IJST (2012) A3 (Special issue-Mathematics): 371-376
Iranian Journal of Science & Technology
http://www.shirazu.ac.ir/en

On compact operators on the Riesz B^m-difference sequence spaces-II

M. Basarir[1]* and E. E. Kara[2]

[1]*Department of Mathematics, Sakarya University, 54187, Sakarya, Turkey*
[2]*Department of Mathematics, Bilecik University, 11210, Bilecik, Turkey*
E-mails: basarir@sakarya.edu.tr & emrah.kara@bilecik.edu.tr

Abstract

The compact operators on the Riesz sequence space $r_p^q(B^m)$ $(1 \leq p < \infty)$ have been studied by Başarır and Kara, "*IJST (2011) A4, 279-285*". In the present paper, we will characterize some classes of compact operators on the normed Riesz sequence spaces $r_0^q(B^m)$ and $r_\infty^q(B^m)$ by using the Hausdorff measure of noncompactness.

Keywords*:* B^m-difference sequence spaces; Hausdorff measure of noncompactness; compact operators

1. Introduction

The Hausdorff measure of noncompactness has various applications in the theory of sequence spaces, one of them is to obtain necessary and sufficient conditions for matrix operators between BK spaces to be compact. Recently, several authors characterized classes of compact operators given by infinite matrices on the some sequence spaces by using this method [1-17].

In this paper, we will continue to study the characterization of compact operators on some Riesz B^m-difference sequence spaces. For this purpose, we will use the Hausdorff measure of noncompactness and some results in [3, 14].

2. Preliminaries and notations

Let ω be the space of all real valued sequences. Any vector subspace of ω is called a sequence space. We write ℓ_∞, c, c_0 and ϕ the sets of all bounded, convergent, null and finite sequences, respectively. Also, by cs, ℓ_1 and ℓ_p $(1 < p < \infty)$, we denote the sequence spaces of all convergent, absolutely and $p-$absolutely convergent series, respectively. Further, we use the conventions that $e = (1,1,\dots)$ and $e^{(k)}$ is the sequence whose only non-zero term is 1 in the kth place for each $k \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,\dots\}$.

The β-dual of a subset X of ω is defined by

$$X^\beta = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x \in X\}.$$

A sequence space X is called a BK space if it is a Banach space with continuous coordinates $p_n: X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. The sequence spaces c_0, c and ℓ_∞ are BK-spaces with the usual sup-norm given by $\|x\|_{\ell_\infty} = \sup_{k\in\mathbb{N}}|x_k|$ and the space ℓ_p is a BK- space with the usual ℓ_p-norm defined by $\|x\|_{\ell_p} = (\sum_{k=0}^\infty |x_k|^p)^{1/p}$, where $1 \leq p < \infty$.

If $(X, \|.\|)$ is a normed sequence space, then we write

$$\|a\|_X^* = \sup_{x\in S_X}\left|\sum_k a_k x_k\right| \qquad (1)$$

for $a \in \omega$ provided the expression on the right hand side exists and is finite, which is the case whenever X is a BK space and $a \in X^\beta$, where S_X is the unit sphere in X, i.e., $S_X = \{x \in X : \|x\| = 1\}$.

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ a real sequence such that

$$Ax = (A_n(x)) = \left(\sum_k a_{nk} x_k\right) \text{ exists for each } n. \qquad (2)$$

Then the sequence $Ax = (A_n(x))$ is called A-transform of x. For two sequence spaces X and Y we say that the matrix A maps X into Y if Ax exists and belongs to Y for all $x \in X$. By (X, Y), we denote the set of all matrices which map X into Y. Also, we write A_n for the sequence in the nth row of A, that is, $A_n = (a_{nk})_{k=0}^\infty$. Thus $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

*Corresponding author
Received: 16 January 2012 / Accepted: 8 April 2012

For basic definitions and notation we refer to [18-20]. The following results are fundamental for our investigation.

Lemma 1.1. [3] Let X and Y be BK-spaces. Then we have $(X,Y) \subset B(X,Y)$, that is, every $A \in (X,Y)$ defines a linear operator $L_A \in B(X,Y)$, where $L_A(x) = Ax$ for all $x \in X$.

