*IJST (2012) A3 (Special issue-Mathematics): 377-382*
***Iranian Journal of Science & Technology***

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# On a class of locally dually flat Finsler metrics with isotropic S-curvature

A. Tayebi[1], E. Peyghan[2]* and H. Sadeghi[1]

[1]*Department of Mathematics and Computer Science, University of Qom, Qom, Iran*
[2]*Department of Mathematics and Computer Science, Arak University, Arak 38156-8-8349, Iran*
*E-mails: akbar.tayebi@gmail.com, sadeghihassan64@gmail.com & epeyghan@gmail.com*

## Abstract

Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structure. In this paper, we prove that every locally dually flat generalized Randers metric with isotropic S-curvature is locally Minkowskian.

**Keywords**: Locally dually flat metric; S-curvature

## 1. Introduction

In [1], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they studied the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory. In Finsler geometry, Shen extends the notion of locally du ally flatness for Finsler metrics [2]. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structure [3-6].

A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat if at any point there is a standard coordinate system $(x^i, y^i)$ in $TM$ which satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}.$$

In this case, the coordinate $(x^i)$ is called an adapted local coordinate system. It is easy to see that every locally Minkowskian metric is locally dually flat. But the converse is not true, generally [3].

The S-curvature is constructed by Shen for given comparison theorems on Finsler manifolds [7]. A

A Finsler metric $F$ on an n-dimensional manifold $M$ is said to have isotropic S-curvature if isotopic $S = (n+1)c(x)F$, for some scalar function $c$ on $M$. It is known that some of Randers metrics are of S-curvature [8, 9]. This is one of our motivations for considering Finsler metrics of isotropic S-curvature.

In this paper, we show that a locally dually flat generalized Randers metric with isotropic S-curvature reduces to a locally Minkowskian metric. More precisely, we prove the following.

**Theorem 1.1.** Let $F^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_1\beta^2$ $(c_1 \neq 0,\ c_2 \neq 0)$ be a non- Randers type and non-Riemannian generalized Randers metric on a manifold M of dimension $n \geq 3$. Then $F$ is locally dually flat with isotropic S-curvature, $S = (n+1)c(x)F$, if and only if it is locally Minlowskian.

## 2. Preliminaries

A Finsler metric on an n-dimensional manifold $M$ is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is $C^\infty$ on $TM_0 := TM \setminus \{0\}$; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM; (iii) for each $y \in T_xM$ the following quadratic form $g_y$ on $T_xM$ is positive definite,

*Corresponding author
Received: 30 October 2011 / Accepted: 17 January 2012

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y+su+tv)]|_{s,t=0}, \quad u,v \in T_xM$$

Given a Finsler manifold $(M,F)$, a global vector field $G$ is induced by $F$ on $TM_0$ which in a standard coordinate $(x_i, y_i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i$ are the coefficients of the spray associated with $F$ and given by the following

$$G^i = \frac{g^{il}}{4}\left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}$$

Indeed, $G$ is called the associated spray to $(M,F)$ [10].

A Finsler metric $F(x,y)$ on an open domain $U \subset R^n$ is said to be locally projectively flat if its geodesic coefficients $G^i$ are in the form $G^i(x,y) = P(x,y)y^i$, where $P: TU = U \times R^n \to R$ is positively homogeneous with degree one, $P(x,\lambda y) = \lambda P(x,y)$, $\lambda > 0$. We call $P(x,y)$ the projective factor of F.

A Finsler metric $F = F(x,y)$ on a manifold $M$ is said to be locally dually flat if at any point there is a standard coordinate system $(x^i, y^i)$ in $TM$ such that $L = F^2(x,y)$ satisfies

$$L_{x^k y^l} y^k = 2L_{x^l}. \tag{1}$$

In this case, the coordinate $(x^i)$ is called an adapted local coordinate system. It is easy to see that every locally Minkowskian metric is satisfied in the above equation, hence it is locally dually flat. In [3], the following is proved.

**Lemma 2.1.** ([3]) Let $F = F(x,y)$ be a Finsler metric on an open subset $U \subset R^n$. Then F is locally flat and projectively flat on $U$ if and only if $F_{x^k} = CFF_{y^k}$, where $C$ is a constant.

