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Phragmén-Lindelöf type results for a class of nonlinear damped wave equations

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Abstract

This paper deals with the behavior at infinity of solutions to a class of wave equations with nonlinear damping terms defined in a semi-infinite cylinder. The spatial behavior of solutions is studied and an alternative of Phragmén-Lindelöf type theorems is obtained in the results. The main point in the contribution is the use of energy method.

Keywords*:* Spatial estimates; viscoelasticity; Saint-Venant principle Phragmén-Lindelöf principle

1. Introduction

In recent years, several papers have been devoted to the study of asymptotic behavior of end effects for partial differential equations and systems. A great number of these studies were motivated by the desire to establish versions of *Saint-Venant's* principle that was initiated by Toupin [1] and developed by Horgan and Knowles [2] and the updated articles by Horgan [3, 4]. The same kind of results can be found in the studies by Knowles [5, 6], Oleinik [7], Flavin [8, 9-11] and Horgan [12]. A number of rigorous mathematical works are devoted to the study of such results for hyperbolic equations. We may recall the pioneer studies by Flavin *et.al* [13] and Chirita *et.al* [14-16]. The common goal in these works has been to construct an energy inequality.

When dissipative terms are present, alternative results can be considered. Quintanilla in [17] established spatial decay estimates for some classes of hyperbolic heat equation and proved same results in nonlinear viscoelasticity [18, 19]. In linear viscoelasticity, Diaz and Quintanilla [20] proved simillar results. In a recent work, Yilmaz [21] obtained the spatial growth and decay estimates for a class of quasilinear equations modelling dynamic viscoelasticity.

The aim of the present work is to establish a spatial decay and growth estimates for solutions to a nonlinear wave equation with nonlinear damping terms defined in a semi-infinite cylinder. We prove some theorems of Phragmén-Lindelöf type when the Neumann boundary condition (2) is imposed on the finite end of the cylinder and the Dirichlet boundary condition (3) is considered on the lateral surface. Our study is inspired by the results of [22], in which Celebi and Kalantarov obtained growth and decay estimates for a class of hyperbolic equations under nonlinear boundary conditions.

More precisely, we are concerned with the initial-boundary value problem

$$u_{tt} + au_t - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t|u_t|^p = div(\nabla u|\nabla u|^p), \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u}{\partial \nu}(x',0,t) = h(x',t), \quad (x',t) \in D_0 \times (0,T), \tag{2}$$

$$u(x,t) = 0, \quad (x,t) \in S_0 \times (0,T), \tag{3}$$

$$u(x,0) = u_t(x,0) = 0, \quad x \in \Omega, \tag{4}$$

where a is a positive constant, $p \geq 1, \nu$ is the outward normal to the boundary and

$h(.,t) \in C^1(D_0)$,

for all $t \in (0,T)$. Ω is the cylinder

$$\Omega = \{x \in R^n : x_n \in R^+, (x', x_n) \in D_{x_n}, n \geq 2\},$$

where

$$D_z = \{(x', x_n) \in \Omega : \ x_n = z\},$$

and

$$S_z = \{x \in R^n : \ x' \in \partial D_{x_n}, z \leq x_n < \infty \}.$$

We also assume that ∂D_z is sufficiently smooth to apply the divergence theorem. In the sequel we use

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$$\Omega_z = \Omega \cap \{x \in R^n : 0 < x_n < z\},$$

$$R_z = \Omega \cap \{x \in R^n : z < x_n < \infty\},$$

and assume that the function g satisfies

$$1 - \int_0^\infty g(s)ds = l > 0, \tag{5}$$

and

$$g(s) \geq 0, \quad g'(s) \leq 0, \quad \forall s \geq 0. \tag{6}$$

In addition, we assume that for functions $v \in L^1[0,T]$, the inequality

$$v(t) \geq (g * v)(t), \tag{7}$$

holds for g, where

$$(g * v)(t) = \int_0^t g(t-\tau)v(\tau)d\tau.$$

2. Spatial estimates

For the solutions of the problem (1)-(4) if $h(x',t) = 0$, we introduce the energy function $E(z)$ given by

$$E(z) = \int_0^T \int_{\Omega_z} (u_t^2 + |\nabla u|^2 + |u_t|^{p+2} + |\nabla u|^{p+2})\, dxdt + \int_0^T (g \circ \nabla u)_{\Omega_z}(t)dt, \tag{8}$$

where

$$(g \circ \nabla u)_D(t) = \int_0^t g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|_D^2 d\tau.$$

