2\right)\sqrt{(1 - t^2)(1 - k^2 t^2)}}.$$

3.1.6. Form-VI

For this form, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq + bq\ln q = \alpha q + \delta q_{xt} + \lambda q_{tt}. \quad (63)$$

Here, from (27) we get

$$\frac{x-vt}{\sqrt{v^2-k^2+\delta v-\lambda v^2}} = \frac{2}{b}\left\{\left(\alpha - a + \frac{b}{2}\right) - b\ln g\right\}^{\frac{1}{2}} \quad (64)$$

Upon simplifying (64), the Gaussons for the perturbed KGE are given by

$$q(x,t) = g(x-vt) = Ae^{-B^2(x-vt)^2}, \quad (65)$$

where the amplitude A and width B of the Gausson are given by

$$A = \exp\left\{\frac{\alpha - a}{b} + \frac{1}{2}\right\} \quad (66)$$

and

$$B = \frac{1}{2}\sqrt{\frac{b}{v^2-k^2+\delta v-\lambda v^2}} \quad (67)$$

which forces the constraint condition given by

$$b\left(v^2-k^2+\delta v-\lambda v^2\right) > 0 \quad (68)$$

to hold in order for the Gaussons to exist.

4. G'/G Method

In this section, the G'/G method will be adopted to carry out the integration of the perturbed KGE. This study will be divided into the following six subsections based on the types of nonlinearity that are being studied.

4.1. Description of the method

The objective of this section is to outline the use of the G'/G-expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose that we have a nonlinear PDE for $u(x,t)$, in the form of

$$P(q, q_x, q_t, q_{xx}q_{xt}, q_{tt}, \cdots) = 0 \quad (69)$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. The wave transformation

$$q(x,t) = Q(\xi), \xi = B(x-vt), \quad (70)$$

reduces Eq. (69) to the ordinary differential equation (ODE)

$$P(Q, BQ', -vBQ', B^2Q'', -vB^2Q'', v^2B^2Q'', \cdots) = 0 \quad (71)$$

and restricts the general solutions on travelling wave solutions, where $U = U(\xi)$, and prime denotes the derivative with respect to ξ. We assume that the solution of Eq. (71) can be expressed by a polynomial in G'/G as follows:

$$Q(\xi) = \sum_{l=0}^{m} a_l \left(\frac{G'}{G}\right)^l, \ a_m \neq 0 \quad (72)$$

where $a_l, l = 0,1,\cdots,m$ are constants to be determined later and $G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda\frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0. \quad (73)$$

where λ and μ are arbitrary constants. Using the general solutions of Eq. (73), we have

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2-4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{c_1\cosh(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + c_2\sinh(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{c_1\sinh(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + c_2\cosh(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2-4\mu < 0, \\ \frac{c_1}{c_1\xi + c_2} - \frac{\lambda}{2}, \ \lambda^2-4\mu = 0 \end{cases}$$

where c_1 and c_2 are arbitrary constants.
To determine q explicitly, we take the following four steps:

Step 1. Determine the integer m by substituting Eq. (72) along with Eq. (73) into Eq. (71), and balance the highest order nonlinear term(s) and the highest order partial derivative.

Step 2. Substitute Eq. (72) with the value of m determined in Step 1, along with Eq. (73) into Eq. (71) and collect all terms with the same order of G'/G together, the left-hand side of Eq. (71) is converted into a polynomial in G'/G. Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for v, B, λ, μ and a_l for $l = 0,1,\cdots,m.$

Step 3. Solve the system of algebraic equations obtained in Step 2, for v, B, λ, μ and a_l, for $l = 0,1,\cdots,m$ by using Maple.

Step 4. Use the results obtained in the above mentioned steps to derive a series of fundamental solutions $Q(\xi)$ of Eq. (71) depending on G'/G, since the solutions of Eq. (73) are well known to us, then we can obtain the exact solutions of Eq. (69).

4.2. Applications to Perturbed KGE

In this section, we will demonstrate the G'/G -expansion method on several forms of the perturbed KGE.

4.2.1. Form-I

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^2 = \alpha q + \beta q_t + \gamma q_x + \delta q_{xt} + \omega q_{tt} \tag{74}$$

which by the wave transformation (70) is converted to

$$(v^2 - k^2 + \delta v - \omega v^2)B^2 Q'' + (a-\alpha)Q - bQ^2 + (\beta v - \gamma)BQ' = 0, \tag{75}$$

where k, a, b, α, β, γ, δ and ω are constants. According to step 1, we get $m+2 = 2m,$ hence $m = 2$.

We then assume that Eq. (21) has the following formal solutions:

$$Q = a_2\left(\frac{G'}{G}\right)^2 + a_1\left(\frac{G'}{G}\right) + a_0, \tag{76}$$

where $a_0, a_1,$ and a_2 are constants which are unknown to be determined by solving the set of algebraic equations obtained by step 2 by use of Maple, we get the following results:

$$a_0 = \frac{-1}{2b\beta^2}(3\lambda^2 B^2 z + \beta^2(a-\alpha)) \tag{77}$$

$$a_1 = \frac{-6}{5b}B(5\lambda B y + \beta v - \gamma) \tag{78}$$

$$a_2 = \frac{-6B^2 y}{b} \tag{79}$$

$$\mu = \frac{\beta^2(a-\alpha) + \beta^2\lambda^2 z}{4B^2 z}. \tag{80}$$

where

$z = \beta^2 k^2 - \gamma\beta\delta + \gamma^2\omega - \gamma^2$, $y = k^2 - v^2 - v\delta + v^2\omega.$

When $\Delta = \lambda^2 - 4\mu = \dfrac{\beta^2(a-\alpha)}{B^2 z} > 0$ we obtain hyperbolic function solutions:

$$q(x,t) = Q(\xi) = d_0 + d_1 f(\xi) + d_2 f(\xi)^2, \tag{81}$$

where

$$d_1 = \frac{-6B(\beta v - \gamma)}{5b}, \tag{82}$$

$$d_2 = \frac{-6B^2 y}{b}, \tag{83}$$

$$d_0 = \frac{15B^2\lambda^2\theta + \beta^2(-5(\alpha - a) + 6B\lambda(\beta v - \gamma))}{10\beta^2 b}, \tag{84}$$

$$f(\xi) = \frac{\varepsilon(c_1\sinh(\varepsilon\xi) + c_2\cosh(\varepsilon\xi))}{(c_1\cosh(\varepsilon\xi) + c_2\sinh(\varepsilon\xi))}, \tag{85}$$

and $\theta = (-\omega+1)(\gamma^2 - \beta^2 v^2) - \beta^2 v\delta + \gamma\beta\delta,$ $\varepsilon = \dfrac{1}{2}\sqrt{\dfrac{\beta^2(a-\alpha)}{B^2 z}}$ and $\xi = x - vt$, $v = \dfrac{\gamma}{\beta}.$

When $\Delta = \lambda^2 - 4\mu = \dfrac{\beta^2(a-\alpha)}{B^2 y} < 0$ we obtain trigonometric function solutions:

$$q(x,t) = Q(\xi) = d_0 + d_1 f(\xi) + d_2 f(\xi)^2, \tag{86}$$

where

$$f(\xi) = \frac{\varepsilon(-c_1\sin(\varepsilon\xi) + c_2\cos(\varepsilon\xi))}{(c_1\cos(\varepsilon\xi) + c_2\sin(\varepsilon\xi))}. \tag{87}$$

When $\Delta = \lambda^2 - 4\mu = \dfrac{\beta^2(a-\alpha)}{B^2 y} = 0$, the rational function solutions obtained as:

$$q(x,t) = Q(\xi) = d_0 + d_1 f(\xi) + d_2 f(\xi)^2, \tag{88}$$

where

$$f(\xi)=\frac{c_2}{c_1+c_2\xi}. \tag{89}$$

and d_0, d_1, d_2, ε in (86) and (88) have the same values as Eq. (82), (83), (84) and λ, c_1, c_2, B are arbitrary constants.