For a Finsler metric $F$ on an n-dimensional manifold $M$, the Busemann- Hausdorff volume form $dV_F = \sigma_F(x)dx^1 ... dx^n$ is defined by

$$\sigma(F) := \frac{Vol(B^n(1))}{Vol\left\{ (y^i) \in R^n \mid F(y^i \frac{\partial}{\partial x^i}|_x) \right\}}.$$

Here Vol denotes the Euclidean volume and $B^n(1)$ denotes the unit ball in $R^n$.

Then the S-curvature is defined by

$$S(y) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}[\ln \sigma_F(x)]$$

where $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ [7]. **S** said to be isotropic if there is a scalar function $c(x)$ on M such that

$$S(x,y) = (n+1)c(x)F(x,y).$$

## 3. Proof of Theorem 1.1.

In this section, we are going to prove the Theorem 1.1. First let us introduce our notations. Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j \theta_i^j$$

where $\theta^i := dx^i$ and $\theta_i^j := \Gamma_{ik}^j dx^k$ denote the Levi-Civita connection form of $\alpha$. Put

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \ s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$$

Clearly, $\beta$ is closed if and only if $s_{ij} = 0$ [11]. Put

$$r_{00} := r_{ij}y^i y^j, \ s_{k0} := s_{km}y^m, \ r_j := b^i r_{ij}, \ s_j := b^i s_{ij}$$

Let $r_{i0} := r_{ij}y^j$, $s_{i0} := s_{ij}y^j$ and $s_0 := s_j y^j$. We have the following identities

$$\alpha_{x^k} = \frac{y_m}{\alpha}\frac{\partial G_\alpha^m}{\partial y^k}, \ \beta_{x^k} = b_{m|k}y^m + b_m \frac{\partial G_\alpha^m}{\partial y^k},$$
$$s_{y^k} = \frac{\alpha b_k - s y_k}{\alpha^2} \tag{2}$$

where $s := \frac{\beta}{\alpha}$ and $y_k := a_{jk}y^j$.

Let $F^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$ be a generalized Randers metric on an open subset $U \subset R^n$, where

$c_i$'s $(i=1,2)$ are non-zero constants [12-14]. To prove the Theorem 1.1, we need the following.

**Theorem 3.1.** ([5]) Let $F=\alpha\phi(s)$, $s=\dfrac{\beta}{\alpha}$ be an $(\alpha,\beta)-$metric on an n- dimensional manifold $M^n$ $(n\geq 3)$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i\neq 0$ is a 1-form on $M$. Suppose that $F$ is not Riemannian and $\phi'(s)\neq 0$. Then F is locally dually flat on $M$ if and only if $\alpha,\beta$ and $\phi=\phi(s)$ satisfy

$$s_{l0}=\frac{1}{3}(\beta\theta_l-\theta b_l),$$
$$r_{00}=\frac{2}{3}\theta\beta+\left[\tau+\frac{2}{3}\left(b^2\tau-\theta_lb^l\right)\right]\alpha^2+\frac{1}{3}\left(3k_2-2-3k_3b^2\right)\tau\beta^2,$$
$$G^l_\alpha=\frac{1}{3}\left[2\theta+(3k_1-2)\tau\beta\right]y^l+\frac{1}{3}\left(\theta^l-\tau b^l\right)\alpha^2+\frac{1}{2}k_3\tau\beta^2b^l,$$
$$\tau\left[s\left(k_2-k_3s^2\right)\left(\phi\phi'-s\phi'^2-s\phi\phi''\right)-\left(\phi'^2+\phi\phi''\right)+k_1\phi\left(\phi-s\phi'\right)\right]=0,$$

where $\tau=\tau(x)$ is a scalar function, $\theta:=\theta_i(x)y^i$ is a 1-form on $M$ and $\theta^l:=a^{lm}\theta_m$ and $k_1:=\Pi(0)$, $k_2:=\dfrac{\Pi'(0)}{Q(0)}$,

$$k_3:=\frac{1}{6Q(0)^2}\left[3Q''(0)\Pi'(0)-6\Pi'(0)^2-Q(0)\Pi'''(0)\right],$$
$$Q:=\frac{\phi'}{\phi-s\phi'},\quad \Pi:=\frac{\phi'^2+\phi\phi''}{\phi\left(\phi-s\phi'\right)}.$$