Lemma 1.2. [6] Let X denote any of the spaces c_0 or ℓ_∞. If $A = (a_{nk}) \in (X,c)$, then we have

$$\alpha_k = \lim_n a_{nk} \text{ exists for every } k \in \mathbb{N},$$

$$\alpha = (\alpha_k) \in \ell_1,$$

$$\sup_n \left(\sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) < \infty,$$

$$\lim_n A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \text{ for all } x = (x_k) \in X.$$

Lemma 1.3. [3] Let $X \supset \phi$ be BK-space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X,Y)$, then

$$\|L_A\| = \|A\|_{(X,\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

Lemma 1.4. [3] Let X denote any of the spaces c_0, c or ℓ_∞. Then, we have $X^\beta = \ell_1$ and $\|a\|_X^* = \|a\|_{\ell_1}$ for all $a = (a_k) \in \ell_1$.

If X and Y are Banach spaces then $B(X,Y)$ is the set of all continuous linear operators $L: X \to Y$; $B(X,Y)$ is a Banach space with the operator norm defined by $\|L\| = \sup\{\|L(x)\|: \|x\| \le 1\}$ for all $L \in B(X,Y)$. A linear operator $L: X \to Y$ is said to be compact if the domain of L is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of such operators by $\mathrm{C}(X,Y)$.

Let M be a subset of a metric space (X,d) and $\varepsilon>0$ Then, a subset A of X is called an ε -net of M in X if for every $x \in M$ there exists $a \in A$ such that $d(x,a) < \varepsilon$. Further, if the set A is finite, then the ε-net A of M is called a finite ε-net of M, and we say that M has a finite ε-net in X. A subset of a metric space is said to be totally bounded if it has a finite ε-net for every $\varepsilon > 0$.

If (X,d) is a metric space, we write $\mathcal{M}_X$ for the class of all bounded subsets of X. If $Q \in \mathcal{M}_X$ then the Hausdorff measure of noncompactness of the set Q, denoted by $\chi(Q)$, is given by

$$\chi(Q) = \inf\{\varepsilon > 0:\ Q \text{ has a finite } \varepsilon-\text{net in } X\}.$$

The function $\chi: \mathcal{M}_X \to [0,\infty)$ is called the Hausdorff measure of noncompactness.

The basic properties of the Hausdorff measure of noncompactness can be found in [20], for example, if Q, Q_1 and Q_2 are bounded subsets of a metric space (X,d), then

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded}$$
$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \le \chi(Q_2).$$

Further, if X is a normed space, then the function χ has some additional properties connected with the linear structure, e.g.

$$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2),$$
$$\chi(\alpha Q) = |\alpha|\chi(Q) \text{ for all } \alpha \in \mathbb{C}.$$

The following lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

Lemma 1.5. [20]. Let X and Y be Banach spaces and $L \in B(X,Y)$. Then we have

$$\|L\|_\chi = \chi(L(S_X)) \tag{3}$$

and

$$L \in \mathrm{C}(X,Y) \text{ if and only if } \|L\|_\chi = 0. \tag{4}$$

Now we give the following two lemmas which show how to compute the Hausdorff measure of noncompactness in the sequence spaces c_0 and c.

Lemma 1.6. [6] Let $Q \in \mathcal{M}_{c_0}$ and $P_r: c_0 \to c_0$ $(r \in \mathbb{N})$ be the operator defined by $P_r(x) = x^{[r]} = (x_0, x_1, x_2, \dots, x_r, 0, 0, \dots)$ for all $x = (x_k) \in c_0$. Then, we have

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{x \in Q} \|(I - P_r)(x)\|_{\ell_\infty} \right),$$

where I is the identity operator on c_0.

Lemma 1.7. [20]. Let $Q \in \mathcal{M}_c$ and $P_r: c \to c$ $(r \in \mathbb{N})$ be the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, \dots, e^{(r)}\}$. Then, we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \left(\sup_{x \in Q} \|(I - P_r)(x)\|_{\ell_\infty} \right) \le \chi(Q)$$
$$\le \lim_{r\to\infty} \left(\sup_{x \in Q} \|(I - P_r)(x)\|_{\ell_\infty} \right),$$

where I is the identity operator on c.

2. The Riesz B^m-difference sequence spaces $r_0^q(B^m)$, $r_c^q(B^m)$ and $r_\infty^q(B^m)$

In this section, by taking some special cases of the paranormed Riesz B^m-difference sequence spaces $r_0^q(p, B^m)$, $r_c^q(p, B^m)$ and $r_\infty^q(p, B^m)$, we obtain BK-spaces and give some results related to these spaces.

