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Characterizations of ternary semigroups by $(\in, \in \vee q_k)$-fuzzy ideals

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Abstract

In this paper we have generalized the concepts of $(\in, \in \vee q)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy bi-ideals by introducing the concepts of $(\in, \in \vee q_k)$-fuzzy ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals in ternary semigroups and several related properties are investigated. Different characterizations of regular and weakly regular ternary semigroups by the properties of these ideals are given.

Keywords: Ternary semigroups; $(\in, \in \vee q_k)$-fuzzy ideals; $(\in, \in \vee q_k)$-fuzzy quasi-ideals; $(\in, \in \vee q_k)$-fuzzy bi-ideals

1. Introduction

Ternary semigroups were introduced by Lehmer [1] in 1932. In [2] Sioson developed the ideal theory of ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized regular ternary semigroups by the properties of their quasi-ideals. Dixit and Dewan in [3] discussed the quasi-ideals and bi-ideals in ternary semigroups. Properties of ideals in regular ternary semigroups and n-ary semigroups are studied in [4]. Applications of ideals to the divisibility theory in ternary and n-ary semigroups and rings can be found in [5]. The fundamental concept of fuzzy set introduced by Zadeh in his seminal paper [6] in 1965, laid the foundation of fuzzy set theory. Extensive applications of fuzzy set theory have been found in various fields such as economics, computer science, control engineering, expert system, information sciences, coding theory, operation research, robotics and many others. The fuzzification of algebraic structures was initiated by Rosenfeld in his pioneering paper [7] in 1971, where he introduced the notion of fuzzy subgroup of a group. The literature on fuzzy set theory has grown rapidly. Kuroki initiated the study of fuzzy semigroups (see [8-11]). A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [12], where theoretical results on fuzzy semigroups and its applications in fuzzy coding, fuzzy finite state machines and fuzzy languages can be found. Murali [13] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. Pu and Liu in [14] introduced the idea of quasi-coincidence of a fuzzy point with a fuzzy set. It is worth mentioning that Bhakat in [15] and Bhakat and Das (see [16-18]) introduced the concept of (α, β)-fuzzy subgroup by using ‘belongs to’ and ‘quasi-coincident with’ between a fuzzy point and a fuzzy subset and introduced the concepts of $(\in, \in \vee q)$-fuzzy normal, quasinormal, maximal subgroup and $(\in, \in \vee q)$-fuzzy subgroup. In fact $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. Many researchers used these concepts to generalize some concepts of algebra (see [19, 20]). Davvaz in [21] discussed $(\in, \in \vee q)$-fuzzy subnear rings and ideals. The present authors in [22] introduced (α, β)-fuzzy ideals in ternary semigroups, where they characterized regular ternary semigroups by the properties of these ideals. Generalizing the concept of quasi-coincidence of a fuzzy point with a fuzzy set, Jun in [23] defined $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras. In [24] Jun et al. discussed $(\in, \in \vee q_k)$-fuzzy h-ideals and $(\in, \in \vee q_k)$-fuzzy k-ideals of a hemiring. Jun et

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al. in [25] introduced the notion of $(\in, \in \vee q_k)$-fuzzy filters in BL-algebra. Shabir et al. in [26] characterized different classes of semigroups by $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals. Recently, Shabir and Mahmood in [27] defined $(\in, \in \vee q_k)$-fuzzy h-subhemirings, $(\in, \in \vee q_k)$-fuzzy h-bi-ideals of a hemiring, where they characterized h-hemiregular and h-intra hemiregular hemirings by the properties of their $(\in, \in \vee q_k)$-fuzzy h-ideals.

In the present paper we introduce the concepts of $(\in, \in \vee q_k)$-fuzzy ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals and $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals in ternary semigroups. We also investigated some related properties of ternary semigroups. We have characterized regular and weakly regular ternary semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals, $(\in, \in \vee q_k)$-fuzzy bi-ideals and $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals.

2. Preliminaries

A ternary semigroup is an algebraic structure $(S, [\ ])$ such that S is a non-empty set and $[\ ] : S^3 \to S$ a ternary operation satisfying the following associative law: $[[(abc)]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in S$. For simplicity we write $[abc]$ as 'abc' and consider the ternary operation $[\ ]$ as '·'. A non-empty subset A of a ternary semigroup S is called a ternary subsemigroup of S if $AAA \subseteq A$. By a left (right, lateral) ideal of a ternary semigroup S we mean a non-empty subset A of S such that $SSA \subseteq A$ $(ASS \subseteq A, SAS \subseteq A)$. If a non-empty subset A of S is a left and right ideal of S, then it is called a two sided ideal of S. If a non-empty subset A of a ternary semigroup S is a left, right and lateral ideal of S, then it is called an ideal of S. A non-empty subset A of a ternary semigroup S is called a quasi-ideal of S if $ASS \cap SAS \cap SSA \subseteq A$ and $ASS \cap SSASS \cap SSA \subseteq A$. A is called a bi-ideal of S if it is a ternary subsemigroup of S and $ASASA \subseteq A$. A non-empty subset A of a ternary semigroup S is called a generalized bi-ideal of S if $ASASA \subseteq A$. It is clear that every left (right, lateral) ideal of S is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal of S. An element a of a ternary semigroup S is called regular if there exists an element $x \in S$ such that $axa = a$. A ternary semigroup S is called regular if every element of S is regular.

Theorem 1. [2] A ternary semigroup S is regular if and only if $R \cap M \cap L = RML$ for every right ideal R, every lateral ideal M and every left ideal L of S.

Theorem 2. [28] The following conditions on a ternary semigroup S are equivalent:
(1) S is regular;
(2) $BSBSB = B$ for every bi-ideal B of S;
(3) $QSQSQ = Q$ for every quasi-ideal Q of S.

Theorem 3. [28] A ternary semigroup S is regular if and only if $R \cap L = RSL$ for every right ideal R and every left ideal L of S.

If $A \subseteq S$, then the characteristic function of A is a function C_A of S into $\{0,1\}$ defined by:

$$C_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

A fuzzy subset f of a universe X is a function from X into the unit closed interval $[0,1]$, that is, $f : X \to [0,1]$. A fuzzy subset f in a universe X of the form:

$$f(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and fuzzy set f in a set X, Pu and Liu [14] introduced the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t q f$) if $f(x) \geq t$ resp. ($f(x) + t > 1$) and in this case, $x_t \in \vee q f$ (resp. $x_t \in \wedge q f$) means that $x_t \in f$ or $x_t q f$ (resp.

$x_t \in f$ and $x_t qf$). To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold.

For any two fuzzy subsets f and g, $f \le g$ means $f(x) \le g(x)$ for all $x \in S$. The symbols $f \wedge g$ and $f \vee g$ have the following meanings:
$(f \wedge g)(x) = f(x) \wedge g(x)$ and
$(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in S$.

Let f, g and h be three fuzzy subsets of a ternary semigroup S. The product $f \circ g \circ h$ is a fuzzy subset of S defined by:

$$(f \circ g \circ h)(a) = \begin{cases} \bigvee\limits_{a=xyz} \{f(x) \wedge g(y) \wedge h(z)\} & \text{if there exist } x, y, z \in S \\ & \text{such that } a = xyz \\ 0 & \text{otherwise.} \end{cases}$$

Let f be a fuzzy subset of a ternary semigroup S. Then the set
$U(f;t) = \{x \in S \ : \ f(x) \ge t\}$, where $t \in [0,1]$, is called a level subset of f.

Definition 1. A fuzzy subset f of a ternary semigroup S is a fuzzy ternary subsemigroup of S if $f(xyz) \ge f(x) \wedge f(y) \wedge f(z)$ for all $x, y, z \in S$.

Definition 2. A fuzzy subset f of a ternary semigroup S is a fuzzy left (right, lateral) ideal of S if $f(xyz) \ge f(z)$ $(f(xyz) \ge f(x), f(xyz) \ge f(y))$ for all $x, y, z \in S$.

Definition 3. A fuzzy subset f of a ternary semigroup S is called a fuzzy generalized bi-ideal of S if $f(xuyvz) \ge f(x) \wedge f(y) \wedge f(z)$ for all $u, v, x, y, z \in S$, and is called a fuzzy bi-ideal of S if it is both fuzzy ternary subsemigroup and fuzzy generalized bi-ideal of S.

3. $(\in, \in \vee q_k)$-fuzzy ideals

Throughout this paper S will denote a ternary semigroup and k be an arbitrary element of $[0,1)$ unless otherwise specified.