Multiplying (1) by u_t and integrating over Ω_z we obtain

$$\frac{d}{dt}\left[\ \frac{1}{2}\|u_t\|_{\Omega_z}^2 + \frac{1}{2}\|\nabla u\|_{\Omega_z}^2 + \frac{1}{p+2}\int_{\Omega_z} |\nabla u|^{p+2}dx\ \right] + a\,\|u_t\|_{\Omega_z}^2 + \int_{\Omega_z} |u_t|^{p+2} - \int_{\Omega_z}\int_0^t g(t-\tau)\,\nabla u_t(t)\nabla u(\tau)d\tau dx$$
$$= \int_{D_z} u_t u_{x_n} dx' + \int_{D_z} u_t u_{x_n}|\nabla u|^p dx' - \int_{D_z}\int_0^t g(t-\tau)\,u_t(t)u_{x_n}(\tau)d\tau dx'. \tag{9}$$

It is not difficult to see

$$\int_{\Omega_z}\int_0^t g(t-\tau)\,\nabla u_t(t)\nabla u(\tau)d\tau dx$$
$$= -\frac{1}{2}\frac{d}{dt}\left[\ (g \circ \nabla u)_{\Omega_z}(t)\ \right] + \frac{1}{2}\frac{d}{dt}\left[\ \int_0^t g(\tau)d\tau\ \|\nabla u(t)\|_{\Omega_z}^2\ \right] + \frac{1}{2}(g' \circ \nabla u)_{\Omega_z}(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_{\Omega_z}^2. \tag{10}$$

Therefore, (9) can be rewritten in the form

$$\frac{d}{dt}\left[\ \frac{1}{2}\|u_t\|_{\Omega_z}^2 + \frac{1}{2}\|\nabla u\|_{\Omega_z}^2 + \frac{1}{2}(g \circ \nabla u)_{\Omega_z}(t) + \frac{1}{p+2}\int_{\Omega_z}|\nabla u|^{p+2}dx - \frac{1}{2}\int_0^t g(\tau)d\tau\,\|\nabla u\|_{\Omega_z}^2\ \right] + a\|u_t\|_{\Omega_z}^2$$
$$+ \frac{1}{2}g(t)\|\nabla u\|_{\Omega_z}^2 - \frac{1}{2}(g' \circ \nabla u)_{\Omega_z}(t) + \int_{\Omega_z}|u_t|^{p+2}dx$$
$$= \int_{D_z} u_t u_{x_n}dx' + \int_{D_z} u_t u_{x_n}|\nabla u|^p dx' - \int_{D_z}\int_0^t g(-\tau)\,u_t(t)u_{x_n}(\tau)d\tau dx'. \tag{11}$$

By taking the scalar product of (1) with ϵu for $\epsilon > 0$, integrating over Ω_z and adding to (11), we find

$$\frac{d}{dt}\left[\ \frac{1}{2}\|u_t\|_{\Omega_z}^2 + \frac{a\epsilon}{2}\|u\|_{\Omega_z}^2 + \epsilon(u\,,u_t)_{\Omega_z} + \frac{1}{p+2}\int_{\Omega_z}|\nabla u|^{p+2}dx\ + \frac{1}{2}\|\nabla u\|_{\Omega_z}^2\left(1 - \int_0^t g(\tau)d\tau\right) + \frac{1}{2}(g \circ \nabla u)_{\Omega_z}(t)\ \right]\ + (a-\epsilon)\|u_t\|_{\Omega_z}^2$$
$$+ \left(\frac{1}{2}g(t) + \epsilon\right)\|\nabla u\|_{\Omega_z}^2 + \int_{\Omega_z}|u_t|^{p+2}dx + \epsilon\int_{\Omega_z} uu_t\,|u_t|^p dx + \epsilon\int_{\Omega_z}|\nabla u|^{p+2}dx - \frac{1}{2}(g' \circ \nabla u)_{\Omega_z}(t)$$
$$-\epsilon\int_{\Omega_z}\int_0^t g(t-\tau)\,\nabla u(t)\nabla u(\tau)d\tau dx = (u_t\,,u_{x_n})_{D_z} + \epsilon(u\,,u_{x_n})_{D_z} + \int_{D_z} u_t u_{x_n}|\nabla u|^p dx'$$
$$+\epsilon\int_{D_z} uu_{x_n}|\nabla u|^p dx' - \int_{D_z}\int_0^t g(t-\tau)\,u_t(t)u_{x_n}(\tau)d\tau dx' - \epsilon\int_{D_z}\int_0^t g(-\tau)\,u(t)u_{x_n}(\tau)d\tau dx'. \tag{12}$$