4.2.2. Form-II

In this case, the perturbed KGE that will be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^3=\alpha q+\beta q_t+\gamma q_x$$
$$+\delta q_{xt}+\omega q_{tt} \tag{90}$$

where by the wave transformation (70) is converted to

$$yB^2Q''+(a-\alpha)Q-bQ^3+(\beta v-\gamma)BQ'=0. \tag{91}$$

Balancing Q'' and Q^3 gives $m+2=3m$, hence $m=1$. Then assuming

$$Q=a_1\frac{G'}{G}+a_0 \tag{92}$$

where $a_0=\pm\frac{\sqrt{-2by}(3By\lambda+v\beta-\gamma)}{6by}$ (93)

$$a_1=\pm\frac{\sqrt{-2by}B}{b} \tag{94}$$

$$\mu=\frac{B^2\lambda^2z+2\beta^2(a-\alpha)}{4zB^2}. \tag{95}$$

Here, $y=-v^2+k^2-v\delta+v^2\omega$ and $z=-\gamma^2+k^2\beta^2-\beta\gamma\delta+\gamma^2\omega.$ Therefore,

$$q(x,t)=Q(\xi)=\pm\frac{\sqrt{-2by}(v\beta-\gamma)}{6by}$$
$$\pm\frac{\sqrt{-2by}B\varepsilon}{b}f(\xi)$$

$$f(\xi)=\begin{cases}\dfrac{c_1\sinh(\varepsilon\xi)+c_2\cosh(\varepsilon\xi)}{c_1\cosh(\varepsilon\xi)+c_2\sinh(\varepsilon\xi)}, \\ \lambda^2-4\mu=\dfrac{-2\beta^2(a-\alpha)zB^2>0}{}, \\ \dfrac{-c_1\sin(\varepsilon\xi)+c_2\cos(\varepsilon\xi)}{c_1\cos(\varepsilon\xi)+c_2\sin(\varepsilon\xi)}, \\ \lambda^2-4\mu=\dfrac{-2\beta^2(a-\alpha)}{zB^2}<0, \\ \dfrac{c_1}{c_1\xi+c_2},\lambda^2-4\mu=\dfrac{-2\beta^2(a-\alpha)}{zB^2}=0,\end{cases}$$

and $\varepsilon=\sqrt{\frac{\beta^2(a-\alpha)}{2zB^2}}$, $\xi=x-\frac{\gamma}{\beta}t.$

4.2.3. Form-III

In this case, the perturbed KGE to be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^n=\alpha q+\beta q_t+\gamma q_x$$
$$+\delta q_{xt}+\lambda q_{tt} \tag{96}$$

and, by the wave transformation (94) is reduced to

$$yB^2Q''+(a-\alpha)Q-bQ^n+(\beta v-\gamma)BQ'=0, \tag{97}$$

where $y=v^2-k^2+\delta v-\omega v^2.$

Balancing Q^n and Q'' yields $m+2=nm$ and hence $m=\frac{2}{n-1}$. We use $Q=W^{\frac{2}{n-1}}$, so we have

$$(n-1)^2\left\{-bW^4+(a-\alpha)W^2\right\}$$
$$+2y(n-1)B^2W''W-2y(n-3)B^2\left(W'\right)^2=0. \tag{98}$$

According to step 1, we get $2m+2=4m$, hence $m=1$.We therefore assume that

$$W=a_1\left(\frac{G'}{G}\right)+a_0, \tag{99}$$

where $a_0=\pm\frac{\sqrt{-2by}(3By\lambda+v\beta-\gamma)}{6by}$ (100)

$$a_1=\pm\frac{\sqrt{b(a-\alpha)}}{b} \tag{101}$$

$$v = \frac{\delta \pm \sqrt{\delta^2 - 4k^2\omega + 4k^2}}{2(\omega - 1)}, \beta = 0, \ \gamma = 0. \quad (102)$$

$$Q = [\pm \frac{\sqrt{-2by}(3By\lambda + v\beta - \gamma)}{6by}$$

$$\pm \frac{\sqrt{b(a-\alpha)}}{b}\left\{\frac{-\lambda}{2} + \varepsilon f(\xi)\right\}]^{\frac{2}{n-1}} \text{ and}$$

$$f(\xi) = \begin{cases} \dfrac{c_1 \sinh(\varepsilon\xi) + c_2 \cosh(\varepsilon\xi)}{c_1 \cosh(\varepsilon\xi) + c_2 \sinh(\varepsilon\xi)}, & \lambda^2 - 4\mu > 0, \\ \dfrac{-c_1 \sin(\varepsilon\xi) + c_2 \cos(\varepsilon\xi)}{c_1 \cos(\varepsilon\xi) + c_2 \sin(\varepsilon\xi)}, & \lambda^2 - 4\mu < 0, \\ \dfrac{c_1}{c_1\xi + c_2}, & \lambda^2 - 4\mu = 0, \end{cases}$$

with $\varepsilon = \dfrac{\sqrt{\delta^2 - 4\mu}}{2}$,

$$\xi = x - \frac{\delta \pm \sqrt{\delta^2 - 4k^2\omega + 4k^2}}{2(\omega - 1)} t.$$

5. Exp-Function Method

This section will integrate the perturbed KGE by the exponential function technique that is abbreviated as exp-function approach. The details are in the following subsection below.

5.1. Description of the method

The exp-function method is based on the assumption that traveling wave solutions of Eq. (21) can be expressed in the following form:

$$Q(\xi) = \frac{\sum_{n=-c}^{d} a_n e^{n\xi}}{\sum_{m=-p}^{q} b_m e^{m\xi}}, \quad (103)$$

where c, d, p, and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. To determine the values of c and p, we balance the linear term of highest order in Eq. (21) with the highest order nonlinear term. Similarly, to determine the values of d and q, we balance the linear term of lowest order in Eq. (21) with the lowest order nonlinear term.

5.1.1. Form-I

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^2 = \alpha q + \beta q_t + \gamma q_x$$
$$+\delta q_{xt} + \omega q_{tt} \quad (104)$$

which by the wave transformation, (102) is converted to

$$\left(v^2 - k^2 + \delta v - \omega v^2\right) B^2 Q'' + (a - \alpha) Q$$
$$-bQ^2 + (\beta v - \gamma) B Q' = 0, \quad (105)$$

where k, a, b, α, β, δ and ω are arbitrary constants. Using the ansatz (101), for the linear term of highest order Q^2 by simple calculation, we have

$$Q'' = \frac{c_1 e^{(c+3p)\xi} + \cdots}{c_2 e^{4p\xi} \cdots} \quad (106)$$

and

$$Q^2 = \frac{c_3 e^{(2c+2p)\xi} + \cdots}{c_4 e^{4p\xi} + \cdots}, \quad (107)$$

where c_i are determined coefficients only for simplicity. Balancing highest order of exp-function in Q^2 and Q'', we have

$$2c + 2p = c + 3p, \quad (108)$$

which leads to the result

$$p = c. \quad (109)$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in Eq. (105)

$$Q'' = \frac{d_1 e^{-(3q+d)\xi} + \cdots}{d_2 e^{-4q\xi} + \cdots} \quad (110)$$

and $$Q^2 = \frac{\cdots + d_3 e^{-(2d+2q)\xi}}{\cdots + d_4 e^{-4q\xi}}, \quad (111)$$

where d_i are determined coefficients only for simplicity. Balancing lowest order of exp-function in Eqs. (110) and (111), we have

$$-(3q + d) = -(2d + 2q), \quad (112)$$

which leads to the result

$$d = q. \tag{113}$$

Since the final solution does not strongly depend upon the choice of values of c and d, for simplicity, we set $p = c = 1$ and $d = q = 1$, the trial function, Eq. (101) becomes

$$Q(\xi) = \frac{a_{-1}e^{-\xi} + a_0 + a_1 e^{\xi}}{b_{-1}e^{-\xi} + b_0 + b_1 e^{\xi}}. \tag{114}$$

Substituting Eq. (114) into Eq. (105), and equating to zero the coefficients of all powers of $e^{n\xi}$ yields a set of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0, v, B and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain the following several sets of solutions.