In [23], Jun defined $x_t q_k f$ if $f(x) + t + k > 1$, $x_t \in \vee q_k f$ if $x_t \in f$ or $x_t q_k f$.

Definition 4. A fuzzy subset f of a ternary semigroup S is called an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S if:
$(T1)$ for all $x, y, z \in S$ and $t, r, s \in (0,1]$, $x_t \in f$, $y_r \in f$, $z_s \in f$ implies $(xyz)_{\min\{t,r,s\}} \in \vee q_k f$.

Definition 5. A fuzzy subset f of a ternary semigroup S is called an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S if:
$(T2)$ for all $x, y, z \in S$ and $t \in (0,1]$, $z_t \in f$ implies $(xyz)_t \in \vee q_k f$ ($(zxy)_t \in \vee q_k f$, $(xzy)_t \in \vee q_k f$).

Definition 6. A fuzzy subset f of a ternary semigroup S is called an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S if it is both $(\in, \in \vee q_k)$-fuzzy left ideal and $(\in, \in \vee q_k)$-fuzzy right ideal of S. A fuzzy subset f of a ternary semigroup S is called an $(\in, \in \vee q_k)$-fuzzy ideal of S if it is an $(\in, \in \vee q_k)$-fuzzy left ideal, $(\in, \in \vee q_k)$-fuzzy right ideal and $(\in, \in \vee q_k)$-fuzzy lateral ideal of S.

Remark 1. Every fuzzy left (right, lateral) ideal of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S, but the converse is not true.

Remark 2. Every $(\in, \in \vee q)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal) of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal) of S, but the converse is not true.

Example 1. Consider the ternary semigroup $S = \{a, b, c, d\}$, with the ternary operation $[xyz] = x$ for all $x, y, z \in S$. Define a fuzzy subset f of S by:

$$f(a) = 0.7, \ \ f(b) = 0.6, \ \ f(c) = 0.8, \ f(d) = 0.55.$$

Then simple calculations show that f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S for $k = 0.2$, but not a fuzzy left ideal of S, because $f(dba) \not\ge f(a)$.

Example 2. Consider $S=\{-i,0,i\}$, where S is a ternary semigroup under the usual multiplication of complex numbers. Define a fuzzy subset f of S by:

$$f(-i)=0.35,\ \ f(i)=0.7,\ \ f(0)=0.8.$$

Then we have

$$U(f;t)=\begin{cases} S & \text{if } t\le 0.35\\ \{0,i\} & \text{if } 0.35<t\le 0.7\\ \{0\} & \text{if } 0.7<t\le 0.8\\ \phi & \text{if } 0.8<t\le 1.\end{cases}$$

Then simple calculations show that f is an $(\in,\in\vee q_k)$-fuzzy ternary subsemigroup of S for $k=0.3$, but not an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S because $i_{0.7}\in f$ but $(iii)_{0.7}\overline{\in}\wedge\overline{q}f$.

Theorem 4. Let A be a ternary subsemigroup (left ideal, right ideal, lateral ideal) of a ternary semigroup S and $\alpha\in\{\in,q,\in\vee q\}$. Then the fuzzy subset f of S defined by:

$$f(x)=\begin{cases}\ge \frac{1-k}{2} & \text{if } x\in A\\ 0 & \text{otherwise,}\end{cases}$$

is an $(\alpha,\in\vee q_k)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal) of S.

Proof: Let A be a ternary subsemigroup of S.
(1) If $\alpha=\in$. Let $x,y,z\in S$ and $t,r,s\in(0,1]$ be such that $x_t\in f$, $y_r\in f$, $z_s\in f$. Then $f(x)\ge t>0,\ f(y)\ge r>0,\ f(z)\ge s>0$. Thus $f(x)\ge\frac{1-k}{2}$, $f(y)\ge\frac{1-k}{2}$, $f(z)\ge\frac{1-k}{2}$. This implies that $x,y,z\in A$. Since A is a ternary subsemigroup of S so, $xyz\in A$. Hence $f(xyz)\ge\frac{1-k}{2}$.

If $\min\{t,r,s\}\le\frac{1-k}{2}$, then $f(xyz)\ge\frac{1-k}{2}\ge\min\{t,r,s\}$. This implies that $(xyz)_{\min\{t,r,s\}}\in f$.

If $\min\{t,r,s\}>\frac{1-k}{2}$, then $f(xyz)+\min\{t,r,s\}+k>\frac{1-k}{2}+\frac{1-k}{2}+k>1$ and so $(xyz)_{\min\{t,r,s\}}q_kf$. Hence f is an $(\in,\in\vee q_k)$-fuzzy ternary subsemigroup of S.

(2) If $\alpha=q$. Let $x,y,z\in S$ and $t,r,s\in(0,1]$ be such that x_tqf, y_rqf, z_sqf. Then, $f(x)+t>1$, $f(y)+r>1$, $f(z)+s>1$. So $x,y,z\in A$. Since A is a ternary subsemigroup of S so, $xyz\in A$. Thus $f(xyz)\ge\frac{1-k}{2}$.

If $\min\{t,r,s\}\le\frac{1-k}{2}$, then $f(xyz)\ge\frac{1-k}{2}\ge\min\{t,r,s\}$. So $(xyz)_{\min\{t,r,s\}}\in f$.

If $\min\{t,r,s\}>\frac{1-k}{2}$, then $f(xyz)+\min\{t,r,s\}+k>1$. So $(xyz)_{\min\{t,r,s\}}q_kf$.

Hence f is a $(q,\in\vee q_k)$-fuzzy ternary subsemigroup of S.

(3) If $\alpha=\in\vee q$, let $x,y,z\in S$ and $t,r,s\in(0,1]$ be such that $x_t\in\vee qf$, $y_r\in\vee qf$, $z_s\in\vee qf$.
This implies that $x_t\in f$ or x_tqf, $y_r\in f$ or y_rqf, $z_s\in f$ or z_sqf.

If x_tqf, $y_r\in f$, z_sqf. This implies that $f(x)+t>1$, $f(y)\ge r$, $f(z)+s>1$. This implies that $x,y,z\in A$. Since A is a ternary subsemigroup of S, $xyz\in A$. This implies that $f(xyz)\ge\frac{1-k}{2}$. Similarly for the other cases. Analogous to (1) and (2) we obtain $(xyz)_{\min\{t,r,s\}}q_kf$. Therefore, f is an $(\in\vee q,\in\vee q_k)$-fuzzy ternary subsemigroup of S.

Corollary 1. [22] Let A be a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of S, and $\alpha\in\{\in,q,\in\vee q\}$. Then the fuzzy subset f of S defined by:

$$f(x)=\begin{cases}\ge\frac{1}{2} & \text{if } x\in A\\ 0 & \text{otherwise,}\end{cases}$$

is an $(\alpha,\in\vee q)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of S.

Theorem 5. Let f be a fuzzy subset of a ternary semigroup S. Then f is an $(\in,\in\vee q_k)$-fuzzy ternary subsemigroup of S if and only if, $f(xyz)\ge\min\{f(x),f(y),f(z),\frac{1-k}{2}\}$.

Proof: Suppose f is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S. On the contrary, assume that there exist $x, y, z \in S$ such that $f(xyz) < \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$.
Choose $t \in (0,1]$ such that $f(xyz) < t \leq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$.
Then
$x_t \in f$, $y_t \in f$, $z_t \in f$. But $f(xyz) < t \Rightarrow (xyz)_t \overline{\in} f$ and $f(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. So $(xyz)_{\min\{t,t,t\}} \overline{\in \vee q_k} f = (xyz)_t \overline{\in \vee q_k} f$. Which is a contradiction. Hence, $f(xyz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$.

Conversely, assume that $f(xyz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$. Let $x_t \in f$, $y_r \in f$, $z_s \in f$ for all $t, r, s \in (0,1]$. Then $f(x) \geq t$, $f(y) \geq r$, $f(z) \geq s$. Now, $f(xyz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\} \geq \min\{t, r, s, \frac{1-k}{2}\}$.

If $\min\{t, r, s\} \leq \frac{1-k}{2}$, then $f(xyz) \geq \min\{t, r, s\}$. So $(xyz)_{\min\{t,r,s\}} \in f$.

If $\min\{t, r, s\} > \frac{1-k}{2}$, then $f(xyz) + \min\{t, r, s\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. This implies that $(xyz)_{\min\{t,r,s\}} q_k f$. Hence f is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S.

Taking $k = 0$ in Theorem 5, we have the following corollary.