**Lemma 3.2.** Let $F^2=c_1\alpha^2+2c_2\alpha\beta+c_1\beta^2$, $\left(c_1\neq 0,\ c_2\neq 0\right)$ be a non- Riemannian generalized Randers metric on a manifold $M$ of dimension $n\geq 3$. Then $F$ is locally dually flat on $M$ if and only if $\alpha,\beta$ and $\phi=\phi(s)$ satisfy

$$s_{l0}=\frac{1}{3}(\beta\theta_l-\theta b_l), \tag{3}$$

$$G^l_\alpha=\frac{1}{3}\left[2\theta+\tau\beta\right]y^l+\frac{1}{3}\left(\theta^l-\tau b^l\right)\alpha^2, \tag{4}$$

$$r_{00}=\frac{2}{3}\theta\beta+\left[\tau+\frac{2}{3}\left(b^2\tau-\theta_lb^l\right)\right]\alpha^2+\frac{5}{3}\tau\beta^2, \tag{5}$$

where $\tau=\tau(x)$ is a scalar function and $\theta=\theta_ky^k$ is a 1-form on $M$.

**Proof:** For a generalized Randers metric $F^2=c_1\alpha^2+2c_2\alpha\beta+c_1\beta^2$, we have the following

$$\phi=\sqrt{c_1+2c_2s+c_1s^2}, \tag{6}$$

$$Q=\frac{c_2+c_1s}{c_1+c_2s}, \tag{7}$$

$$Q'=\frac{c_1^2-c_2^2}{\left(c_1+c_2s\right)^2},$$

$$Q''=\frac{-2c_2\left(c_1^2-c_2^2\right)}{\left(c_1+c_2s\right)^3}, \tag{8}$$

$$\Pi=\frac{c_1}{c_1+c_2s},$$

$$\Pi'=-\frac{c_1c_2}{\left(c_1+c_2s\right)^2}, \tag{9}$$

$$\Pi''=\frac{2c_1c_2^2}{\left(c_1+c_2s\right)^3},$$

$$\Pi'''=-\frac{6c_1c_2^3}{\left(c_1+c_2s\right)^4}, \tag{10}$$

$$k_1=1,\ k_2=-1,\ k_3=0. \tag{11}$$

Using (6)-(11), we get:

$$s\left(k_2-k_3s^2\right)\left(\phi\phi'-s\phi'^2-s\phi\phi''\right)-\left(\phi'^2+\phi\phi''\right)+k_1\phi\left(\phi-s\phi'\right)=0$$

Then by Theorem 3.1, we get the proof.

Now, let $\phi=\phi(s)$ be a positive $C^\infty$ function on $(-b_0,b_0)$. For a number $b\in[0,b_0)$, let

$$\Phi:=-\left(Q-sQ'\right)\left\{n\Delta+1+sQ\right\}-\left(b^2-s^2\right)\left(1+sQ\right)Q'' \tag{12}$$

where

$$\Delta := 1 + sQ + \left(b^2 - s^2\right)Q' \quad (13)$$

By considering (7), the relation (12) can be written as follows:

$$\Phi = -\left(Q - sQ'\right)\left(n+1\right)\Delta + \left(b^2 - s^2\right)\left\{\left(Q - sQ'\right)Q' - \left(1 + sQ\right)Q''\right\}. \quad (14)$$

**Remark 3.1.** By a direct computation, we can obtain a formula for mean Cartan torsion of an $(\alpha, \beta)-$metric as follows

$$I_i = -\frac{\Phi\left(\phi - s\phi'\right)}{2\Delta\phi\alpha^2}\left(\alpha b_i - sy_i\right).$$

Clearly, an $(\alpha, \beta)-$metric $F = \alpha\phi(s),\ s = \dfrac{\beta}{\alpha}$ is Riemannian if and only if $\Phi = 0.$

In [8], Cheng-Shen study the class of $(\alpha, \beta)-$metrics of non-Randers type $\phi \neq \sqrt{1 + t_2 s^2} + t_3 s$ with isotropic S-curvature and obtain the following.