For a sequence $x=(x_k)$, we denote the difference sequence by $\Delta x = x_k - x_{k-1}$. Let (q_k) be a sequence of positive numbers, $Q_n = \sum_{k=0}^{n} q_k$; $(n \in \mathbb{N})$ and $r, s \neq 0$. Then the matrices $B^m = (b_{nk}^m)$ and $R^q.B^m = T^m = (t_{nk}^m)$ $(m \in \mathbb{N})$ are defined by

$$b_{nk}^m = \begin{cases} \binom{m}{n-k} r^{m-n+k} s^{n-k} & (\max\{0, n-m\} \leq k \leq n) \\ 0 & (0 \leq k < \max\{0, n-m\}) \text{ or } (k > n) \end{cases}$$

and

$$t_{nk}^m = \begin{cases} \frac{1}{Q_n} \sum_{i=k}^{n} \binom{m}{i-k} r^{m-i+k} s^{i-k} q_i & (k < n) \\ \frac{r^m}{Q_n} & (k = n) \\ 0 & (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$, where the matrix $R^q = (r_{nk})$ is the Riesz mean, that is,

$$r_{nk}^q = \begin{cases} \frac{q_k}{Q_n} & (0 \leq k \leq n) \\ 0 & (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. It is obvious that the matrix B^m reduced the difference matrix Δ^m in case $r = 1$ and $s = -1$, where $\Delta^m = \Delta(\Delta^{m-1})$.

Recently, the generalized B^m- Riesz difference sequence spaces $r_0^q(p, B^m)$, $r_c^q(p, B^m)$ and $r_\infty^q(p, B^m)$ have been introduced by Başarır and Öztürk [21] as follows:

$$r_0^q(p, B^m) = \{x = (x_k) \in \omega : T^m x \in c_0(p)\},$$
$$r_c^q(p, B^m) = \{x = (x_k) \in \omega : T^m x \in c(p)\}$$

and

$$r_\infty^q(p, B^m) = \{x = (x_k) \in \omega : T^m x \in \ell_\infty(p)\},$$

where $p = (p_n)$ is a bounded sequence of strictly positive real numbers and $c_0(p)$, $c(p)$, $\ell_\infty(p)$ are the paranormed sequence spaces defined by Maddox.

If we take $p_n = 1$ for all $n \in \mathbb{N}$, then we have that

$$r_0^q(B^m) = \{x = (x_k) \in \omega : T^m x \in c_0\},$$
$$r_c^q(B^m) = \{x = (x_k) \in \omega : T^m x \in c\}$$

and

$$r_\infty^q(B^m) = \{x = (x_k) \in \omega : T^m x \in \ell_\infty\}.$$

It is obvious that if we put $q_k = \lambda_k - \lambda_{k-1}$ for all k, then the spaces $r_0^q(B^m)$, $r_c^q(B^m)$ and $r_\infty^q(B^m)$ are reduced to the spaces $c_0^\lambda(B^m)$, $c^\lambda(B^m)$ and $\ell_\infty^\lambda(B^m)$, respectively; where c_0^λ, c^λ and ℓ_∞^λ are the spaces defined by Mursaleen and Noman in [22-24].

Let X be any of the spaces $r_0^q(B^m)$, $r_c^q(B^m)$ or $r_\infty^q(B^m)$. It is obvious that X is BK-spaces with the norm given by

$$\|x\|_X = \|T^m x\|_\infty = \sup_n |T_n^m(x)|. \quad (5)$$

Throughout, for any sequence $x = (x_k)$, we define the associated sequence $y = (y_k)$, which will be frequently used as the T^m-transform of x, that is, $y = T^m x$ and so

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k} \binom{m}{i-j} r^{m-i+j} s^{i-j} q_i \right] x_j + \frac{r^m}{Q_k} q_k x_k; \; (k \in \mathbb{N}). \quad (6)$$

Obviously, if the sequences x and y are connected by the relation (6), then $x \in r_0^q(B^m)$, $r_c^q(B^m)$ or $r_\infty^q(B^m)$ if and only if $y \in c_0$, c or ℓ_∞, respectively. Further, if $x \in X$, then $\|x\|_X = \|y\|_\infty$, where $X = r_0^q(B^m)$, $r_c^q(B^m)$ or $r_\infty^q(B^m)$.

We shall write throughout for brevity that

$$\nabla(i,j,k) = (-1)^{j-k} \frac{s^{j-i}}{r^{m+j-i}} \binom{m+j-i-1}{j-i} \frac{1}{q_i}; \quad (j,k,m \in \mathbb{N})$$

for all $r, s \neq 0$ and $q_i > 0$ $(i \in \mathbb{N})$.
The following results will be needed in establishing our results.