Corollary 2. [22] Let f be a fuzzy subset of a ternary semigroup S. Then f is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if, $f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\}$.

Theorem 6. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S if and only if the level subset $U(f;t)(\neq \phi)$ is a ternary subsemigroup of S for all $t \in (0, \frac{1-k}{2}]$.

Proof: Suppose f is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S and $x, y, z \in U(f;t)$ for some $t \in (0, \frac{1-k}{2}]$. Then $f(x) \geq t$, $f(y) \geq t$, $f(z) \geq t$. By Theorem 5, $f(xyz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} = t$. So $f(xyz) \geq t$. This implies that $xyz \in U(f;t)$. Hence $U(f;t)$ is a ternary subsemigroup of S.

Conversely, assume that $U(f;t)(\neq \phi)$ is a ternary subsemigroup of S for all $t \in (0, \frac{1-k}{2}]$. Suppose, on the contrary, that there exist $x, y, z \in S$ such that $f(xyz) < \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$. Choose $t \in (0, \frac{1-k}{2}]$ such that $f(xyz) < t \leq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$. Then $x, y, z \in U(f;t)$, but $xyz \notin U(f;t)$, which is a contradiction. Thus $f(xyz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$. Hence f is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S.

Taking $k = 0$ in Theorem 6, we have the following corollary.

Corollary 3. [22] A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if the level subset $U(f;t)(\neq \phi)$ is a ternary subsemigroup of S, for all $t \in (0, 0.5]$.

Theorem 7. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S if and only if $f(xyz) \geq \min\{f(z), \frac{1-k}{2}\}$ $(f(xyz) \geq \min\{f(x), \frac{1-k}{2}\}, f(xyz) \geq \min\{f(y), \frac{1-k}{2}\})$, for all $x, y, z \in S$.

Proof: Suppose f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Suppose, on the contrary, that there exist $x, y, z \in S$ such that $f(xyz) < \min\{f(z), \frac{1-k}{2}\}$. Choose $t \in (0,1]$ such that $f(xyz) < t \leq \min\{f(z), \frac{1-k}{2}\}$. Then $z_t \in f$ but $(xyz)_t \overline{\in} f$. Also, $f(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. This implies that $(xyz)_t \overline{\in \vee q_k} f$, which contradicts the hypothesis. Hence $f(xyz) \geq \min\{f(z), \frac{1-k}{2}\}$.

Conversely, assume that $f(xyz) \geq \min\{f(z), \frac{1-k}{2}\}$. Let $x, y, z \in S$ and $t \in (0,1]$ be such that $z_t \in f$. Then $f(z) \geq t$. This implies that $f(xyz) \geq \min\{f(z), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$.

If $t > \frac{1-k}{2}$, then $f(xyz) \geq \frac{1-k}{2}$. So $f(xyz) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. This implies that $(xyz)_t q_k f$.

If $t \leq \frac{1-k}{2}$, then $f(xyz) \geq t$. This implies that

$(xyz)_t \in f$. Thus $(xyz)_t \in \vee q_k f$. Hence f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S.

Corollary 4. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy ideal of S if and only if $f(xyz) \geq \min\{f(z), \frac{1-k}{2}\}$, $f(xyz) \geq \min\{f(x), \frac{1-k}{2}\}$, and $f(xyz) \geq \min\{f(y), \frac{1-k}{2}\}$.

If we take $k = 0$ in Theorem 7, we obtain the following corollary.

Corollary 5. [22] A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal of S if and only if $f(xyz) \geq min\{f(z), o.5\}\left(\begin{array}{l} f(xyz) \geq min\{f(x), o.5\}, \\ f(xyz) \geq min\{f(y), o.5\} \end{array}\right)$.

Theorem 8. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S if and only if $U(f;t)(\neq \phi)$ is a left (right, lateral) ideal of S.

Proof: The proof is similar to the proof of Theorem 6.

Theorem 9. If f is an $(\in, \in \vee q_k)$-fuzzy left ideal, g an $(\in, \in \vee q_k)$-fuzzy lateral ideal and h an $(\in, \in \vee q_k)$-fuzzy right ideal of a ternary semigroup S, then $f \circ g \circ h$ is an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S.

Proof: The proof is straightforward.

Lemma 1. The intersection of any family of $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideals of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S.

Proof: Let $\{f_i\}_{i \in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy left ideals of a ternary semigroup S and $x, y, z \in S$. Then, $\left(\underset{i \in I}{\wedge} f_i\right)(xyz) = \underset{i \in I}{\wedge}\left(f_i(xyz)\right)$.

Since each f_i is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, $f_i(xyz) \geq f_i(z) \wedge \frac{1-k}{2}$ for all $i \in I$. Thus,

$$\left(\underset{i \in I}{\wedge} f_i\right)(xyz) = \underset{i \in I}{\wedge}\left(f_i(xyz)\right) \geq \underset{i \in I}{\wedge}\left(f_i(z) \wedge \tfrac{1-k}{2}\right)$$

$$= \left(\underset{i \in I}{\wedge} f_i(z)\right) \wedge \tfrac{1-k}{2} = \left(\underset{i \in I}{\wedge} f_i\right)(z) \wedge \tfrac{1-k}{2}.$$

Hence, $\underset{i \in I}{\wedge} f_i$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S.

In a similar fashion we can prove the following lemma.

Lemma 2. The union of any family of $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideals of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S.

Definition 7. An $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup f of a ternary semigroup S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if for all $u, v, x, y, z \in S$ and $t, r, s \in (0,1]$ the following condition holds:

$(T3)$ $x_t \in f$, $y_r \in f$, and $z_s \in f$ implies $(xuyvz)_{\min\{t,r,s\}} \in \vee q_k f$.

Theorem 10. Let B be a bi-ideal of a ternary semigroup S and $\alpha \in \{\in, q, \in \vee q\}$. Then the fuzzy subset f of S defined by:

$$f(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in B \\ 0 & \text{otherwise,} \end{cases}$$

is an $(\alpha, \in \vee q_k)$-fuzzy bi-ideal of S.

Proof: The proof is similar to the proof of Theorem 4.

Theorem 11. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if it satisfies the following conditions:

(1) $f(xyz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$

(2) $f(xuyvz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$ for all $u, v, x, y, z \in S$ and $k \in [0,1)$.

Proof: The proof is similar to the proof of Theorem 5.

Theorem 12. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if $U(f;t)(\neq \phi)$ is a bi-ideal of S

for all $t \in (0, \frac{1-k}{2}]$.

Proof: The proof is similar to the proof of Theorem 6.

Definition 8. A fuzzy subset f of a ternary semigroup S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if $(T3)$ holds.

Theorem 13. Let G be a generalized bi-ideal of a ternary semigroup S and $\alpha \in \{\in, q, \in \vee q\}$. Then the fuzzy subset f of S defined by:

$$f(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in G \\ 0 & \text{otherwise,} \end{cases}$$

is an $(\alpha, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

Proof: The proof is similar to the proof of Theorem 4.

Theorem 14. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if it satisfies the following condition:

$(T4)$ $\quad f(xuyvz) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$

for all $u, v, x, y, z \in S$ and $k \in [0,1)$.

Proof: The proof is similar to the proof of Theorem 5.

Theorem 15. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if $U(f;t)(\neq \phi)$ is a generalized bi-ideal of S for all $t \in (0, \frac{1-k}{2}]$.

Proof: The proof is similar to the proof of Theorem 6.

Definition 9. A fuzzy subset f of a ternary semigroup S is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S if it satisfies:

(1) $f(x) \geq \min\{(f \circ \mathbf{S} \circ \mathbf{S})(x), (\mathbf{S} \circ f \circ \mathbf{S})(x), (\mathbf{S} \circ \mathbf{S} \circ f)(x), \frac{1-k}{2}\}$;

(2) $f(x) \geq \min\{(f \circ \mathbf{S} \circ \mathbf{S})(x), (\mathbf{S} \circ \mathbf{S} \circ f \circ \mathbf{S} \circ \mathbf{S})(x), (\mathbf{S} \circ \mathbf{S} \circ f)(x), \frac{1-k}{2}\}$,

where $\mathbf{S}$ is the fuzzy subset of S mapping every element of S on 1.

Theorem 16. Let f be an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a ternary semigroup S. Then the set $f_0 = \{x \in S \ : \ f(x) > 0\}$ is a quasi-ideal of S.

Proof: We show that $SSf_0 \cap Sf_0S \cap f_0SS \subseteq f_0$ and $SSf_0 \cap SSf_0SS \cap f_0SS \subseteq f_0$.