**Theorem 3.3.** ([8]) Let $F = \alpha\phi(s),\ s = \dfrac{\beta}{\alpha}$ be an non-Riemannian $(\alpha, \beta)-$ metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that $\phi \neq t_1\sqrt{1 + t_2 s^2} + t_3 s$ for any constant $t_1 > 0,\ t_2$ and $t_3$. Then $F$ is of isotropic S-curvature $S = (n+1)cF$, if and only if one of the following holds

$(i)$ $\beta$ satisfies

$$r_{ij} = c\left\{b^2 a_{ij} - b_i b_j\right\},\quad s_j = 0, \quad (15)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $c = c(x)$ satisfies

$$\Phi = -2\left(n+1\right)k\,\frac{\phi\Delta^2}{b^2 - s^2} \quad (16)$$

where $k$ is a constant. In this case, $S = (n+1)cF$ with $c = k\,\varepsilon.$

(ii) $\beta$ satisfies

$$r_{ij} = 0,\quad s_j = 0 \quad (17)$$

In this case $S = 0,$ regardless of choices of a particular $\phi.$

Using the Theorem 3.3, we are going to consider locally dually flat generalized Randers metrics with isotropic S-curvature.

**Proposition 3.1.** Let $F^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_1\beta^2,\ (c_1 \neq 0,\ c_2 \neq 0)$ be a locally dually flat non-Randers on a manifold $M$ of dimension $n \geq 3.$ Suppose that $F$ is of isotropic S-curvature, type and non-Riemannian $S = (n+1)cF$, where $c = c(x)$ is a scalar function on $M$. Then $F$ is a locally generalized Randers metric projectively flat in adapted coordinate systems with $G^i = 0.$

**Proof:** Let $G^i = G^i(x, y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x, y)$ denote the coefficients of F and $\alpha$ respectively in the same coordinate system. By definition, we have

$$G^i = \bar{G}^i_\alpha + Py^i + Q^i \quad (18)$$

where

$$P = \alpha^{-1}\Theta\left\{-2Q\alpha s_0 + r_{00}\right\}, \quad (19)$$

$$Q^i = \alpha Q s^i_0 + \Psi\left\{-2Q\alpha s_0 + r_{00}\right\}b^i, \quad (20)$$

$$\Theta = \frac{\phi\phi' - s\left(\phi\phi'' + \phi'\phi'\right)}{2\phi\left(\left(\phi - s\phi'\right) + \left(b^2 - s^2\right)\phi''\right)},$$

$$\Psi = \frac{1}{2}\frac{\phi''}{\left(\phi - s\phi'\right) + \left(b^2 - s^2\right)\phi''}.$$

First, we suppose that the case (i) of the Theorem 3.3 holds. It is remarkable that, for a generalized Randers metric $F^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_1\beta^2$, the following relations hold

$$\phi = \sqrt{c_1 + 2c_2 s + c_1 s^2},$$

$$Q = \frac{c_2 + c_1 s}{c_1 + c_2 s},$$

$$Q' = \frac{d}{\left(c_1 + c_2 s\right)^2},$$

$$Q''=\frac{-2c_2 d}{\left(c_1+c_2 s\right)^3}.$$

Where $d:=c_1^2-c_2^2$. Thus we have

$$1+sQ=\frac{\phi^2}{c_1+c_2 s} \tag{21}$$

and then

$$\Delta:=1+sQ+\left(b^2-s^2\right)Q'$$

$$=\frac{\left(c_1+c_2 s\right)\phi^2+d\left(b^2-s^2\right)}{\left(c_1+c_2 s\right)^2} \tag{22}$$

By (22), it follows that $\left(c_1+c_2 s\right)^2\Delta$ is a polynomial in s of degree 3. On the other hand, we have

$$\phi\,\Delta^2=\frac{\phi\left[\left(c_1+c_2 s\right)\phi^2+d\left(b^2-s^2\right)\right]^2}{\left(c_1+c_2 s\right)^4} \tag{23}$$