Lemma 2.1. Let X denote any of the spaces $r_0^q(B^m)$ or $r_\infty^q(B^m)$. If $a = (a_k) \in X^\beta$, then $\bar{a} = (\bar{a}_k) \in \ell_1$ and the equality

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \bar{a}_k y_k \quad (7)$$

holds for every $x = (x_k) \in X$, where $y = T^m x$ is given by (6) and

$$\tilde{a}_k = Q_k \left(\frac{a_k}{r^m q_k} + \sum_{j=k+1}^{\infty} \left[\sum_{i=k}^{k+1} \nabla(i,j,k) \right] a_j \right); \; (k \in \mathbb{N}).$$

Proof: This follows immediately by [25, Theorem 5.6].

Lemma 2.2. Let X denote any of the spaces $r_0^q(B^m)$ or $r_\infty^q(B^m)$. Then, we have

$$\|a\|_X^* = \sum_{k=0}^{\infty} |\bar{a}_k| < \infty$$

for all $a = (a_k) \in X^\beta$, where $\bar{a} = (\bar{a}_k)$ is as in Lemma 2.1.

Proof: Let $Y = c_0, \ell_\infty$ and take any $a = (a_k) \in X^\beta$. Then, we have by Lemma 2.1 that $\bar{a} = (\bar{a}_k) \in \ell_1$ and the equality (7) holds for all sequences $x = (x_k) \in X$ and $y = (y_k) \in Y$ which are connected by the relation (6). Moreover, it follows by (5) that $x \in S_X$ if and only if $y \in S_Y$. Therefore, it follows by (1) and (7) that

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right| = \sup_{y \in S_Y} \left| \sum_{k=0}^{\infty} \bar{a}_k y_k \right| = \|\bar{a}\|_Y^*$$

and since $\bar{a} \in \ell_1$, we obtain from Lemma 1.4

$$\|a\|_X^* = \|\bar{a}\|_Y^* = \|\bar{a}\|_{\ell_1} < \infty$$

which concludes the proof.

Now, for an infinite matrix $A = (a_{nk})$, we shall write

$$\bar{a}_{nk} = Q_k \left(\frac{a_{nk}}{r^m q_k} + \sum_{j=k+1}^{\infty} \left[\sum_{i=k}^{k+1} \nabla(i,j,k) \right] a_{nj} \right) \quad (8)$$

for all $n, k \in \mathbb{N}$ provided the convergence of the series.

Lemma 2.3. Let X be any of the spaces $r_0^q(B^m)$ or $r_\infty^q(B^m)$, Y the respective one of the spaces c_0 or ℓ_∞, Z a sequence space and $A = (a_{nk})$ an infinite matrix. If $A \in (X, Z)$, then $\bar{A} \in (Y, Z)$ such that $Ax = \bar{A}y$ for all $x \in X$ and $y \in Y$ which are connected by the relation (6), where the matrix $\bar{A} = (\bar{a}_{nk})$ is defined as in (8).

Proof: This result can be proved using a similar method in [6, Lemma 2.3].

Lemma 2.4. Let X be any of the spaces $r_0^q(B^m)$or $r_\infty^q(B^m)$, $A = (a_{nk})$ an infinite matrix and $\bar{A} = (\bar{a}_{nk})$ the matrix in (8). If A is in any of the classes (X, c_0), (X, c) or (X, ℓ_∞), then

$$\|L_A\| = \|A\|_{(X, \ell_\infty)} = \sup_n \left(\sum_k |\bar{a}_{nk}| \right) < \infty.$$

Proof: This is immediate by combining Lemmas 1.3 and 2.2.

3. Compact operators on the spaces $r_0^q(B^m)$ and $r_\infty^q(B^m)$

In this final section, we establish some identities or estimates for the Hausdorff measures of noncompactness of certain matrix operators on the spaces $r_0^q(B^m)$ and $r_\infty^q(B^m)$. Further, we deduce the necessary and sufficient (or only sufficient) conditions for such operators to be compact. For the most recent work on this topic, we refer to [3, 6].

Now, let $A = (a_{nk})$ be an infinite matrix and $\bar{A} = (\bar{a}_{nk})$ the associated matrix defined by (8). Then we have following results.