Let $a \in SSf_0 \cap Sf_0S \cap f_0SS$. This implies that $a \in SSf_0$, $a \in Sf_0S$ and $a \in f_0SS$. Thus there exist $x, y, z \in f_0$ and $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = xs_1t_1$, $a = s_2yt_2$, $a = s_3t_3z$.

Now,

$$\begin{aligned}(\mathbf{S} \circ \mathbf{S} \circ f)(a) &= \bigvee_{a=pqr} \{\mathbf{S}(p) \wedge \mathbf{S}(q) \wedge f(r)\} \\ &\geq \mathbf{S}(s_3) \wedge \mathbf{S}(t_3) \wedge f(z) = f(z)\end{aligned}$$

This implies that $(\mathbf{S} \circ \mathbf{S} \circ f)(a) \geq f(z)$.

Similarly, $(\mathbf{S} \circ f \circ \mathbf{S})(a) \geq f(y)$ and $(f \circ \mathbf{S} \circ \mathbf{S})(a) \geq f(x)$, so

$$\begin{aligned}f(a) &\geq \min\left\{(f \circ \mathbf{S} \circ \mathbf{S})(a), (\mathbf{S} \circ f \circ \mathbf{S})(a), (\mathbf{S} \circ \mathbf{S} \circ f)(a), \frac{1-k}{2}\right\} \\ &\geq \min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\} > 0.\end{aligned}$$

This implies that $a \in f_0$.

Thus, $SSf_0 \cap Sf_0S \cap f_0SS \subseteq f_0$.

Again, let $a \in SSf_0 \cap SSf_0SS \cap f_0SS$. Then $a \in SSf_0$ and $a \in SSf_0SS$ and $a \in f_0SS$. Thus $a = xs_1t_1$, $a = s_2t_2ys_4t_4$, $a = s_3t_3z$ for some $x, y, z, s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4 \in S$. For $a = s_3t_3z$ and $a = xs_1t_1$, discussed above.

Now,

$$f(a) \geq \min\{(f \circ \mathbf{S} \circ \mathbf{S})(a), (\mathbf{S} \circ \mathbf{S} \circ f \circ \mathbf{S} \circ \mathbf{S})(a), (\mathbf{S} \circ \mathbf{S} \circ f)(a), \tfrac{1-k}{2}\},$$

and by the above arguments

$(\mathbf{S} \circ \mathbf{S} \circ f)(a) \geq f(z)$, $\ (f \circ \mathbf{S} \circ \mathbf{S})(a) \geq f(x)$.

Now,

$$(\mathbf{S}\circ\mathbf{S}\circ f\circ\mathbf{S}\circ\mathbf{S})(a)=\left(\bigvee_{a=pqr}((\mathbf{S}\circ\mathbf{S}\circ f)(p)\wedge\mathbf{S}(q)\wedge\mathbf{S}(r))\right)$$
$$=\left(\bigvee_{a=pqr}\left(\bigvee_{p=uvw}\mathbf{S}(u)\wedge\mathbf{S}(v)\wedge f(w)\right)\wedge\mathbf{S}(q)\wedge\mathbf{S}(r)\right)$$
$$=\left(\bigvee_{a=(uvw)qr}\mathbf{S}(u)\wedge\mathbf{S}(v)\wedge f(w)\wedge\mathbf{S}(q)\wedge\mathbf{S}(r)\right)$$
$$\geq\mathbf{S}(s_2)\wedge\mathbf{S}(t_2)\wedge f(y)\wedge\mathbf{S}(s_4)\wedge\mathbf{S}(t_4)=f(y).$$

Now,

$$f(a)\geq\min\left\{(f\circ\mathbf{S}\circ\mathbf{S})(a),(\mathbf{S}\circ\mathbf{S}\circ f\circ\mathbf{S}\circ\mathbf{S})(a),(\mathbf{S}\circ\mathbf{S}\circ f)(a),\frac{1-k}{2}\right\}$$
$$\geq\min\left\{f(x),f(y),f(z),\frac{1-k}{2}\right\}$$
$$>0\qquad\because f(x),f(y),f(z)>0,\frac{1-k}{2}>0.$$

This implies that $a\in f_0$. Hence $SSf_0\cap SSf_0SS\cap f_0SS\subseteq f_0$.
Therefore, f_0 is a quasi-ideal of S.

Lemma 3. A non-empty subset Q of a ternary semigroup S is a quasi-ideal of S if and only if the characteristic function C_Q, of Q, is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S.

Proof: The proof is straightforward.
In a similar fashion we can prove the following lemma.

Lemma 4. A non-empty subset A of a ternary semigroup S is a left (right, lateral, two sided) ideal of S if and only if the characteristic function C_A, of A, is an $(\in,\in\vee q_k)$-fuzzy left (right, lateral, two sided) ideal of S.

Theorem 17. Every $(\in,\in\vee q_k)$-fuzzy left (right, lateral) ideal of a ternary semigroup S is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S.

Proof: The proof is straightforward.

Remark 3. The converse of the above theorem does not hold in general.

Example 3. Let
$$S=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$
be a ternary semigroup under ternary matrix multiplication. Then $Q=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ be the quasi-ideal of S which is neither left, nor right nor a lateral ideal of S (see [3]). Then by Lemma 3, C_Q is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S and C_Q is neither $(\in,\in\vee q_k)$-fuzzy left ideal, nor $(\in,\in\vee q_k)$-fuzzy right, nor an $(\in,\in\vee q_k)$-fuzzy lateral ideal of S.

Theorem 18. Every $(\in,\in\vee q_k)$-fuzzy quasi-ideal of a ternary semigroup S is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S.

Proof: Suppose f is an $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S and $u,v,x,y,z\in S$.
Now,

$$f(xyz)\geq(f\circ\mathbf{S}\circ\mathbf{S})(xyz)\wedge(\mathbf{S}\circ f\circ\mathbf{S})(xyz)\wedge(\mathbf{S}\circ\mathbf{S}\circ f)(xyz)\wedge\frac{1-k}{2}$$
$$=\left(\bigvee_{xyz=abc}\{f(a)\wedge\mathbf{S}(b)\wedge\mathbf{S}(c)\}\right)\wedge\left(\bigvee_{xyz=pqr}\{\mathbf{S}(p)\wedge f(q)\wedge\mathbf{S}(r)\}\right)$$
$$\wedge\left(\bigvee_{xyz=lmn}\{\mathbf{S}(l)\wedge\mathbf{S}(m)\wedge f(n)\}\right)\wedge\frac{1-k}{2}$$
$$\geq(f(x)\wedge\mathbf{S}(y)\wedge\mathbf{S}(z))\wedge(\mathbf{S}(x)\wedge f(y)\wedge\mathbf{S}(z))$$
$$\wedge(\mathbf{S}(x)\wedge\mathbf{S}(y)\wedge f(z))\wedge\frac{1-k}{2}=f(x)\wedge f(y)\wedge f(z)\wedge\frac{1-k}{2}$$

So $f(xyz)\geq\min\{f(x),f(y),f(z),\frac{1-k}{2}\}$.
Also,

$$f(xuyvz)\geq(f\circ\mathbf{S}\circ\mathbf{S})(xuyvz)\wedge(\mathbf{S}\circ\mathbf{S}\circ f\circ\mathbf{S}\circ\mathbf{S})(xuyvz)$$
$$\wedge(\mathbf{S}\circ\mathbf{S}\circ f)(xuyvz)\wedge\frac{1-k}{2}$$
$$=\left(\bigvee_{xuyvz=abc}\{f(a)\wedge\mathbf{S}(b)\wedge\mathbf{S}(c)\}\right)$$
$$\wedge\left(\bigvee_{xuyvz=rst}\left\{\left(\bigvee_{r=s_1s_2s_3}\mathbf{S}(s_1)\wedge\mathbf{S}(s_2)\wedge f(s_3)\right)\wedge\mathbf{S}(s)\wedge\mathbf{S}(t)\right\}\right)$$
$$\wedge\left(\bigvee_{xuyvz=lmn}\{\mathbf{S}(l)\wedge\mathbf{S}(m)\wedge f(n)\}\right)\wedge\frac{1-k}{2}$$
$$\geq(f(x)\wedge\mathbf{S}(uyv)\wedge\mathbf{S}(z))\wedge((\mathbf{S}(x)\wedge\mathbf{S}(u)\wedge f(y)\wedge\mathbf{S}(v)\wedge\mathbf{S}(z)))$$
$$\wedge(\mathbf{S}(x)\wedge\mathbf{S}(uyv)\wedge f(z))\wedge\frac{1-k}{2}$$
$$=f(x)\wedge f(y)\wedge f(z)\wedge\frac{1-k}{2}.$$

This implies that
$f(xuyvz)\geq\min\{f(x),f(y),f(z),\frac{1-k}{2}\}$.
Therefore, f is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S.