Set 1.

$$a_{-1} = 0,\ a_1 = 0,\ a_0 = \frac{3b_0(a-\alpha)}{b},$$
$$b_{-1} = \frac{b_0^2}{4b_1}, v = \frac{\gamma}{\beta}, B = h\beta, \tag{115}$$

so

$$q(x,t) = Q(\xi) = \frac{3b_0(a-\alpha)}{b(b_1e^{\xi} + b_0 + \frac{1}{4}\frac{b_0^2 e^{-\xi}}{b_1})}, \tag{116}$$

where b, β, $b_1 \neq 0$, $\xi = x - \frac{\gamma}{\beta}t$ and h are the root of

$$\left(-\gamma^2 - \delta\gamma\beta + k^2\beta^2 + \omega\gamma^2\right)Z^2 - a + \alpha = 0. \tag{117}$$

and b_0, b_1 are free parameters, Here and the following cases a, b, k, α, β, γ, ω are free parameters.

Set 2. $a_{-1} = a_1 = b_{-1} = 0, v = h,$

$$a_0 = \frac{b_0(a-\alpha)}{b}, B = \frac{a-\alpha}{-\gamma + \beta h}, \tag{118}$$

so

$$q(x,t) = Q(\xi) = \frac{b_0(a-\alpha)}{b(b_1e^{\xi} + b_0)}, \tag{119}$$

where in the second, third and fourth set of solutions h is the root of

$$(-1+\omega)Z^2 - \delta Z + k^2 = 0 \tag{120}$$

and b_0, b_1 are free parameters with $b \neq 0$ and $-\gamma + \beta h \neq 0$ conditions.

Set 3.

$$a_1 = a_0 = b_1 = 0,\ a_{-1} = \frac{b_{-1}(a-\alpha)}{b},\ v = h, \tag{121}$$

so that

$$q(x,t) = Q(\xi) = \frac{b_{-1}(a-\alpha)e^{-\xi}}{b(b_{-1}e^{-\xi} + b_0)}, \tag{122}$$

where b_0, b_{-1} are free parameters and $b \neq 0$.

Set 4. $a_1 = a_{-1} = b_1 = 0, v = h,$

$$a_0 = \frac{b_0(a-\alpha)}{b}. \tag{123}$$

So $$q(x,t) = Q(\xi) = \frac{b_0(a-\alpha)}{b(b_{-1}e^{-\xi} + b_0)}, \tag{124}$$

where b_0, b_{-1} are free parameters and $b \neq 0$.

5.1.2. Form-II

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^3 = \alpha q + \beta q_t + \gamma q_x$$
$$+\delta q_{xt} + \omega q_{tt} + \sigma q_{xxt} + \theta q_{xxxx} \tag{125}$$

which by the wave transformation (70) is converted to

$$yB^2Q'' + (a-\alpha)Q - bQ^3 + (\beta v - \gamma)BQ'$$
$$-v\sigma B^3 Q''' + \theta B^4 Q'''' = 0, \tag{126}$$

where

$$y = v^2 - k^2 + \delta v - \omega v^2$$

and k, a, b, α, β, γ, θ, δ and ω are all arbitrary constants. Using the ansatz (101) and balancing highest order of exp-function in Q^3 and Q'' leads to the result

$$p = c. \tag{127}$$

Similarly, balancing the linear term of lowest order in Q^3 and Q'' leads to the result which leads to the result

$$d = q. \tag{128}$$

Since the final solution does not strongly depend upon the choice of values of c and d, for simplicity, we set $p = c = 1$, $d = q = 1$, the trial function, Eq. (101) becomes

$$Q(\xi) = \frac{a_{-1}e^{-\xi} + a_0 + a_1 e^{\xi}}{b_{-1}e^{-\xi} + b_0 + b_1 e^{\xi}}. \tag{129}$$

Substituting Eq. (129) into Eq. (126), and equating to zero the coefficients of all powers of $e^{n\xi}$ yields a set of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0, and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain the following several sets of solutions.

Set 1. $a_{-1} = a_0 = 0, b_1 = b_{-1} = 0,$

$$a_1 = a_1, b_0 = b_0, v = \frac{h}{B}. \tag{130}$$

So

$$q(x,t) = Q(\xi) = \frac{a_1 e^{\xi}}{b_0}, \tag{131}$$

where b, $b_0 \neq 0$ and h is a root of

$$(\omega - 1)Z^2 + (-\sigma B^2 - \delta B - \beta)Z - a + B^2k^2 + \alpha + B\gamma + B^4\theta = 0, \tag{132}$$

where a_1, b_0 are free parameters.

Set 2.

$$a_{-1} = a_1 = b_1 = \sigma = c = 0, v = h, \tag{133}$$

$$b_0 = \pm\sqrt{\frac{b}{a-\alpha}}a_0, B = \frac{-3(a-\alpha)}{2(\beta h - \gamma)}, \tag{134}$$

where $b_{-1}, a_0 \neq 0$ and h is the root of

$$\left\{9(\omega-1)(a-\alpha) + 2\beta^2\right\}Z^2 + (-4\gamma\beta + 9\delta(\alpha - a))Z + 2\gamma^2 + 9k^2(a-\alpha) = 0 \tag{135}$$

and $\frac{b}{a-\alpha} \geq 0, \beta h - \gamma \neq 0.$ (136)

So,

$$q(x,t) = Q(\xi) = \frac{a_0}{\pm\sqrt{\frac{b}{a-\alpha}}a_0 + b_{-1}e^{-\xi}}. \tag{137}$$

Set 3.

$$a_{-1} = a_1 = b_0 = 0, b_{-1} = \frac{ba_0^2}{8b_1(a-\alpha)}, \sigma = 0, \theta = 0 \tag{138}$$

$$v = \frac{\gamma}{\beta}, B = h\beta, \tag{139}$$

where a_0, b_1 are free parameters and h is the root of

$$\left(k^2\beta^2 - \gamma^2 - \delta\gamma\beta + \omega\gamma^2\right)Z^2 - a + \alpha = 0, \tag{140}$$

and

$$q(x,t) = Q(\xi) = \frac{a_0}{b_1 e^{\xi} + \frac{ba_0^2}{8b_1(a-\alpha)}e^{-\xi}}. \tag{141}$$

Set 4. $a_{-1} = a_1 = b_1 = \sigma = c = 0, v = h,$

$$b_0 = \pm\sqrt{\frac{b}{a-\alpha}}a_0 \tag{142}$$

$$B = \frac{-3(a-\alpha)}{2(\beta h - \gamma)}, \tag{143}$$

where

$$h = \left\{9(\omega-1)(a-\alpha) + 2\beta^2\right\}Z^2 + (-4\gamma\beta + 9\delta(\alpha - a))Z + 2\gamma^2 + 9k^2(a-\alpha) \tag{144}$$

and a_0, b_0, b_{-1} are free parameters. So

$$q(x,t)=Q(\xi)=\frac{a_0}{\pm\sqrt{\frac{b}{a-\alpha}}a_0+b_{-1}e^{-\xi}}. \quad (145)$$