Using the Hölder and Young's inequalities, for the last integral in the right hand side of (12) we have

$$\int_{D_z}\int_0^t g(t-\tau)\,u(t)u_{x_n}(\tau)d\tau dx'$$
$$=\int_{D_z}\int_0^t g(t-\tau)\,u(t)[\,u_{x_n}(\tau)-u(t)\,]d\tau dx'$$
$$+\int_0^t g(t-\tau)\,\|u\|^2_{D_z}d\tau$$
$$\geq -\frac{1}{2}\ \ [\ \int_0^t g(t-\tau)\,\left\|u_{x_n}(\tau)-u(t)\right\|^2_{D_z}d\tau$$
$$+\int_0^t g(\tau)d\tau\,\|u\|^2_{D_z}\ \ ]\ +\int_0^t g(\tau)\,d\tau\|u\|^2_{D_z}. \qquad (13)$$

Analogously,

$$\int_{D_z}\int_0^t g(t-\tau)\,u_t(t)u_{x_n}(\tau)d\tau dx'$$
$$\geq -\frac{1}{2}\ \ [\ \int_0^t g(t-\tau)\,\left\|u_{x_n}(\tau)-u_t(t)\right\|^2_{D_z}d\tau$$
$$+\int_0^t g(\tau)d\tau\,\|u_t\|^2_{D_z}\ \ ]+\int_0^t g(\tau)\,d\tau\|u_t\|^2_{D_z}. \qquad (14)$$

Also, the last integral in the left hand side of (12) can be written in the form

$$\int_{\Omega_z}\int_0^t g(t-\tau)\,\nabla u(t)\nabla u(\tau)d\tau dx$$
$$=\frac{1}{2}\left(\int_0^t g(\tau)\,d\tau\right)\|\nabla u\|^2_{\Omega_z}+\frac{1}{2}\int_0^t g(t-\tau)\|\nabla u(\tau)\|^2_{\Omega_z}\,d\tau$$
$$-\frac{1}{2}(g\circ\nabla u)_{\Omega_z}(t). \qquad (15)$$

After integrating (12) with respect to t over $(0,T)$ and using (13)-(15), the conditions (5)-(6) and the inequality

$$\epsilon(u,u_t)_{\Omega_z}\geq -\epsilon^2\|u\|^2_{\Omega_z}-\frac{1}{4}\|u_t\|^2_{\Omega_z},$$

taking $\epsilon<\frac{a}{2}$, one can find

$$(a-\epsilon)\int_0^T\|u_t\|^2_{\Omega_z}\,dt+\frac{\epsilon l}{2}\int_0^T\|\nabla u\|^2_{\Omega_z}\,dt$$
$$+\int_0^T\int_{\Omega_z}|u_t|^{p+2}dx\,dt+\ \epsilon\int_0^T\int_{\Omega_z}|\nabla u|^{p+2}dx\,dt$$
$$+\frac{\epsilon}{2}\int_0^T(g\circ\nabla u)_{\Omega_z}(t)dt+\ \epsilon\int_0^T\int_{\Omega_z}uu_t\,|u_t|^p dxdt$$
$$+\frac{\epsilon}{2}\int_0^T\left(\|\nabla u\|^2_{\Omega_z}-\int_0^t g(t-\tau)\,\|\nabla u(\tau)\|^2_{\Omega_z}d\tau\right)dt$$
$$\leq\int_0^T(u_t\,,u_{x_n})_{D_z}dt+\epsilon\int_0^T(u\,,u_{x_n})_{D_z}dt$$
$$+\int_0^T\int_{D_z}u_t u_{x_n}|\nabla u|^p dx'\,dt$$
$$+\epsilon\int_0^T\int_{D_z}uu_{x_n}|\nabla u|^p dx'\,dt$$
$$+\frac{1}{2}\int_0^T\int_0^t g(t-\tau)\,\left\|u_{x_n}(\tau)-u_t(t)\right\|^2_{D_z}d\tau dt$$
$$+\frac{\epsilon}{2}\int_0^T\int_0^t g(t-\tau)\,\left\|u_{x_n}(\tau)-u(t)\right\|^2_{D_z}d\tau dt$$
$$+\frac{1-l}{2}\int_0^T(\|u_t\|^2_{D_z}+\epsilon\|u\|^2_{D_z})\,dt. \qquad (16)$$