4. Regular Ternary Semigroups

In this section we characterize regular ternary semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals, $(\in, \in \vee q_k)$-fuzzy bi-ideals.

Definition 10. Let f, g and h be fuzzy subsets of a ternary semigroup S. We define the fuzzy subsets f_k, $f \wedge_k g$, $f \vee_k g$, and $f \circ_k g \circ_k h$ as follows:

(1) $f_k(x) = f(x) \wedge \frac{1-k}{2}$

(2) $(f \wedge_k g)(x) = (f \wedge g)(x) \wedge \frac{1-k}{2}$

(3) $(f \vee_k g)(x) = (f \vee g)(x) \wedge \frac{1-k}{2}$,

(4) $(f \circ_k g \circ_k h)(x) = (f \circ g \circ h)(x) \wedge \frac{1-k}{2}$ for all $x \in S$.

Lemma 5. Let f, g and h be fuzzy subsets of a ternary semigroup S. Then the following hold:

(1) $f \wedge_k g = f_k \wedge g_k$;

(2) $f \vee_k g = f_k \vee g_k$;

(3) $f \circ_k g \circ_k h = f_k \circ g_k \circ h_k$.

Theorem 19. Let f be an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of a ternary semigroup S. Then f_k is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S.

Proof: Assume that f is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of a ternary semigroup S and $x, y, z \in S$. Then

$$\begin{aligned}(f_k)(xyz) &= f(xyz) \wedge \frac{1-k}{2} \geq \left(\min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}\right) \wedge \frac{1-k}{2} \\ &= \min\left\{f(x) \wedge \frac{1-k}{2}, f(y) \wedge \frac{1-k}{2}, f(z) \wedge \frac{1-k}{2}, \frac{1-k}{2}\right\} \\ &= \min\left\{f_k(x), f_k(y), f_k(z), \frac{1-k}{2}\right\}.\end{aligned}$$

This shows that f_k is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S.

Theorem 20. Let f be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of a ternary semigroup S. Then f_k is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Proof: Let $u, v, x, y, z \in S$. Then

$$\begin{aligned}(f_k)(xuyvz) &= f(xuyvz) \wedge \frac{1-k}{2} \geq \left(\min\left\{f(x), f(y), f(z), \frac{1-k}{2}\right\}\right) \wedge \frac{1-k}{2} \\ &= \min\left\{f(x) \wedge \frac{1-k}{2}, f(y) \wedge \frac{1-k}{2}, f(z) \wedge \frac{1-k}{2}, \frac{1-k}{2}\right\} \\ &= \min\left\{f_k(x), f_k(y), f_k(z), \frac{1-k}{2}\right\}.\end{aligned}$$

Also, f_k is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S by Theorem 19.

Therefore, f_k is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

In a similar fashion we can prove the following theorems.

Theorem 21. Let f be an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of a ternary semigroup S. Then f_k is an $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S.

Theorem 22. Let f be an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a ternary semigroup S. Then f_k is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.

Lemma 6. Let A, B, and C be non-empty subsets of a ternary semigroup S. Then the following hold:

(1) $(C_A \wedge_k C_B) = (C_{A \cap B})_k$;

(2) $(C_A \vee_k C_B) = (C_{A \cup B})_k$;

(3) $(C_A \circ_k C_B \circ_k C_c) = (C_{ABC})_k$, where C_A is the characteristic function of A.

Lemma 7. A non-empty subset A of a ternary semigroup S is a left (right, lateral, two sided) ideal of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (right, lateral, two sided) ideal of S.

Proof: Let A be a left ideal of S. Then by Theorem 4, $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S.

Conversely, assume that $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Let $z \in A$. Then $(C_A)_k(z) = \frac{1-k}{2}$. This implies that $z_{\frac{1-k}{2}} \in (C_A)_k$.

Since $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, so $(xyz)_{\frac{1-k}{2}} \in \vee q_k (C_A)_k$. This implies that

$(xyz)_{\frac{1-k}{2}} \in (C_A)_k$ or $(xyz)_{\frac{1-k}{2}} q_k (C_A)_k$. Thus $(C_A)_k(xyz) \geq \frac{1-k}{2}$ or $(C_A)_k(xyz) + \frac{1-k}{2} + k > 1$. If $(C_A)_k(xyz) + \frac{1-k}{2} + k > 1$, then $(C_A)_k(xyz) > \frac{1-k}{2}$. Thus $(C_A)_k(xyz) \geq \frac{1-k}{2}$. Hence $(C_A)_k(xyz) = \frac{1-k}{2}$. This implies that $xyz \in A$. Hence A is a left ideal of S.

Definition 11. An $(\in, \in \vee q_k)$-fuzzy ideal f of S is called idempotent if $f \circ_k f \circ_k f = f_k$.

Lemma 8. Let Q be a non-empty subset of a ternary semigroup S. Then Q is a quasi-ideal of S if and only if $(C_Q)_k$ is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.

Proof: The proof is similar to the proof of Lemma 7.

Next we show that if f is an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S, then f_k is a fuzzy left (right) ideal of S.

Proposition 1. Let f be an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of a ternary semigroup S. Then f_k is a fuzzy left (right) ideal of S.

Proof: Let f be an $(\in, \in \vee q_k)$-fuzzy left ideal of a ternary semigroup S. Then for all $x, y, z \in S$ we have $f(xyz) \geq f(z) \wedge \frac{1-k}{2}$. This implies that $f(xyz) \wedge \frac{1-k}{2} \geq f(z) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2}$. So $f_k(xyz) \geq f_k(z)$. Hence f_k is a fuzzy left ideal of S.

Similarly, if f is an $(\in, \in \vee q_k)$-fuzzy right ideal of S, then f_k is a fuzzy right ideal of S.

Next we show that every fuzzy left ideal of S is not of the form f_k for some $(\in, \in \vee q_k)$-fuzzy left ideal f of S.

Example 4. Let $S = \{a, b, c, d\}$, and $[xyz] = a$ for all $x, y, z \in S$. Then S is a ternary semigroup. Define a fuzzy subset f of S by: $f(a) = 1$, $f(b), f(c), f(d) > \frac{1-k}{2}$. Then f is a fuzzy left ideal of S for all $k \in [0,1)$, but this is not of the form f_k for some $(\in, \in \vee q_k)$-fuzzy left ideal f of S.

Lemma 9. Let f be an $(\in, \in \vee q_k)$-fuzzy right ideal and g an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Then $f \circ_k \mathbf{S} \circ_k g \leq f \wedge_k g$.

Proof: Let f be an $(\in, \in \vee q_k)$-fuzzy right ideal and g an $(\in, \in \vee q_k)$-fuzzy left ideal of S and $a \in S$. Consider, $(f \circ_k \mathbf{S} \circ_k g)(a) = (f \circ \mathbf{S} \circ g)(a) \wedge \frac{1-k}{2}$
$= \left(\bigvee_{a=lmn} f(l) \wedge \mathbf{S}(m) \wedge g(n) \right) \wedge \frac{1-k}{2} = \left(\bigvee_{a=lmn} f(l) \wedge g(n) \right) \wedge \frac{1-k}{2}$
$\leq \left(\bigvee_{a=lmn} f(lmn) \wedge g(lmn) \right) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).$
This implies that $f \circ_k \mathbf{S} \circ_k g \leq f \wedge_k g$.

Theorem 23. For a ternary semigroup S, the following conditions are equivalent:
(1) S is regular;
(2) $f \wedge_k g \wedge_k h = f \circ_k g \circ_k h$ for every $(\in, \in \vee q_k)$-fuzzy right ideal f, every $(\in, \in \vee q_k)$-fuzzy lateral ideal g and every $(\in, \in \vee q_k)$-fuzzy left ideal h of S.