Set 5. $a_1=b_1=\sigma=c=0,$

$$b_0=\frac{a_0 b}{\pm(a-\alpha)\sqrt{\frac{b}{a-\alpha}}}, \quad (146)$$

$$b_{-1}=\pm\sqrt{\frac{b}{a-\alpha}}a_{-1},\ v=\frac{h}{B}. \quad (147)$$

Here h is the root of

$$(-1+\omega)Z^2+(-\delta B+\beta)Z$$
$$-\gamma B+k^2B^2-2\alpha+2a=0 \quad (148)$$

and a_0, a_{-1} are free parameters and

$$q(x,t)=Q(\xi)$$
$$=\frac{a_0+a_{-1}e^{-\xi}}{\pm\sqrt{\frac{b}{a-\alpha}}a_{-1}e^{-\xi}\pm\frac{ba_0}{(a-\alpha)\sqrt{\frac{b}{a-\alpha}}}}. \quad (149)$$

5.1.3. Form-III

In this case, the perturbed KGE that will be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^n=\alpha q+\beta q_t+\gamma q_x$$
$$+\delta q_{xt}+\omega q_{tt} \quad (150)$$

and by $q-Q(\xi),\xi=B(x-vt)$ (151)

is reduced to

$$\left(v^2-k^2+\delta v-\omega v^2\right)B^2Q''+aQ-bQ^n$$
$$=\alpha Q+v\,\beta BQ'+\gamma BQ'. \quad (152)$$

By $q=W^{\frac{2}{n-1}}$, this is converted to

$$2y(n-1)B^2W''W-2y(-3+n)B^2W'^2$$
$$-2(-\gamma+\beta v)(n-1)BWW'+bW^4(n-1)^2$$
$$-W^2(n-1)^2(a-\alpha)=0, \quad (153)$$

here

$$y=k^2-\delta v+\omega v^2-v^2. \quad (154)$$

and $k,a,b,\alpha,\beta,\gamma,\delta$ and ω are arbitrary constants. Using the ansatz (101) and balancing highest order of exp-function in WW'' and W^4 leads to the result

$$p=c. \quad (155)$$

Similarly, balancing the linear term of lowest order in WW'' and W^4 leads to the result which leads to the result

$$d=q. \quad (156)$$

Since the final solution does not strongly depend upon the choice of values of c and d, for simplicity we set $p=c=1$, $d=q=1$, the trial function, Eq. (101) becomes

$$Q(\xi)=\frac{a_{-1}e^{-\xi}+a_0+a_1e^{\xi}}{b_{-1}e^{-\xi}+b_0+b_1e^{\xi}}. \quad (157)$$

Substituting Eq. (157) into Eq. (152), and equating to zero the coefficients of all powers of $e^{n\xi}$ yields a set of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0, and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain the following several sets of solutions.

Set 1. $a_{-1}=a_1=b_0=0,$

$$b_{-1}=\frac{ba_0^2}{2(n+1)(a-\alpha)b_1} \quad (158)$$

$$v=\frac{\gamma}{\beta}, B=h(n-1)\beta. \quad (159)$$

Here h is root of

$$\left(-4\gamma^2-4\delta\gamma\beta+4\omega\gamma^2+4k^2\beta^2\right)Z^2$$
$$+\alpha-a=0. \quad (160)$$

So, $q(x,t)=W^{\frac{2}{n-1}}$

$$= \left\{ \frac{a_0}{b_1 e^{\xi} + \frac{b a_0^2}{2(n+1)(a-\alpha)b_1} e^{-\xi}} \right\}^{\frac{2}{n-1}} \qquad (161)$$

where a_0 and b_1 are arbitrary constants.

Set 2. $a_1 = b_1 = 0,\ v = \frac{\gamma}{\beta}, a_0 = \pm\sqrt{\frac{a-\alpha}{b}} b_0,$

$a_{-1} = \frac{\pm(a-\alpha)b_{-1}}{\sqrt{\frac{a-\alpha}{b}} b}$, (162) and $q(x,t) = W^{\frac{2}{n-1}}$

$$= \left\{ \frac{\pm\sqrt{\frac{a-\alpha}{b}} b_0 \pm \frac{(a-\alpha)b_{-1}}{\sqrt{\frac{a-\alpha}{b}} b} e^{-\xi}}{b_0 + b_{-1} e^{-\xi}} \right\}^{\frac{2}{n-1}} \qquad (163)$$

Set 3. $a_1 = b_1 = 0, v = \frac{h}{B}, b_0 = \frac{\pm b a_0}{(a-\alpha)\sqrt{\frac{b}{a-\alpha}}},$

$$b_{-1} = \pm\sqrt{\frac{b}{a-\alpha}} a_{-1}, \qquad (164)$$

and

$q(x,t) = W^{\frac{2}{n-1}}$

$$= \left\{ \frac{a_0 + a_{-1} e^{-\xi}}{\pm \frac{b a_0}{(a-\alpha)\sqrt{\frac{b}{a-\alpha}}} \pm \sqrt{\frac{b}{a-\alpha}} a_{-1} e^{-\xi}} \right\}^{\frac{2}{n-1}} \qquad (165)$$

where h is the root of

$$(\omega-1)Z^2 + (-\delta B + \beta)Z + (-\alpha + a)(n-1) + k^2 B^2 - \gamma B = 0, \qquad (166)$$

and a_0 and a_{-1} are free parameters.

5.1.4. Form-IV

In this case, the perturbed KGE that will be studied is given by

$$q_{tt} - k^2 q_{xx} + aq - bq^n + cq^{2n-1} = \alpha q + \beta q_t + \gamma q_x + \delta q_{xt} + \omega q_{tt} \qquad (167)$$

which, by $q = W^{\frac{1}{n-1}}$ is reduced to

$$(n-1)^2 \left\{(a-\alpha)W^2 - bW^3 + cW^4\right\} + B^2 W'^2 (n-2) + BWW'(-\gamma + \beta v)(n-1) - y(n-1)B^2 WW'' = 0, \qquad (168)$$

here $y = -v^2 + k^2 - \delta v + \omega v^2$ and $k, a, b, \alpha, \beta, \gamma, c, \delta$ and ω are arbitrary constants. Using the ansatz (101) and balancing highest order of exp-function in WW'' and W^4 leads to the result

$$p = c. \qquad (169)$$

Similarly, balancing the linear term of lowest order in WW'' and W^4 leads to the result which leads to the result

$$d = q. \qquad (170)$$

Since the final solution does not strongly depend upon the choice of values of c and d, so for simplicity, we set $p = c = 1$, $d = q = 1$, the trial function, Eq. (101) becomes

$$Q(\xi) = \frac{a_{-1}e^{-\xi} + a_0 + a_1 e^{\xi}}{b_{-1}e^{-\xi} + b_0 + b_1 e^{\xi}} \qquad (171)$$

Substituting Eq. (171) into Eq. (168), and equating to zero the coefficients of all powers of $\exp(n\xi)$ yields a set of algebraic equations for $a_1, a_0, a_{-1}, b_1, b_0,$ and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain the following several sets of solutions.

Set 1.

$$a_{-1} = a_1 = 0, v = \frac{\gamma}{\beta}, \qquad (172)$$

$$a_0 = \frac{(a-\alpha)(n+1)b_0}{b},$$

$$b_{-1} = -\frac{(c(n+1)^2(a-\alpha)-b^2n)b_0^2}{4(nb^2b_1)},$$

$$B = \pm(n-1)\times\beta\sqrt{\frac{a-\alpha}{-\gamma^2+\omega\gamma^2-\delta\gamma\beta+k^2\beta^2}}.$$

So $q(x,t) = W^{\frac{1}{n-1}} =$

$$\left\{\frac{\frac{(a-\alpha)(n+1)b_0}{b}}{-\frac{(c(n+1)^2(a-\alpha)-b^2n)b_0^2}{4(nb^2b_1)}e^{-\xi}+b_0+b_1e^{\xi}}\right\}^{\frac{1}{n-1}} \quad (173)$$

where b_0 and b_1 are free parameters.