Using Young and Poincaré inequalities, we obtain the following estimates

$$\int_0^T\int_{\Omega_z}uu_t\,|u_t|^p dxdt$$
$$\geq -c(\delta)\int_0^T\int_{\Omega_z}|u_t|^{p+2}dxdt$$
$$-\delta\int_0^T\int_{\Omega_z}|u|^{p+2}dxdt$$
$$\geq -c(\delta)\int_0^T\int_{\Omega_z}|u_t|^{p+2}dxdt$$
$$-\delta C_p\int_0^T\int_{\Omega_z}|\nabla u|^{p+2}dxdt, \qquad (17)$$

$$\int_0^T\int_{D_z}uu_{x_n}|\nabla u|^p dx'$$
$$\leq\frac{1}{p+2}\int_0^T\int_{D_z}|u|^{p+2}dx'dt$$
$$+\frac{p+1}{p+2}\int_0^T\int_{D_z}|\nabla u|^{p+2}dx'dt$$
$$\leq\left(\frac{C'_p+p+1}{p+2}\right)\int_0^T\int_{D_z}|\nabla u|^{p+2}dx'dt, \qquad (18)$$

$$\int_0^T\int_{D_z}u_t u_{x_n}|\nabla u|^p dx'$$
$$\leq\frac{1}{p+2}\int_0^T\int_{D_z}|u_t|^{p+2}dx'dt$$
$$+\frac{p+1}{p+2}\int_0^T\int_{D_z}|\nabla u|^{p+2}dx'dt, \qquad (19)$$

where C_p and C'_p are positive constants depending on Ω_z and D_z and δ is an arbitrary positive constant. Now, using the estimates (17)-(19) and (7) we find from (16) that

$$cE(z)\leq\int_0^T(u_t\,,u_{x_n})_{D_z}dt+\epsilon\int_0^T(u\,,u_{x_n})_{D_z}dt$$
$$+\frac{1-l}{2}\int_0^T(\|u_t\|^2_{D_z}+\epsilon\|u\|^2_{D_z})\,dt$$

$$+\frac{1}{p+2}\int_0^T\int_{D_z}|u_t|^{p+2}dx'dt$$
$$+M_1\int_0^T\int_{D_z}|\nabla u|^{p+2}dx'dt$$
$$+\frac{1}{2}\int_0^T\int_0^t g(t-\tau)\left\|u_{x_n}(\tau)-u_t(t)\right\|_{D_z}^2 d\tau dt$$
$$+\frac{\epsilon}{2}\int_0^T\int_0^t g(t-\tau)\left\|u_{x_n}(\tau)-u(t)\right\|_{D_z}^2 d\tau dt, \qquad (20)$$

where

$$c=\min\left\{a-\epsilon,\frac{\epsilon l}{2},1-\epsilon c(\delta),\epsilon\left(1-\delta C_p\right)\right\},$$
$$M_1=\frac{\epsilon C_p'+(p+1)(\epsilon+1)}{p+2},$$

and the constant δ is chosen such that $\delta<\frac{1}{C_p}$ and $\epsilon<\frac{1}{c(\delta)}$. For the last integral in the right hand side of the inequality (20) we have

$$\frac{1}{2}\int_0^T\int_0^t g(t-\tau)\left\|u_{x_n}(\tau)-u(t)\right\|_{D_z}^2 d\tau dt$$
$$\leq\int_0^T(g\circ\nabla u)_{D_z}(t)dt$$
$$+2(1-l)\int_0^T\left(\|\nabla u\|_{D_z}^2+\|u\|_{D_z}^2\right)dt, \qquad (21)$$

and similarly

$$\frac{1}{2}\int_0^T\int_0^t g(t-\tau)\left\|u_{x_n}(\tau)-u_t(t)\right\|_{D_z}^2 d\tau dt$$
$$\leq\int_0^T(g\circ\nabla u)_{D_z}(t)dt$$
$$+2(1-l)\int_0^T\left(\|\nabla u\|_{D_z}^2+\|u_t\|_{D_z}^2\right)dt. \qquad (22)$$