Proof: $(1) \Rightarrow (2)$: Let f be an $(\in, \in \vee q_k)$-fuzzy right ideal, g an $(\in, \in \vee q_k)$-fuzzy lateral ideal and h an $(\in, \in \vee q_k)$-fuzzy left ideal of S and $a \in S$. Then,

$$
\begin{aligned}
(f \circ_k g \circ_k h)(a) &= (f \circ g \circ h)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a=pqr} f(p) \wedge g(q) \wedge h(r) \right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pqr} \begin{pmatrix} (f(p) \wedge \frac{1-k}{2}) \wedge (g(q) \wedge \frac{1-k}{2}) \\ \wedge (h(r) \wedge \frac{1-k}{2}) \end{pmatrix} \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=pqr} (f(pqr) \wedge g(pqr) \wedge h(pqr)) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} = (f \wedge_k g \wedge_k h)(a).
\end{aligned}
$$

So $f \circ_k g \circ_k h \leq f \wedge_k g \wedge_k h$.

Now, since S is regular, for any $a \in S$ there exists $x \in S$ such that $a = axa = a(xax)a$. Now,

$$(f \circ_k g \circ_k h)(a) = (f \circ g \circ h)(a) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a=pqr} f(p) \wedge g(q) \wedge h(r) \right) \wedge \frac{1-k}{2}$$
$$\geq (f(a) \wedge g(xax) \wedge h(a)) \wedge \frac{1-k}{2}$$
$$= [f(a) \wedge \left(g(xax) \wedge \frac{1-k}{2} \right) \wedge h(a)] \wedge \frac{1-k}{2}$$
$$\geq (f(a) \wedge g(a) \wedge h(a)) \wedge \frac{1-k}{2} = (f \wedge_k g \wedge_k h)(a).$$

This implies that $f \circ_k g \circ_k h \geq f \wedge_k g \wedge_k h$. Hence $f \wedge_k g \wedge_k h = f \circ_k g \circ_k h$.

$(2) \Rightarrow (1)$: Let R, M and L be the right ideal, lateral ideal and left ideal of S, respectively. Then by Lemma 4, C_R, C_M and C_L are $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$-fuzzy lateral ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of S, respectively. Thus by hypothesis, $C_R \wedge_k C_M \wedge_k C_L = (C_R \circ_k C_M \circ_k C_L)$

$$(C_{R \cap M \cap L})_k = (C_{RML})_k.$$

Thus $R \cap M \cap L = RML$. Hence by Theorem 1, S is regular.

Theorem 24. The following conditions are equivalent for a ternary semigroup S:

(1) S is regular;

(2) $f \wedge_k g = f \circ_k \mathbf{S} \circ_k g$ for every $(\in, \in \vee q_k)$-fuzzy right ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g of S.

Proof: The proof follows from Theorem 3 and Theorem 23.

Theorem 25. For a ternary semigroup S, the following conditions are equivalent:

(1) S is regular;

(2) $f_k = f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f of S;

(3) $f_k = f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f of S;

(4) $f_k = f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f$ for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal f of S.

Proof: $(1) \Rightarrow (2)$: Let f be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S and $a \in S$. Since S is regular, so there exists $x \in S$ such that $a = axa = axaxa$. Now,

$$(f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f)(a) = (f \circ \mathbf{S} \circ f \circ \mathbf{S} \circ f)(a) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a=rst} (f \circ \mathbf{S} \circ f)(r) \wedge \mathbf{S}(s) \wedge f(t) \right)$$
$$= \left(\begin{array}{c} \bigvee_{a=rst} \left(\bigvee_{r=lmn} f(l) \wedge \mathbf{S}(m) \wedge f(n) \right) \\ \wedge \mathbf{S}(s) \wedge f(t) \end{array} \right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a=(lmn)st} f(l) \wedge \mathbf{S}(m) \wedge f(n) \wedge \mathbf{S}(s) \wedge f(t) \right) \wedge \frac{1-k}{2}$$
$$\geq (f(a) \wedge \mathbf{S}(x) \wedge f(a) \wedge \mathbf{S}(x) \wedge f(a)) \wedge \frac{1-k}{2}$$
$$= f(a) \wedge \frac{1-k}{2} = f_k(a).$$

This implies that $f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f \geq f_k$. Since f is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S,

$$(f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f)(a) = (f \circ \mathbf{S} \circ f \circ \mathbf{S} \circ f)(a) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a=pvz} (f \circ \mathbf{S} \circ f)(p) \wedge \mathbf{S}(v) \wedge f(z) \right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a=pvz} \left(\bigvee_{p=xuy} f(x) \wedge \mathbf{S}(u) \wedge f(y) \right) \wedge \mathbf{S}(v) \wedge f(z) \right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a=xuyvz} f(x) \wedge \mathbf{S}(u) \wedge f(y) \wedge \mathbf{S}(v) \wedge f(z) \right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a=xuyvz} f(x) \wedge f(y) \wedge f(z) \right) \wedge \frac{1-k}{2}$$
$$\leq \bigvee_{a=xuyvz} f(xuyvz) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} = f_k(a).$$

Thus $f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f \leq f_k$. Hence $f_k = f \circ_k \mathbf{S} \circ_k f \circ_k \mathbf{S} \circ_k f$.

$(2) \Rightarrow (3) \Rightarrow (4)$ Straightforward, because every $(\in, \in \vee q_k)$-fuzzy quasi-ideal is an $(\in, \in \vee q_k)$-fuzzy bi-ideal and every $(\in, \in \vee q_k)$-fuzzy bi-ideal is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

$(4) \Rightarrow (1)$: Let Q be any quasi-ideal of S. Then by Lemma 3, C_Q is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Thus by hypothesis

$$(C_Q)_k = (C_Q \circ_k C_S \circ_k C_Q \circ_k C_S \circ_k C_Q).$$

This implies that $(C_Q)_k = (C_{QSQSQ})_k$. Thus $Q = QSQSQ$. Hence by Theorem 2, S is regular.

Theorem 26. For a ternary semigroup S, the following conditions are equivalent:
(1) S is regular;
(2) $f \wedge_k g \leq f \circ_k \mathbf{S} \circ_k g$ for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g of S;
(3) $f \wedge_k g \leq f \circ_k \mathbf{S} \circ_k g$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g of S;
(4) $f \wedge_k g \leq f \circ_k \mathbf{S} \circ_k g$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g of S.

Proof: $(1) \Rightarrow (4)$: Let f be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and g an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Since S is regular, so for any $a \in S$ there exists $x \in S$ such that $a = axa$. Now,

$$\begin{aligned}(f \circ_k \mathbf{S} \circ_k g)(a) &= (f \circ \mathbf{S} \circ g)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a=pqr} f(p) \wedge \mathbf{S}(q) \wedge g(r)\right) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge \mathbf{S}(x) \wedge g(a) \wedge \frac{1-k}{2} \\ &= (f \wedge g)(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

This implies that $f \circ_k \mathbf{S} \circ_k g \geq f \wedge_k g$.
$(4) \Rightarrow (3) \Rightarrow (2)$ Straightforward.
$(2) \Rightarrow (1)$ Let f be an $(\in, \in \vee q_k)$-fuzzy right ideal and g an $(\in, \in \vee q_k)$-fuzzy left ideal of S, since every $(\in, \in \vee q_k)$-fuzzy right ideal is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Therefore, $f \wedge_k g \leq f \circ_k \mathbf{S} \circ_k g$. Also by Lemma 9, $f \wedge_k g \geq f \circ_k \mathbf{S} \circ_k g$. Thus
$f \wedge_k g = f \circ_k \mathbf{S} \circ_k g$.
Hence by Theorem 24 S is regular.

Theorem 27. The following conditions are equivalent for a ternary semigroup S:
(1) S is regular;
(2) $f \wedge_k g \leq (f \circ_k \mathbf{S} \circ_k g) \wedge (g \circ_k \mathbf{S} \circ_k f)$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals f and g of S;
(3) $f \wedge_k g \leq (f \circ_k \mathbf{S} \circ_k g) \wedge (g \circ_k \mathbf{S} \circ_k f)$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideals f and g of S;
(4) $f \wedge_k g \leq (f \circ_k \mathbf{S} \circ_k g) \wedge (g \circ_k \mathbf{S} \circ_k f)$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f and every $(\in, \in \vee q_k)$-fuzzy quasi-ideal g of S;
(5) $f \wedge_k g \leq (f \circ_k \mathbf{S} \circ_k g) \wedge (g \circ_k \mathbf{S} \circ_k f)$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g of S;
(6) $f \wedge_k g \leq (f \circ_k \mathbf{S} \circ_k g) \wedge (g \circ_k \mathbf{S} \circ_k f)$ for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g of S;
(7) $f \wedge_k g \leq (f \circ_k \mathbf{S} \circ_k g) \wedge (g \circ_k \mathbf{S} \circ_k f)$ for every $(\in, \in \vee q_k)$-fuzzy right ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g of S.