5.1.5. Form-V

For this form, the perturbed KGE that will be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^{1-n}+c_1q^{n+1}=\alpha q+\beta q_t \quad (174)$$

$$+\gamma q_x+\delta q_{xt}+\omega q_{tt}+\sigma q_{xxt}+\theta q_{xxxx} \quad (175)$$

by $q(x,t)=Q(\xi),\ \xi=B(x-vt)$ (176)

is converted to

$$B^2\left(v^2-k^2+\delta v-\omega v^2\right)Q''+aQ$$
$$-bQ^{1-n}+cQ^{n+1}-\alpha Q+B(\beta v-\gamma)Q'$$
$$+v\sigma B^3Q'''-\theta B^4Q''''=0. \quad (177)$$

By

$$Q=W^{\frac{4}{n}} \quad (178)$$

is reduced to $-12n^2\sigma v(n-4)B^3W^2W'W''$

$$+16\theta(n-4)n^2B^4W'W'''W^2+an^4W^4$$
$$-\alpha n^4W^4-bn^4+cn^4W^8$$
$$+8\sigma vn(n-2)(n-4)B^3W\left(W'\right)^3$$
$$-4n^3\left(-v^2+k^2-\delta v+\omega v^2\right)B^2W^3W''$$
$$+4\sigma vn^3B^3W^3W'''-4\theta n^3B^4W''''W^3$$
$$-48\theta n(n-2)(n-4)B^4\left(W'\right)^2W''W \quad (179)$$

$$+8\theta(n-2)(3n-4)(n-4)B^4\left(W'\right)^4$$
$$+4n^2\left(-v^2+k^2-\delta v+\omega v^2\right)(n-4)B^2W^2$$
$$\times\left(W'\right)^2+4n^3(\beta v-\gamma)BW^3W'$$
$$+12\theta n^2(n-4)B^4\left(W''\right)^2W^2=0.$$

Here k, a, b, α, β, γ, c, θ, δ and ω are arbitrary constants. Using the ansatz (101) and balancing highest order of exp-function in WW'' and W^4 leads to the result

$$p=c. \quad (180)$$

Similarly, balancing the linear term of lowest order in WW'' and W^4 leads to the result, which leads to the result

$$d=q. \quad (181)$$

Since the final solution does not strongly depend upon the choice of values of c and d, for simplicity, we set $p=c=1$, $d=q=1$, the trial function, Eq. (101) becomes

$$Q(\xi)=\frac{a_{-1}e^{-\xi}+a_0+a_1e^{\xi}}{b_{-1}e^{-\xi}+b_0+b_1e^{\xi}} \quad (182)$$

Substituting Eq. (182) into Eq. (177), and equating to zero the coefficients of all powers of $e^{n\xi}$ yields a set of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0 and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain the following set of solutions.

$$a_1=a_{-1}=b_1=b_{-1}=0,\ b_0=ha_0, \quad (183)$$

where h is root of

$$bZ^8-c2+(\alpha-a)Z^4=0, \quad (184)$$

where a_0 is an arbitrary constant.

5.1.6. Form-VI

For this form, the perturbed KGE that will to be studied is given by

$$q_{tt}-k^2q_{xx}+aq+bq\ln q=\alpha q+\beta q_t$$
$$+\gamma q_x+\delta q_{xt}+\omega q_{tt} \quad (185)$$

with $Q(x,t)=Q(\xi),\ \xi=B(x-vt)$ is converted to

$$B^2\left(v^2-k^2+\delta v-\omega v^2\right)Q''+(a-\alpha)Q$$

$$+bQ\ln Q+(\beta v-\gamma)BQ'=0 \quad (186)$$

and by $\ln q=u$ it is reduced to

$$B^2\left(v^2-k^2+\delta v-\omega v^2\right)\left\{u''+\left(u'\right)^2\right\}$$
$$+(a-\alpha)+bu+B(\beta v-\gamma)u'=0, \quad (187)$$

here k, a, b, α, β, γ, δ and ω are arbitrary constants. Using the ansatz (101) and balancing highest order of exp-function in WW'' and W^4 leads to the result

$$p=c. \quad (188)$$

Similarly, balancing the linear term of lowest order in u'' and $\left(u'\right)^2$ leads to the result, which leads to the result

$$d=q. \quad (189)$$

Since the final solution does not strongly depend upon the choice of values of c and d, for simplicity, we set $p=c=1$, $d=q=1$, the trial function, Eq. (101) becomes

$$Q(\xi)=\frac{a_{-1}e^{-\xi}+a_0+a_1e^{\xi}}{b_{-1}e^{-\xi}+b_0+b_1e^{\xi}}. \quad (190)$$

Substituting Eq. (190) into Eq. (186), and equating to zero the coefficients of all powers of $e^{n\xi}$ yields a set of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0 and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain the following set of solutions.

Set 1.

$$a_1=b_1=b_{-1}=0, v=h, a_0=\frac{-(a-\alpha)b_0}{b},$$
$$B=\frac{b}{\beta h-\gamma}. \quad (191)$$

Here h is the root of

$$(-1+\omega)Z^2-Z\delta+k^2=0 \quad (192)$$

and a_{-1} and b_0 are arbitrary constants. So

$$q(x,t)=Q(\xi)=e^{u(\xi)}=e^{\frac{\frac{-(a-\alpha)b_0}{b}+a_{-1}e^{-\xi}}{b_0}}. \quad (193)$$

Set 2.

$$a_1=b_1=b_0=0, v=h, B=\frac{b}{\beta h-\gamma},$$
$$a_{-1}=\frac{-(a-\alpha)b_{-1}}{b}. \quad (194)$$

Here a_0, b_{-1} are arbitrary constants and h is the roots of

$$(-1+\omega)Z^2-Z\delta+k^2=0. \quad (195)$$

So $q(x,t)=Q(\xi)=e^{u(\xi)}=e^{\frac{a_0+\frac{-(a-\alpha)b_{-1}}{b}e^{-\xi}}{b_{-1}e^{-\xi}}}$, (196)

where $b\neq 0$ and $\beta h\neq\gamma$.

6. Mapping Methods

Now, we solve eq. (19) by a mapping method and a modified mapping method which generates a variety of periodic wave solutions (PWSs) in terms of squared Jacobi elliptic functions (JEFs) and we subsequently derive their infinite period counterparts in terms of hyperbolic functions which are solitary wave solutions (SWSs), shock wave solutions or singular solutions. For solving by these methods, we set in eq. (21) $\beta=\gamma=\sigma=\nu=0$. Thus eq. (21) reduces to

$$\left(v^2-k^2+\delta v-\lambda v^2\right)g''+F(g)-\alpha g=0. \quad (197)$$

6.1. Form-I

In this case, the perturbed KGE that will be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^2=\alpha q+\delta q_{xt}+\lambda q_{tt}. \quad (198)$$

Using the travelling wave hypotheses (20) and (22), eq. (198) reduces to

$$Ag''+Bg+Cg^2=0, \quad (199)$$

where

$$A=v^2-k^2+\delta v-\lambda v^2,\ B=a-\alpha, C=-b. \quad (200)$$

6.1.1. Mapping Method

Here, we assume that eq. (199) has a solution in the form

$$g = A_0 + A_1 f, \tag{201}$$

where

$$f'' = P + Qf + Rf^2,$$
$$\left(f'\right)^2 = 2Pf + Qf^2 + \frac{2}{3}Rf^3. \tag{202}$$