Thus if $\Phi=-2\left(n+1\right)k\,\frac{\phi\Delta^2}{b^2-s^2}$ holds, then by (23) it results that

$$\left(b^2-s^2\right)\left(c_1+c_2 s\right)^4\Phi$$
$$=-2\left(n+1\right)k\,\phi\left[\left(c_1+c_2 s\right)\phi^2+d\left(b^2-s^2\right)\right]^2. \tag{24}$$

By (24), it follows that $\left(b^2-s^2\right)\left(c_1+c_2 s\right)^4\Phi$ is not a polynomial in s (if $k=0,$ then by considering the Remark 3.1, we get a contradiction). Indeed, if we put

$$\phi\Delta^2=\frac{\bar{\Delta}}{\left(c_1+c_2 s\right)^4},$$

where

$$\bar{\Delta}=\sqrt{c_1 s^2+2c_2 s+c_1}\left\{\left(c_1+c_2 s\right)\phi^2+d\left(b^2-s^2\right)\right\}^2,$$

then $\bar{\Delta}$ is a polynomial in s and b if and only if $\phi=c_1\left(\alpha+\beta\right),\ \left(c_1\geq 0\right).$ But by assumption $F$ is not a Randers-type metric. So $\bar{\Delta}$ is not a polynomial in s, and then $\left(b^2-s^2\right)\left(c_1+c_2 s\right)^4\Phi$ is not a polynomial in s.

Now, we consider another formula for $\Phi$:

$$\Phi=-\left(Q-sQ'\right)\left(n+1\right)\Delta$$
$$+\left(b^2-s^2\right)\left\{\left(Q-sQ'\right)Q'-\left(1+sQ\right)Q''\right\}. \tag{25}$$

We have

$$Q-sQ'=\frac{c_2\phi^2}{\left(c_1+c_2 s\right)^2}. \tag{26}$$

by (8), (21), (25) and (26), it follows that

$$\Phi=-\frac{\left(n+1\right)c_2\phi^2}{\left(c_1+c_2 s\right)^2}\Delta+\left(b^2-s^2\right)\left[\frac{c_2 d\,\phi^2}{\left(c_1+c_2 s\right)^4}+\frac{2c_2 d\,\phi^2}{\left(c_1+c_2 s\right)^4}\right]$$

$$=-\frac{\left(n+1\right)c_2\phi^2\left[\left(c_1+c_2 s\right)\phi^2+d\left(b^2-s^2\right)\right]}{\left(c_1+c_2 s\right)^4}$$

$$+\left[\frac{3c_2 d\left(b^2-s^2\right)\phi^2}{\left(c_1+c_2 s\right)^4}\right]$$

$$=\frac{\left(2-n\right)c_2 d\left(b^2-s^2\right)\phi^2-\left(n+1\right)c_2\left(c_1+c_2 s\right)\phi^4}{\left(c_1+c_2 s\right)^4} \tag{27}$$

By (27), it results that for the $\Phi$ defined by (25), the relation $\left(b^2-s^2\right)\left(c_1+c_2 s\right)^4\Phi$ is a polynomial in $s$ and $b$ of degree 7 and 4, respectively. The coefficient of $s^7$ is $\left(n+1\right)c_1^2 c_2^2$. Thus, $\Phi=0$ is impossible because $c_1^2 c_2^2\neq 0.$ Thus, we can conclude that (16) does not hold. Therefore, the case (ii) of the Theorem 3.3 holds. In this case, we have

$$r_{00}=0, \tag{28}$$

$$s_j=0. \tag{29}$$

By (5) and (28), we obtain

$$\left[\tau+\frac{2}{3}\left(b^2\tau-b_m\theta^m\right)\right]\alpha^2=\beta\left[-\frac{2}{3}\theta+\frac{5}{3}\beta\tau\right] \tag{30}$$