Proof: $(1) \Rightarrow (2)$: Let f and g are $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals of S and $a \in S$. Since S is regular, so there exists $x \in S$ such that $a = axa$. Now,

$$\begin{aligned}(f \circ_k \mathbf{S} \circ_k g)(a) &= (f \circ \mathbf{S} \circ g)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a=rst} f(r) \wedge \mathbf{S}(s) \wedge g(t)\right) \wedge \frac{1-k}{2} \\ &\geq (f(a) \wedge \mathbf{S}(x) \wedge g(a)) \wedge \frac{1-k}{2} \\ &= (f \wedge g)(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a)\end{aligned}$$

Thus $f \wedge_k g \leq f \circ_k \mathbf{S} \circ_k g$. Also,

$$\begin{aligned}(g \circ_k \mathbf{S} \circ_k f)(a) &= (g \circ \mathbf{S} \circ f)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a=lmn} (g(l) \wedge \mathbf{S}(m) \wedge f(n)\right) \wedge \frac{1-k}{2} \\ &\geq (g(a) \wedge \mathbf{S}(x) \wedge f(a)) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

Thus, $f \wedge_k g \leq g \circ_k \mathbf{S} \circ_k f$. Therefore, $f \wedge_k g \leq (f \circ_k \mathbf{S} \circ_k g) \wedge (g \circ_k \mathbf{S} \circ_k f)$.
It is clear that $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$.

$(7)\Rightarrow(1)$: Let $f\wedge_k g\le(f\circ_k \mathbf{S}\circ_k g)\wedge(g\circ_k \mathbf{S}\circ_k f)$ for every $(\in,\in\vee q_k)$-fuzzy right ideal f and every $(\in,\in\vee q_k)$-fuzzy left ideal g of S. Now, (7) implies $f\wedge_k g\le(f\circ_k \mathbf{S}\circ_k g)\wedge(g\circ_k \mathbf{S}\circ_k f)\le(f\circ_k \mathbf{S}\circ_k g)$ and by Lemma 9, $(f\circ_k \mathbf{S}\circ_k g)\le f\wedge_k g$. Consequently, $f\wedge_k g=(f\circ_k \mathbf{S}\circ_k g)$. Thus by Theorem 24, S is regular.

Theorem 28. The following assertions are equivalent for a ternary semigroup S:
(1) S is regular;
(2) $f\wedge_k g\le(f\circ_k \mathbf{S}\circ_k g)$ for every $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal f and every $(\in,\in\vee q_k)$-fuzzy left ideal g of S;
(3) $f\wedge_k g\le(f\circ_k \mathbf{S}\circ_k g)$ for every $(\in,\in\vee q_k)$-fuzzy bi-ideal f and every $(\in,\in\vee q_k)$-fuzzy left ideal g of S;
(4) $f\wedge_k g\le(f\circ_k \mathbf{S}\circ_k g)$ for every $(\in,\in\vee q_k)$-fuzzy quasi-ideal f and every $(\in,\in\vee q_k)$-fuzzy left ideal g of S;
(5) $f\wedge_k g\le(f\circ_k \mathbf{S}\circ_k g)$ for every $(\in,\in\vee q_k)$-fuzzy right ideal f and every $(\in,\in\vee q_k)$-fuzzy bi-ideal g of S;
(6) $f\wedge_k g\le(f\circ_k \mathbf{S}\circ_k g)$ for every $(\in,\in\vee q_k)$-fuzzy right ideal f and every $(\in,\in\vee q_k)$-fuzzy quasi-ideal g of S.

Proof: $(1)\Rightarrow(2)$: Assume that f is an $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal and g an $(\in,\in\vee q_k)$-fuzzy left ideal of S. Since S is regular, so for any $a\in S$ there exists $x\in S$ such that $a=axa$, and

$$\begin{aligned}(f\circ_k \mathbf{S}\circ_k g)(a)&=(f\circ\mathbf{S}\circ g)(a)\wedge\frac{1-k}{2}\\&=\left(\bigvee_{a=pqr} f(p)\wedge\mathbf{S}(q)\wedge g(r)\right)\wedge\frac{1-k}{2}\\&\ge(f(a)\wedge\mathbf{S}(x)\wedge g(a))\wedge\frac{1-k}{2}\\&=(f\wedge g)(a)\wedge\frac{1-k}{2}=(f\wedge_k g)(a)\end{aligned}$$

This implies that $f\wedge_k g\le f\circ_k \mathbf{S}\circ_k g$.
It is clear that $(2)\Rightarrow(3)\Rightarrow(4)$.
$(4)\Rightarrow(1)$: Suppose that $f\wedge_k g\le f\circ_k \mathbf{S}\circ_k g$ for every $(\in,\in\vee q_k)$-fuzzy quasi-ideal f and every $(\in,\in\vee q_k)$-fuzzy left ideal g of S. Let h be any $(\in,\in\vee q_k)$-fuzzy right ideal of S. Take $f=h$, then by (4), $h\wedge_k g\le h\circ_k \mathbf{S}\circ_k g$ and by Lemma 9, $h\circ_k \mathbf{S}\circ_k g\le h\wedge_k g$. Thus $h\wedge_k g=h\circ_k \mathbf{S}\circ_k g$. Therefore, by Theorem 24, S is regular.
$(1)\Rightarrow(5)$: Let f be an $(\in,\in\vee q_k)$-fuzzy right ideal and g an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S. Since S is regular, so for any $a\in S$ there exists $x\in S$ such that $a=axa$, and

$$\begin{aligned}(f\circ_k \mathbf{S}\circ_k g)(a)&=(f\circ\mathbf{S}\circ g)(a)\wedge\frac{1-k}{2}=\left(\bigvee_{a=pqr} f(p)\wedge\mathbf{S}(q)\wedge g(r)\right)\wedge\frac{1-k}{2}\\&\ge f(a)\wedge\mathbf{S}(x)\wedge g(a)\wedge\frac{1-k}{2}=(f(a)\wedge g(a))\wedge\frac{1-k}{2}\\&=(f\wedge_k g)(a).\end{aligned}$$

Thus $f\wedge_k g\le f\circ_k \mathbf{S}\circ_k g$.
It is clear that $(5)\Rightarrow(6)$.
$(6)\Rightarrow(1)$: Suppose that $f\wedge_k g\le f\circ_k \mathbf{S}\circ_k g$ for every $(\in,\in\vee q_k)$-fuzzy right ideal f and every $(\in,\in\vee q_k)$-fuzzy quasi-ideal g of S. Let h be any $(\in,\in\vee q_k)$-fuzzy left ideal of S. Take $h=g$, then by (6), $f\wedge_k h\le f\circ_k \mathbf{S}\circ_k h$ and by Lemma 9, $f\circ_k \mathbf{S}\circ_k h\le f\wedge_k h$. Thus $f\wedge_k h=f\circ_k \mathbf{S}\circ_k h$. Therefore, by Theorem 24, S is regular.

5. Weakly regular Ternary Semigroups

In this section we characterize right weakly regular ternary semigroups by the properties of their $(\in,\in\vee q_k)$-fuzzy right ideals, $(\in,\in\vee q_k)$-fuzzy two sided ideals and $(\in,\in\vee q_k)$-fuzzy generalized bi-ideals.

Definition 12. [28] A ternary semigroup S is said to be right (resp. left) weakly regular, if $x\in(xSS)^3$ $\left(\text{resp.}\ x\in(SSx)^3\right)$ for all $x\in S$.

Example 5. Let X be a countably infinite set and let S be the set of one-one maps $\alpha : X\to X$

with the property that $X-\alpha(X)$ is infinite. Then S is a ternary semigroup with respect to the composition of functions, that is, for $\alpha,\beta,\gamma\in S$, $[\alpha\beta\gamma]=\alpha\circ\beta\circ\gamma.$
This ternary semigroup is not regular but right weakly regular.

Lemma 10. [28] A ternary semigroup S is right weakly regular if and only if $R\cap I=RII$, for every right ideal R and every two sided ideal I of S.

Theorem 29. For a ternary semigroup S, the following assertions are equivalent:
(1) S is right weakly regular;
(2) $f\wedge_k g=f\circ_k g\circ_k g$ for every $(\in,\in\vee q_k)$-fuzzy right ideal f and every $(\in,\in\vee q_k)$-fuzzy two sided ideal g of S.