Theorem 3.1. Let X denote any of the spaces $r_0^q(B^m)$ or $r_\infty^q(B^m)$. Then, we have
(i) If $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \limsup_{n \to \infty} \left(\sum_k |\bar{a}_{nk}| \right). \quad (9)$$

(ii) If $A \in (X, c)$, then

$$\frac{1}{2} \cdot \limsup_{n \to \infty} \left(\sum_k |\bar{a}_{nk} - \bar{\alpha}_k| \right) \le \|L_A\|_\chi \le \limsup_{n \to \infty} \left(\sum_k |\bar{a}_{nk} - \bar{\alpha}_k| \right), \quad (10)$$

where $\bar{\alpha}_k = \lim_{n \to \infty} \bar{a}_{nk}$ for all $k \in \mathbb{N}$.
(iii) If $A \in (X, \ell_\infty)$, then

$$0 \le \|L_A\|_\chi \le \limsup_{n \to \infty} \left(\sum_k |\bar{a}_{nk}| \right). \quad (11)$$

Proof: The limits in (9)-(11) obviously exist.
We write $S = S_X$. Then, we obtain by (3) and Lemma 1.1 that

$$\|L_A\|_\chi = \chi(AS). \quad (12)$$

(i) Let $A \in (X, c_0)$. Then, we have $AS \in \mathcal{M}_{c_0}$ and so it follows by using Lemma 1.6 that

$$\chi(AS) = \lim_{r \to \infty} \left(\sup_{x \in S} \|(I - P_r)(Ax)\|_{\ell_\infty} \right), \quad (13)$$

where $P_r: c_0 \to c_0$ $(r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, \dots, x_r, 0, 0, \dots)$ for all $x = (x_k) \in c_0$. This yields that $\|(I - P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r} |\sum_k a_{nk} x_k|$ for all $x \in X$ and every $r \in \mathbb{N}$. Thus, by combining (1), (2) and Lemma 2.2, we have

$$\sup_{x \in S} \|(I - P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r} \|A_n\|_X^* = \sup_{n>r} \left(\sum_k |\bar{a}_{nk}| \right) \quad (14)$$

for every $r \in \mathbb{N}$.
(13) and (14) imply that

$$\chi(AS) = \lim_{r \to \infty} \left(\sup_{n>r} \left(\sum_k |\bar{a}_{nk}| \right) \right) = \limsup_{n \to \infty} \left(\sum_k |\bar{a}_{nk}| \right).$$

Now, (9) follows from (12).
(ii) Let $A \in (X, c)$. Then, we have $AS \in \mathcal{M}_c$. Also, we know that every $z = (z_n) \in c$ has a unique representation $z = le + \sum_{n=0}^{\infty} (z_n - l) e^{(n)}$, where

$l = \lim_{n\to\infty} z_n$. Thus, we define the projectors $P_r: c \to c$ $(r \in \mathbb{N})$ by $P_0(z) = le$ and $P_r(z) = le + \sum_{n=0}^{r-1}(z_n - l)e^{(n)}$ for $r \geq 1$. Then, we have that $(I - P_r)(z) = \sum_{n=r}^{\infty}(z_n - l)e^{(n)}$ for every $r \in \mathbb{N}$ and whence

$$\|(I - P_r)(z)\|_{\ell_\infty} = \sup_{n\geq r}|z_n - l| \tag{15}$$

for all $z \in c$ and every $r \in \mathbb{N}$. So, one can easily see that $\|I - P_r\| = 2$ for all $r \in \mathbb{N}$. Hence, we obtain from (12) and Lemma 1.7 that

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{x\in S}\|(I - P_r)(Ax)\|_{\ell_\infty}\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{x\in S}\|(I - P_r)(Ax)\|_{\ell_\infty}\right). \tag{16}$$

On the other hand, it is given that $X = r_0^q(B^m)$ or $r_\infty^q(B^m)$, and let Y be respective of the spaces c_0 or ℓ_∞. Also, for every given $x \in X$, let $y \in Y$ be the associated sequence defined by (6). Since $A \in (X, c)$, we have by Lemma 2.3 that $\bar{A} \in (Y, c)$ and $Ax = \bar{A}y$. Further, it follows from Lemma 1.2 that the limits $\bar{\alpha}_k = \lim_{n\to\infty}\bar{a}_{nk}$ for all k, $\bar{\alpha} = (\bar{\alpha}_k) \in Y^\beta = \ell_1$ and $\lim_{n\to\infty}\sum_k \bar{a}_{nk}y_k = \sum_k \bar{\alpha}_k y_k$. Consequently, we derive from (15) that