Using the Poincaré and Young inequalities and the estimates (20), (21) and (22), we obtain

$$cE(z)\leq\frac{1}{p+2}\int_0^T\int_{D_z}|u_t|^{p+2}dx'dt$$
$$+M_1\int_0^T\int_{D_z}|\nabla u|^{p+2}\,dx'dt$$
$$+(1+\epsilon)\int_0^T(g\circ\nabla u)_{D_z}(t)dt$$
$$+\left(\frac{6-5l}{2}\right)\int_0^T\|u_t\|_{D_z}^2dt$$
$$+M_2\int_0^T\|\nabla u\|_{D_z}^2\,dt, \qquad (23)$$

where

$$M_2=\frac{1}{2}\left[(1+\epsilon)(5-4l)+\epsilon\lambda^{-1}(6-5l)\right],$$

In which $\lambda=inf_z\,\lambda_z$ where λ_z is the Poincaré constant. Finally, due to (23) we can summarize the result in the following theorem.

Theorem 1. Let u be a nontrivial solution of (1)-(4) under the conditions (5)-(7) and $h(x',t)=0$. Then

$$\lim_{z\to\infty}\inf\,E(z)\exp\left(-\frac{c}{\gamma}z\right)>0,$$

where

$$\gamma=\max\left\{\frac{1}{p+2},1+\epsilon,M_1,\frac{6-5l}{2},M_2\right\}.$$

Theorem 2. Let u be a nontrivial solution of (1). Under the hypotheses of Theorem 1 with $\frac{\partial u}{\partial \nu}(x',0,t)=h(x',t)$ for $x_n=0$, if $E(+\infty)$ is finite then there is $\alpha>0$ such that

$$\lim_{z\to\infty}\exp(\alpha z)\quad\left\{\int_0^T\left(\|u_t\|_{R_z}^2+\|\nabla u\|_{R_z}^2\right)dt\right.$$
$$+\int_0^T\int_{R_z}|u_t|^{p+2}dxdt$$
$$+\int_0^T\int_{R_z}|\nabla u|^{p+2}dxdt$$
$$\left.+\int_0^T(g\circ\nabla u)_{R_z}(t)\,dt\right\}=0. \qquad (24)$$

Proof: Using the Young and Poincaré inequalities, we find

$$\int_{D_z}\int_0^t g(t-\tau)\,u(t)u_{x_n}(\tau)d\tau dx'$$
$$\leq\frac{1}{2}(1-l)\|u\|_{D_z}^2$$
$$+\frac{1}{2}\int_{D_z}\int_0^t g(t-\tau)\big|u_{x_n}(\tau)-u_{x_n}(t)+u_{x_n}(t)\big|^2 d\tau dx'$$
$$\leq(1-l)\left(1+\frac{\lambda^{-1}}{2}\right)\|\nabla u\|_{D_z}^2$$
$$+(g\circ\nabla u)_{D_z}(t), \qquad (25)$$

and

$$\int_{D_z}\int_0^t g(t-\tau)\,u_t(t)u_{x_n}(\tau)d\tau dx'$$
$$\leq\frac{1}{2}(1-l)\|u_t\|_{D_z}^2+(1-l)\|\nabla u\|_{D_z}^2$$
$$+(1-l)(g\circ\nabla u)_{D_z}(t). \qquad (26)$$

With the same manner followed in Theorem 1 and using (18), (19), (25) and (26) we deduce

$$\tilde{E}(z) \le -\frac{\tilde{\gamma}}{\sigma}\tilde{E}'(z), \qquad (27)$$

where

$$\tilde{E}(z) = \int_0^T \int_{R_z} (u_t^2 + |\nabla u|^2 + |u_t|^{p+2}$$
$$+|\nabla u|^{p+2})dxdt + \int_0^T (g \circ \nabla u)_{R_z}(t)dt,$$

$$\sigma = \min \{ a-\epsilon, \frac{\epsilon l}{2}, 1-\epsilon\eta, \epsilon\big(1-c(\eta)\tilde{C}_p\big)\},$$

and

$$\tilde{\gamma} = \max \{ \frac{2-l}{2}, \frac{(3-2l)(1+\epsilon(\lambda^{-1}+1))}{2}, \frac{1}{p+2}, M_1 \},$$

where $\tilde{C}_p$ is a positive constant which depends on the domain R_z and η is an arbitrary positive constant. We select η such that $c(\eta) < 1/\tilde{C}_p$. Then by choosing

$$\epsilon < \min\{\eta^{-1}, a\},$$

(24) follows from (27).

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