Proof: $(1)\Rightarrow(2)$: Let f be an $(\in,\in\vee q_k)$-fuzzy right ideal and g an $(\in,\in\vee q_k)$-fuzzy two sided ideal of S and $a\in S$. Then

$$(f\circ_k g\circ_k g)(a)=(f\circ g\circ g)(a)\wedge\tfrac{1-k}{2}$$
$$=\left(\bigvee_{a=pqr} f(p)\wedge g(q)\wedge g(r)\right)\wedge\tfrac{1-k}{2}$$
$$\leq\bigvee_{a=pqr}\left(f(pqr)\wedge g(q)\wedge g(pqr)\right)\wedge\tfrac{1-k}{2}\leq\bigvee_{a=pqr}\left(f(pqr)\wedge g(pqr)\right)\wedge\tfrac{1-k}{2}$$
$$\leq\left(f(a)\wedge g(a)\right)\wedge\tfrac{1-k}{2}=(f\wedge_k g)(a).$$

This implies that $f\circ_k g\circ_k g\leq f\wedge_k g$.
Now, to show $f\wedge_k g\leq f\circ_k g\circ_k g$. Let $a\in S$. Since S is right weakly regular so there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)$. Thus

$$(f\wedge_k g)(a)=(f\wedge g)(a)\wedge\frac{1-k}{2}$$
$$=(f(a)\wedge g(a)\wedge g(a))\wedge\frac{1-k}{2}$$
$$\leq\left(f(as_1t_1)\wedge g(as_2t_2)\wedge g(as_3t_3)\right)\wedge\frac{1-k}{2}$$
$$\leq\left(\bigvee_{a=xyz}\left(f(x)\wedge g(y)\wedge g(z)\right)\right)\wedge\frac{1-k}{2}$$
$$=(f\circ g\circ g)(a)\wedge\frac{1-k}{2}=(f\circ_k g\circ_k g)(a).$$

This implies that $f\wedge_k g\leq f\circ_k g\circ_k g$. Thus $f\wedge_k g=f\circ_k g\circ_k g$.

$(2)\Rightarrow(1)$ Let R be the right ideal and I two sided ideal of S. Then by Lemma 4, C_R and C_I are $(\in,\in\vee q_k)$-fuzzy right ideal and $(\in,\in\vee q_k)$-fuzzy two sided ideal of S, respectively. Thus by hypothesis

$$C_R\wedge_k C_I=C_R\circ_k C_I\circ_k C_I$$
$$(C_{R\cap I})_k=(C_{RII})_k.$$

This implies that $R\cap I=RII$. Thus by Lemma 10, S is right weakly regular.
The proof of the following theorem is straightforward and hence omitted.

Theorem 30. For a ternary semigroup S, the following assertions are equivalent:
(1) S is right weakly regular;
(2) Each $(\in,\in\vee q_k)$-fuzzy right ideal f of S is idempotent.

Theorem 31. For a ternary semigroup S, the following assertions are equivalent:
(1) S is right weakly regular;
(2) $f\wedge_k g\wedge_k h=f\circ_k g\circ_k h$ for every $(\in,\in\vee q_k)$-fuzzy bi-ideal f, every $(\in,\in\vee q_k)$-fuzzy two sided ideal g and every $(\in,\in\vee q_k)$-fuzzy right ideal h of S;
(3) $f\wedge_k g\wedge_k h=f\circ_k g\circ_k h$ for every $(\in,\in\vee q_k)$-fuzzy quasi-ideal f, every $(\in,\in\vee q_k)$-fuzzy two sided ideal g and every $(\in,\in\vee q_k)$-fuzzy right ideal h of S.

Proof: $(1)\Rightarrow(2)$: Let f be an $(\in,\in\vee q_k)$-fuzzy bi-ideal, g an $(\in,\in\vee q_k)$-fuzzy two sided ideal and h an $(\in,\in\vee q_k)$-fuzzy right ideal of S. Let $a\in S$. Since S is right weakly regular, for each $a\in S$ there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)=a(s_1t_1as_2t_2)(as_3t_3)$.
Consider

$$\begin{aligned}(f \wedge_k g \wedge_k h)(a) &= (f \wedge g \wedge h)(a) \wedge \frac{1-k}{2}\\ &= (f(a) \wedge g(a) \wedge h(a)) \wedge \frac{1-k}{2}\\ &\le (f(a) \wedge g(s_1t_1as_2t_2) \wedge h(as_3t_3)) \wedge \frac{1-k}{2}\\ &= \left(\bigvee_{a=pqr} (f(p) \wedge g(q) \wedge h(r))\right) \wedge \frac{1-k}{2}\\ &= (f \circ g \circ h)(a) \wedge \frac{1-k}{2} = (f \circ_k g \circ_k h)(a)\end{aligned}$$

This implies that $f \wedge_k g \wedge_k h \le f \circ_k g \circ_k h$.

$(2) \Rightarrow (3)$: Straightforward, because every $(\in, \in \vee q_k)$-fuzzy quasi-ideal is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

$(3) \Rightarrow (1)$: Let f be an $(\in, \in \vee q_k)$-fuzzy right ideal and g an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S. Take $h = g$. Since every $(\in, \in \vee q_k)$-fuzzy right ideal is also $(\in, \in \vee q_k)$-fuzzy quasi-ideal. Thus by hypothesis $f \wedge_k g \wedge_k g \le f \circ_k g \circ_k g$. This implies that $f \wedge_k g \le f \circ_k g \circ_k g$ and $f \circ_k g \circ_k g \le f \wedge_k g$ is straightforward. Thus $f \wedge_k g = f \circ_k g \circ_k g$. Therefore, by Theorem 29, S is right weakly regular.

Theorem 32. For a ternary semigroup S, the following assertions are equivalent:

(1) S is right weakly regular;

(2) $f \wedge_k g \le f \circ_k g \circ_k g$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q_k)$-fuzzy two sided ideal g of S;

(3) $f \wedge_k g \le f \circ_k g \circ_k g$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f and every $(\in, \in \vee q_k)$-fuzzy two sided ideal g of S;

(4) $f \wedge_k g \le f \circ_k g \circ_k g$ for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal f and every $(\in, \in \vee q_k)$-fuzzy two sided ideal g of S.

Proof: Let f be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and g an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S. Let $a \in S$. Since S is right weakly regular,

$a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$

for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Consider,

$$\begin{aligned}(f \wedge_k g)(a) &= (f \wedge g)(a) \wedge \tfrac{1-k}{2} = (f(a) \wedge g(a)) \wedge \tfrac{1-k}{2}\\ &\le (f(a) \wedge g(s_1t_1as_2t_2) \wedge g(as_3t_3)) \wedge \tfrac{1-k}{2}\\ &= \left(\bigvee_{a=lmn} (f(l) \wedge g(m) \wedge g(n))\right) \wedge \tfrac{1-k}{2}\\ &= (f \circ g \circ g)(a) \wedge \tfrac{1-k}{2} = (f \circ_k g \circ_k g)(a).\end{aligned}$$

This implies that $f \wedge_k g \le f \circ_k g \circ_k g$.

$(2) \Rightarrow (3) \Rightarrow (4)$: Straightforward, because every $(\in, \in \vee q_k)$-fuzzy quasi-ideal is an $(\in, \in \vee q_k)$-fuzzy bi-ideal and every $(\in, \in \vee q_k)$-fuzzy bi-ideal is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

$(4) \Rightarrow (1)$: Let f be an $(\in, \in \vee q_k)$-fuzzy right ideal and g an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S. Since every $(\in, \in \vee q_k)$-fuzzy right ideal is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S, thus by hypothesis

$f \wedge_k g \le f \circ_k g \circ_k g$ and $f \circ_k g \circ_k g \le f \wedge_k g$ is straightforward. Hence $f \wedge_k g = f \circ_k g \circ_k g$. Thus by Theorem 29, S is right weakly regular.

6. Conclusion

In the study of fuzzy algebraic system, we notice that the (fuzzy) ideals with special properties always play a prominent role.

In this paper we study $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideals, $(\in, \in \vee q_k)$-fuzzy quasi (bi-, generalized bi-) ideals of ternary semigroups and give several characterizations of regular and right weakly regular ternary semigroups in terms of these notions.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy ternary semigroups and their applications in other branches of algebra. In the future study of fuzzy ternary semigroups, perhaps the following topics are worth to be considered:

(1) To characterize other classes of ternary semigroups by using this notion;

(2) To apply this notion to some other algebraic structures;

(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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