Since $\alpha^2$ is irreducible polynomial of $y^i$, then (30) reduces to the following

$$\tau+\frac{2}{3}\left(b^2\tau-b_m\theta^m\right)=0, \tag{31}$$

$$-\frac{2}{3}\theta+\frac{5}{3}\beta\tau=0. \tag{32}$$

By (3) we have

$$s_0=-\frac{1}{3}\left(\theta b^2-\beta b_m\theta^m\right). \tag{33}$$

It follows from (29) that $s_0=0.$ Then (33) reduces to

$$\theta b^2-\beta b_m\theta^m=0 \tag{34}$$

By (32) and (34), we obtain

$$\frac{2}{3}\left(1-b^2\right)\theta=\frac{2}{3}\left(1-b^2\right)\tau\beta+\left\{\tau+\frac{2}{3}\left(b^2-b_m\theta^m\right)\right\}\beta. \tag{35}$$

Then it follows from (31) and (35) that

$$\theta=\tau\beta \tag{36}$$

By (36) we have

$$\tau b^2-b^j\theta_j=0. \tag{37}$$

By (31) and (37), it follows that $\tau=0,$ and by considering (36), we get $\theta=0.$ Therefore (3), (4) and (5) reduce to the following

$$s_{ij}=0, \tag{38}$$

$$G^l_\alpha=0, \tag{39}$$

$$r_{00}=0. \tag{40}$$

Since $s_0=r_{00}=0,$ then (19) and (20) reduce to

$$P=Q^i=0 \tag{41}$$

By (18), it follows that $G^i_\alpha=0.$ This completes the proof.

**Proof of Theorem 1.1.** By the Proposition 3.1, we conclude that $F$ is dually flat and projectively flat in any adapted coordinate system. By Lemma 2.1, we have

$$F_{x^k}=CFF_{y^k}.$$

The spray coefficients $G^i=Py^i$ are given by $P=\frac{1}{2}CF.$ Since $G^i=0,$ then $P=0$ and thus

$$C=0.$$

It implies that $F_{x^k}=0$ and then $F$ is a locally Minkowskian metric in the adapted coordinated system. This completes the proof.

## References

[1] Amari, S. I. & Nagaoka, H. (2000). *Method of information Geometry*. AMS Translation of Math. Monograph, Oxford University press.

[2] Shen, Z. (2006). Riemann-Finsler geometry with applications to information geometry. *Chin. Ann. Math., 27*(B1), 73-94.

[3] Cheng, X., Shen, Z. & Zhou, Y. (2010). On a class of Locally dually flat Finsler metrics. *Int. J. Math., 21*(11), 1-13.

[4] Cheng, X. & Tian, Y. (2011). Locally dually flat Finsler metrics with special curvature properties. *Diff. Geom. Appl. doi:10.1016/j.difgeo.04.014.*

[5] Xia, Q. (2011). On locally dually flat $(\alpha,\beta)-$metrics. *Diff. Geom. Appl., 29*(2), 233-243.

[6] Xia, Q. (2011). On a class of locally dually flat Finsler metrics of isotropic flag curvature. *Publ. Math. Debrecen. 78*(1), 169-190.

[7] Shen, Z. (1997). Volume comparison and its applications in Riemann-Finsler geometry. *Advances in Math., 128*, 306-328.

[8] Cheng, X. & Shen, Z. (2009). A class of Finsler metrics with isotropic S-curvature. *Israel J. Math., 169*, 317-340.

[9] Tayebi, A. & Rafie-Rad, M. (2008). S-curvature of isotropic Berwald metrics. *Science in China, Series A: Math., 51*, 2198-2204.

[10] Peyghan, E., Ahmadi, A. & Tayebi, A. (2011). Regarding the Kohler- Einstein structure on Cartan spaces with Berwald connection. *IJST, Trans. A, 35*(A2), 89-99.

[11] Matsomoto, M. (1992). Theory of Finsler spaces with $(\alpha,\beta)-$metric. *Rep. Math. Phys., 31*, 43-84.

[12] Tayebi, A. & Peyghan, E. (2009). On a special class of Finsler metrics. *IJST, Trans. A, 33*(A2), 179-186.

[13] Tayebi, A. & Peyghan, E. (2010). On Ricci tensors of Randers metrics. *J. Geom. Phys., 60*, 1665-1670.

[14] Tayebi, A. & Najafi, B. (2012). On Isotropic Berwald Metrics. *Annales Polonici Mathematici, 103*, 109-121.