Eq. (201) is the mapping relation between the solution to eq. (199) and that of eq. (202). We substitute eq. (201) into eq. (199), use eq. (202) and equate the coefficients of like powers of f to zero so that we arrive at the set of equations

$$AA_1R + CA_1^2 = 0, \tag{203}$$

$$AA_1Q + BA_1 + 2CA_0A_1 = 0, \tag{204}$$

$$AA_1P + BA_0 + 2CA_0^2 = 0. \tag{205}$$

From eqs. (203) and (204), we obtain $A_1 = -\frac{AR}{C}$ and $A_0 = -\frac{AQ+B}{2C}$. Eq. (205) gives rise to the constraint relation

$$A^2(Q^2 - 4PR) = B^2. \tag{206}$$

Case 1. $P = 2, Q = -4(1+m^2), R = 6m^2$. Now eq. (202) has two solutions $f(s) = \mathrm{sn}^2(s)$ and $f(s) = \mathrm{cd}^2(s)$. So, we obtain the PWSs of eq. (199) as

$$g(s) = \frac{a - \alpha - 4(1+m^2)(v^2 - k^2 + \delta v - \lambda v^2)}{2b} + \frac{6m^2(v^2 - k^2 + \delta v - \lambda v^2)}{b}\mathrm{sn}^2(s), \tag{207}$$

and

$$g(s) = \frac{a - \alpha - 4(1+m^2)(v^2 - k^2 + \delta v - \lambda v^2)}{2b} + \frac{6m^2(v^2 - k^2 + \delta v - \lambda v^2)}{b}\mathrm{cd}^2(s). \tag{208}$$

When $m \to 1$, the eq. (207) gives rise to the SWS

$$g(s) = \frac{a - \alpha - 8(v^2 - k^2 + \delta v - \lambda v^2)}{2b} + \frac{6(v^2 - k^2 + \delta v - \lambda v^2)}{b}\tanh^2(s). \tag{209}$$

Case 2. $P = 2(1-m^2), Q = -4(2m^2-1), R = -6m^2$. In this case, eq. (202) has the solution $f(s) = \mathrm{cn}^2(s)$. Thus we obtain the PWS of eq. (199) as

$$g(s) = \frac{a - \alpha - 4(2m^2-1)(v^2 - k^2 + \delta v - \lambda v^2)}{2b} - \frac{6m^2(v^2 - k^2 + \delta v - \lambda v^2)}{b}\mathrm{cn}^2(s). \tag{210}$$

When $m \to 1$, the eq. (210) gives rise to the SWS (209).

Case3. $P = -2(1-m^2), Q = 4(2-m^2), R = -6$. Here, eq. (202) has the solution $f(s) = \mathrm{dn}^2(s)$. So, we obtain the PWS of eq. (199) as

$$g(s) = \frac{a - \alpha + 4(2-m^2)(v^2 - k^2 + \delta v - \lambda v^2)}{2b} - \frac{6(v^2 - k^2 + \delta v - \lambda v^2)}{b}\mathrm{dn}^2(s). \tag{211}$$

When $m \to 1$, the SWS (209) is retained.

Case 4. $P = 2m^2, Q = -4(1+m^2), R = 6$. Now, eq. (202) has two solutions $f(s) = \mathrm{ns}^2(s)$ and $f(s) = \mathrm{dc}^2(s)$. In this case, we obtain the two PWSs of eq. (199) as

$$g(s) = \frac{a - \alpha - 4(1+m^2)(v^2 - k^2 + \delta v - \lambda v^2)}{2b} + \frac{6(v^2 - k^2 + \delta v - \lambda v^2)}{b}\mathrm{ns}^2(s), \tag{212}$$

and

$$g(s) = \frac{a - \alpha - 4(1+m^2)(v^2 - k^2 + \delta v - \lambda v^2)}{2b} + \frac{6(v^2 - k^2 + \delta v - \lambda v^2)}{b}\mathrm{dc}^2(s). \tag{213}$$

When $m \to 1$, eq. (212) leads us to the singular solution

$$g(s) = \frac{a - \alpha - 8(v^2 - k^2 + \delta v - \lambda v^2)}{2b}$$

$$+\frac{6(v^2-k^2+\delta v-\lambda v^2)}{b}\coth^2(s). \qquad (214)$$

Case5. $P=-2m^2, Q=4(2m^2-1), R=6(1-m^2).$ So, eq. (202) has the solution $f(s)=\mathrm{nc}^2(s)$. Thus we obtain the PWS of eq. (199) as

$$g(s)=\frac{a-\alpha+4(2m^2-1)(v^2-k^2+\delta v-\lambda v^2)}{2b}+\frac{6(1-m^2)(v^2-k^2+\delta v-\lambda v^2)}{b}\mathrm{nc}^2(s). \qquad (215)$$

Case 6. $P=-2, Q=4(2-m^2), R=-6(1-m^2).$ Here, eq. (202) has the solution $f(s)=\mathrm{nd}^2(s)$. So, we obtain the PWS of eq. (199) as

$$g(s)=\frac{a-\alpha+4(2-m^2)(v^2-k^2+\delta v-\lambda v^2)}{2b}-\frac{6(1-m^2)(v^2-k^2+\delta v-\lambda v^2)}{b}\mathrm{nd}^2(s). \qquad (216)$$

Case 7. $P=2, Q=4(2-m^2), R=6(1-m^2).$ Here, eq. (202) has the solution $f(s)=\mathrm{sc}^2(s)$. Thus the PWS of eq. (199) is

$$g(s)=\frac{a-\alpha+4(2-m^2)(v^2-k^2+\delta v-\lambda v^2)}{2b}+\frac{6(1-m^2)(v^2-k^2+\delta v-\lambda v^2)}{b}\mathrm{sc}^2(s). \qquad (217)$$

Case8. $P=2, Q=4(2m^2-1), R=-6m^2(1-m^2).$ In this case, eq. (202) has the solution $f(s)=\mathrm{sd}^2(s)$. Thus the PWS of eq. (199) is

$$g(s)=\frac{a-\alpha+4(2m^2-1)(v^2-k^2+\delta v-\lambda v^2)}{2b}-\frac{6m^2(1-m^2)(v^2-k^2+\delta v-\lambda v^2)}{b}\mathrm{sd}^2(s) \qquad (218)$$

Case 9. $P=2(1-m^2), Q=4(2-m^2), R=6.$ Here, eq. (202) has the solution $f(s)=\mathrm{cs}^2(s)$. Thus the PWS of eq. (199) is

$$g(s)=\frac{a-\alpha+4(2-m^2)(v^2-k^2+\delta v-\lambda v^2)}{2b}+\frac{6(v^2-k^2+\delta v-\lambda v^2)}{b}\mathrm{cs}^2(s). \qquad (219)$$

When $m\to 1,$ the PWS (219) will give rise to the singular solution (214).

Case10. $P=-2m^2(1-m^2), Q=4(2m^2-1), R=6.$ Thus eq. (202) has the solution $f(s)=\mathrm{ds}^2(s)$. So, the PWS of eq. (199) is

$$g(s)=\frac{a-\alpha+4(2m^2-1)(v^2-k^2+\delta v-\lambda v^2)}{2b}+\frac{6(v^2-k^2+\delta v-\lambda v^2)}{b}\mathrm{ds}^2(s). \qquad (220)$$

When $m\to 1,$ the PWS (220) leads to the same singular solution (214).

It is evident from the constraint relation (206) that Q^2-4PR should always be positive with our choices of P, Q and R for real solutions to exist. In all the cases considered, we can see that Q^2-4PR is equal to $16m^4-16m^2+16$ which is always positive for $0\le m\le 1.$ Thus all our solutions are valid with the constraint relation (206).

6.1.2. Modified Mapping Method

In this case, we assume that eq. (199) has a solution in the form

$$g=A_0+A_1 f+B_1 f^{-1}, \qquad (221)$$

where f satisfies eq. (202). Eq. (221) is the mapping relation between the solution to eq. (202) and that of eq. (199).

We substitute eq. (221) into eq. (199), use eq. (202) and equate the coefficients of like powers of f to zero so that we will obtain a set of equations giving rise to the solutions

$$A_0=-\frac{AQ+B}{2C},\ A_1=-\frac{AR}{C},\ B_1=-\frac{3PA}{C}, \qquad (222)$$

and the constraint relation

$$A^2(Q^2+16PR)=B^2. \qquad (223)$$

Case 1. $P=2, Q=-4(1+m^2), R=6m^2.$ Here, eq. (202) has two solutions

$f(s)=\text{sn}^2(s)$ and $f(s)=\text{cd}^2(s)$.

The PWSs of eq. (199) are

$$g(s)=\frac{a-\alpha}{2b}-\frac{(v^2-k^2+\delta v-\lambda v^2)}{2b}\times[4(1+m^2)-12m^2\text{sn}^2(s)-12m^2\text{ns}^2(s)], \quad (224)$$

and

$$g(s)=\frac{a-\alpha}{2b}-\frac{(v^2-k^2+\delta v-\lambda v^2)}{2b}\times\left[4(1+m^2)-12m^2\text{cd}^2(s)-12m^2\text{dc}^2(s)\right]. \quad (225)$$

When $m\to 1$, eq. (224) degenerates to the solution

$$g(s)=\frac{a-\alpha}{2b}-\frac{(v^2-k^2+\delta v-\lambda v^2)}{2b}\times\left[8-12\left\{\tanh^2(s)+\coth^2(s)\right\}\right]. \quad (226)$$

Case 2. $P=-2(1-m^2), Q=4(2-m^2), R=-6.$
In this case, eq. (202) has the solution $f(s)=\text{dn}^2(s)$. So, the PWS of eq. (199) is

$$g(s)=\frac{a-\alpha}{2b}+\frac{(v^2-k^2+\delta v-\lambda v^2)}{2b}\times[4(2-m^2)-12\text{dn}^2(s)-12(1-m^2)\text{nd}^2(s)]. \quad (227)$$

When $m\to 1$, eq. (227) leads us to the SWS

$$g(s)=\frac{a-\alpha}{2b}-\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}\times\left[1-3\,\text{sech}^2(s)\right]. \quad (228)$$

Case 3. $P=2, Q=4(2-m^2), R=6(1-m^2).$

Thus eq. (202) has the solution $f(s)=\text{sc}^2(s)$. So, the PWS of eq. (199) is

$$g(s)=\frac{a-\alpha}{2b}+\frac{(v^2-k^2+\delta v-\lambda v^2)}{2b}\times\left[4(2-m^2)+12(1-m^2)\text{sc}^2(s)+12\text{cs}^2(s)\right]. \quad (229)$$

When $m\to 1$, eq. (229) degenerates to the singular solution (214).

6.2. Form-II

In this case, the perturbed KGE that will be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^3=\alpha q+\delta q_{xt}+\lambda q_{tt}+\nu q_{xxxx}. \quad (230)$$

Using the travelling wave hypotheses (20) and (22), eq. (230) reduces to

$$Ag''+Bg+Cg^3=0, \quad (231)$$

Where,

$$A=v^2-k^2+\delta v-\lambda v^2,\; B=a-\alpha,\; C=-b. \quad (232)$$

6.2.1. Mapping Method

Here, we assume that eq. (231) has a solution in the form

$$g=A_0+A_1 f, \quad (233)$$

where

$$f''=Pf+Qf^3,\; f'^2=Pf^2+\frac{1}{2}Qf^4+r. \quad (234)$$

Eq. (233) is the mapping relation between the solution to eq. (231) and that of eq. (234).We substitute eq. (233) into eq. (231), use eq. (234) and equate the coefficients of like powers of f to zero so that we arrive at the set of equations

$$QAA_1+CA_1^3=0, \quad (235)$$

$$3CA_0A_1^2=0, \quad (236)$$

$$(PA+B)A_1+3CA_0^2A_1=0, \quad (237)$$

$$BA_0+CA_0^3=0, \quad (238)$$

from which we obtain

$$A_0=0,\; A_1=\pm\sqrt{\frac{QB}{PC}},\quad PA+B=0. \quad (239)$$

Case 1. $P=-2, Q=2, R=1.$
In this case, the solution of eq. (234) is $f(s)=\tanh(s)$. So, we have the shock wave solution of eq. (231) as

$$g(s)=\pm\sqrt{\frac{a-\alpha}{b}}\tanh(s). \quad (240)$$

Case 2. $P=1, Q=-2, R=0.$

So, the solution of eq. (234) is $f(s)=\operatorname{sech}(s)$. Thus the SWS of eq. (231) is

$$g(s)=\pm\sqrt{\frac{2(a-\alpha)}{b}}\operatorname{sech}(s). \tag{241}$$

Case 3. $P=-(1+m^2), Q=2m^2, R=1.$

Here, the solutions of eq. (234) are $f(s)=\operatorname{sn}(s)$ and $f(s)=\operatorname{cd}(s)$. So, the PWSs of eq. (231) are

$$g(s)=\pm\sqrt{\frac{2(a-\alpha)}{b(1+m^2)}}\,m\operatorname{sn}(s), \tag{242}$$

and

$$g(s)=\pm\sqrt{\frac{2(a-\alpha)}{b(1+m^2)}}\,m\operatorname{cd}(s). \tag{243}$$

When $m\to 1$, eq. (242) reduces to the shock wave solution (240).

Case 4. $P=2-m^2, Q=-2, R=m^2-1.$

In this case, the solution of eq. (234) is $f(s)=\operatorname{dn}(s)$. Here, the PWS of eq. (231) is

$$g(s)=\pm\sqrt{\frac{2(a-\alpha)}{b(2-m^2)}}\operatorname{dn}(s). \tag{244}$$

When $m\to 1$, the SWS (241) is recovered from eq. (244).

Case 5. $P=-(1+m^2), Q=2, R=m^2.$

The solutions of eq. (234) are $f(s)=\operatorname{ns}(s)$ and $f(s)=\operatorname{dc}(s)$. So, the PWSs of eq. (231) are

$$g(s)=\pm\sqrt{\frac{2(a-\alpha)}{b(1+m^2)}}\operatorname{ns}(s), \tag{245}$$

and

$$g(s)=\pm\sqrt{\frac{2(a-\alpha)}{b(1+m^2)}}\operatorname{dc}(s). \tag{246}$$

When $m\to 1$, eq. (245) degenerates to the singular solution

$$g(s)=\pm\sqrt{\frac{a-\alpha}{b}}\coth(s). \tag{247}$$

6.2.2. Modified Mapping Method

Now, we use the modified mapping method in which we assume a solution of eq. (231) in the form

$$g=A_0+A_1 f+B_1 f^{-1} \tag{248}$$

where f satisfies eq. (234).

We substitute eq. (248) into eq. (231), use eq. (234) and equate the coefficients of like powers of f to zero to arrive at a set of equations fro which it can be found that

$$A_0=0,\ A_1=\pm\sqrt{-\frac{QA}{C}} \tag{249}$$

$$B_1=\pm\sqrt{-\frac{2RA}{C}},\ PA+B+3CA_1B_1=0. \tag{250}$$

Thus for real solutions of eq. (231) to exist, when Q and R are both positive, A and C should be of opposite signs and when Q and R are both negative, A and C should be of the same signs.

Case 1. $P=-2, Q=2, R=1.$

In this case, the solution of eq. (234) is $f(s)=\tanh(s)$. So, we have the solution of eq. (231) as $g(s)=\sqrt{\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}}$

$$\times\{\pm\tanh(s)\pm\coth(s)\}. \tag{251}$$

Case 2. $P=-(1+m^2), Q=2m^2, R=1.$

Here, the solutions of eq. (234) are $f(s)=\operatorname{sn}(s)$ and $f(s)=\operatorname{cd}(s)$. Thus the PWSs of eq. (231) are

$$g(s)=\sqrt{\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}}$$

$$\times\{\pm m\operatorname{sn}(s)\pm\operatorname{ns}(s)\}, \tag{252}$$

and

$$g(s)=\sqrt{\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}}$$

$$\times\{\pm m\operatorname{cd}(s)\pm\operatorname{dc}(s)\}. \tag{253}$$

When $m\to 1$, eq. (252) gives the same solution (251).

Case 3. $P=2-m^2, Q=2, R=1-m^2$.

So, the solution of eq. (234) is $f(s)=\text{cs}(s)$. Thus we have the PWS of eq. (231) as

$$g(s)=\sqrt{\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}}\times\left\{\pm\text{cs}(s)\pm\sqrt{1-m^2}\,\text{sc}(s)\right\}. \quad (254)$$

When $m\to1$, eq. (254) leads to the singular solution

$$g(s)=\pm\sqrt{\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}}\times\text{csch}(s). \quad (255)$$

Case 4. $P=2-m^2, Q=-2, R=-(1-m^2)$.

Here, the solution of eq. (234) is $f(s)=\text{dn}(s)$. So, we have the PWS of eq. (231) is

$$g(s)=\sqrt{-\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}}\times\left\{\pm\text{dn}(s)\pm\sqrt{1-m^2}\,\text{nd}(s)\right\}. \quad (256)$$

When $m\to1$, eq. (256) gives rise to the SWS

$$g(s)=\pm\sqrt{-\frac{2(v^2-k^2+\delta v-\lambda v^2)}{b}}\times\text{sech}(s). \quad (257)$$

7. Lie Group Analysis

In this section, we study some forms of the perturbed KGE by the aid of Lie group analysis. Initially, the method will be described in a succinct manner and will subsequently be applied to solve a couple of forms of the perturbed KGE.

7.1. Description of the method

The partial differential equation $E(t,x,q,q_t,q_x,q_{tt},q_{tx},q_{xx})=0$ admits the symmetry generator X in the form of prolongation given by

$$X=\xi^1(t,x,q)\partial_t+\xi^2(t,x,q)\partial_x+\eta(t,x,q)\partial_q+\zeta_i\partial_{q_i}+\zeta_{i_1i_2}\partial_{q_{i_1i_2}}, \quad (258)$$

if $XE|_{E=0}=0,$ (259)

where the summation convention is used whenever appropriate. In (258), the additional coefficients $\zeta_i, \zeta_{i_1i_2}$ are determined uniquely by the prolongation formulae $\zeta_i=D_i(W)+\xi^j q_{ij}$,

$$\zeta_{i_1i_2}=D_{i_1}D_{i_2}(W)+\xi^j q_{ji_1i_2}, i,j=1,2, \quad (260)$$

in which W is the Lie characteristic function defined by $W=\eta-\xi^i q_i$. Here the differential operators D_t and D_x are given by $D_1=D_t=\partial_t+q_t\partial_q+\dots$ and $D_2=D_x=\partial_x+q_x\partial_q+\dots$.

7.1.1. Form-I

In this case, the perturbed KGE that will be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^2-\beta q_t-\gamma q_x-\delta q_{xt}=0. \quad (261)$$

The equation (261) admits the following two Lie point symmetries

$$X_1=\partial_t,\quad X_2=\partial_x. \quad (262)$$

The group-invariant solution corresponding to the combination of symmetries X_1+cX_2, where c is a constant is given by

$$q=h(\rho), \quad (263)$$

where $\rho=x-ct$. The group-invariant solution (263) reduces the equation (261), choosing $\gamma=\beta c$, to the following nonlinear second-order ordinary differential equation (ODE)

$$h''+Ah+Bh^2=0, \quad (264)$$

where

$$A:=a/(c^2-k^2+\delta c),\ B:=-b/(c^2-k^2+\delta c)$$

and `prime' denotes differentiation with respect to ρ. Integrating the equation (264) with respect to ρ again and letting the constant of integration be zero we obtain the following nonlinear first-order ODE

$$h'^2=-Ah^2-\frac{2B}{3}h^3. \quad (265)$$

The equation (265) has the following solution after substituting the values of A and B

$$h(\rho)=\frac{3a}{2b}sech^2\left(d\pm\sqrt{\frac{a}{4(k^2-c^2-\delta c)}}\rho\right), \quad (266)$$

where d is a constant of integration. Hence we obtain the following solitary wave exact group-invariant solutions for the equation (261), in which $\gamma=\beta c$, given by

$$q(x,t)=\frac{3a}{2b}sech^2(d\pm\sqrt{\frac{a}{4(k^2-c^2-\delta c)}} \times(x-ct)). \quad (267)$$

7.1.2. Form-IV

For this form, the perturbed KGE that will be studied is given by

$$q_{tt}-k^2q_{xx}+aq-bq^n+dq^{2n-1} -\delta q_{xt}-\lambda q_{tt}=0,\ n>1. \quad (268)$$

Again the equation (268) admits the following two Lie point symmetries

$$X_1=\partial_t,\quad X_2=\partial_x. \quad (269)$$

The X_1+cX_2-group-invariant solution corresponding to the combination of symmetries X_1 and X_2, where c is a constant is given by

$$q=h(\rho), \quad (270)$$

where $\rho=x-ct$.

The reduced nonlinear second-order ODE resulting from substituting the group-invariant solution (270) into the equation (268) is given by

$$h''-Ah+Bh^n-Ch^{2n-1}=0, \quad (271)$$

where

$$A:=a/(\lambda c^2-c^2+k^2-\delta c),$$
$$B:=b/(\lambda c^2-c^2+k^2-\delta c),$$
$$C:=d/(\lambda c^2-c^2+k^2-\delta c)$$

and `prime' denotes differentiation with respect to ρ. Integrating the equation (271) with respect to ρ and letting the constant of integration be zero yields the following nonlinear first-order ODE

$$h'^2=Ah^2-\frac{2B}{(n+1)}h^{n+1}+\frac{C}{n}h^{2n}. \quad (272)$$

Further integration of the equation (253) and then substituting the values of A, B and C we obtain the following solution for the equation (272)

$$h(\rho)=\frac{D}{[E+\cosh(e\pm F\rho)]^{\frac{1}{n-1}}}, \quad (273)$$

where e is a constant of integration,

$$D=\left(\frac{an(n+1)}{\sqrt{n[nb^2-ad(n+1)^2]}}\right)^{\frac{1}{n-1}}, \quad (274)$$

$$E=\frac{bn}{\sqrt{n[nb^2-ad(n+1)^2]}}, \quad (275)$$

and

$$F=(n-1)\sqrt{\frac{a}{(\lambda c^2-c^2+k^2-\delta c)}}. \quad (276)$$

Thus we get the following solitary travelling wave exact group-invariant solutions for the equation (268) given by

$$q(x,t)=\frac{D}{[E+\cosh(e\pm F(x-ct)]^{\frac{1}{n-1}}}, \quad (277)$$

where the constants D, E and F are given by the equations (274), (275) and (276), respectively.

8. Conclusions

This paper studied the KGE with six forms of nonlinearity including the log law nonlinearity. The perturbation terms are taken from the theory of long Josephson junctions, modeled by the sine-Gordon equation and its type, and we were justified in studying them in the context of KGE. Thus, the perturbed KGE was integrated in the presence of these strong perturbation terms by the aid of several integration tools. In particular, the traveling wave solutions were obtained, G'/G method approach was used, exp-function method was carried out, mapping method and its versions were also applied and finally the Lie symmetry approach was also utilized to extract several forms of solution to the perturbed KGE.

In addition to soliton solutions, cnoidal waves, snoidal waves, other PWS and rational solutions were obtained. The limiting cases of these PWS are also given. These solutions are going to be extremely useful in the context of relativistic quantum mechanics where KGE arises.

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