$$\begin{aligned}\|(I - P_r)(Ax)\|_{\ell_\infty} &= \|(I - P_r)(\bar{A}y)\|_{\ell_\infty} \\ &= \sup_{n\geq r}\left|\bar{A}_n y - \sum_k \bar{\alpha}_k y_k\right| \\ &= \sup_{n\geq r}\left|\sum_k(\bar{a}_{nk} - \bar{\alpha}_k)y_k\right|\end{aligned}$$

for all $r \in \mathbb{N}$. Moreover, since $x \in S = S_X$ if and only if $y \in S_Y$, we obtain by (1) and Lemma 1.4 that

$$\begin{aligned}&\sup_{x\in S}\|(I - P_r)(Ax)\|_{\ell_\infty} \\ &= \sup_{n\geq r}\left(\sup_{x\in S_Y}\left|\sum_k(\bar{a}_{nk} - \bar{\alpha}_k)y_k\right|\right) \\ &= \sup_{n\geq r}\|\bar{A}_n - \bar{\alpha}\|_Y^* \\ &= \sup_{n\geq r}\|\bar{A}_n - \bar{\alpha}\|_{\ell_1}\end{aligned}$$

for all $r \in \mathbb{N}$. Hence, we get (10) from (16).
Finally, to prove (iii) we define the operators $P_r: \ell_\infty \to \ell_\infty$ $(r \in \mathbb{N})$ as in the proof of part (i) for all $x = (x_k) \in \ell_\infty$. Then, we have

$$AS \subset P_r(AS) + (I - P_r)(AS);\ (r \in \mathbb{N}).$$

Thus, it follows by the elementary properties of the function χ that

$$\begin{aligned}0 \leq \chi(AS) &\leq \chi\big(P_r(AS)\big) + \chi((I - P_r)(AS)) \\ &= \chi((I - P_r)(AS)) \\ &\leq \sup_{x\in S}\|(I - P_r)(Ax)\|_{\ell_\infty} \\ &= \sup_{n>r}\|\bar{A}_n\|_{\ell_1}\end{aligned}$$

for all $r \in \mathbb{N}$ and hence,

$$\begin{aligned}0 \leq \chi(AS) &\leq \lim_{r\to\infty}\left(\sup_{n>r}\|\bar{A}_n\|_{\ell_1}\right) \\ &= \limsup_{n\to\infty}\|\bar{A}_n\|_{\ell_1}.\end{aligned}$$

This and (12) together imply (11) and completes the proof.

Corollary 3.2. Let X denote any of the spaces $r_0^q(B^m)$ or $r_\infty^q(B^m)$. Then, we have
(i) If $A \in (X, c_0)$, then

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_k|\bar{a}_{nk}|\right) = 0.$$

(ii) If $A \in (X, c)$, then

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_k|\bar{a}_{nk} - \bar{\alpha}_k|\right) = 0,$$

where $\bar{\alpha}_k = \lim_{n\to\infty}\bar{a}_{nk}$ for all $k \in \mathbb{N}$.
(iii) If $A \in (X, \ell_\infty)$, then

$$L_A \text{ is compact if } \lim_{n\to\infty}\left(\sum_k|\bar{a}_{nk}|\right) = 0.$$

Proof: This result follows from Theorem 3.1 by using (4).
Since the matrix T^m is a triangle we have the following observation from [14, Corollaries 6.9 and 6.11].

Corollary 3.3. For every matrix $A \in (r_\infty^q(B^m), c_0)$ or $A \in (r_\infty^q(B^m), \ell_\infty)$, the operator L_A is compact.

Acknowledgments

We would like to express our gratitude to the reviewer for his/her careful reading which improved the presentation of the paper.

References

[1] Başarır, M. & Kara, E. E. (2011). On compact operators on the Riesz B^m-difference sequence space, *IJST, Trans. A*, A4, 279-285.

[2] Djolovic, I. & Malkowsky, E. (2008). Matrix transformations and compact operators on some new mth-order difference sequences. *Appl. Math. Comput.*, *198(2)*, 700-714.

[3] Mursaleen, M. & Noman, A. K. (2010). Compactness by the Hausdorff measure of noncompactness. *Nonlinear Analysis*, *73*(8), 2541-2557.

[4] Mursaleen, M. & Noman, A. K. (2010). Applications of the Hausdorff measure of noncompactness in some

sequence spaces of weighted means. *Comput. Math. Appl*. *60*(5), 1245-1258.

[5] Mursaleen, M. & Noman, A. K. (2011). On σ-conservative matrices and compact operators on the space V_σ. *Appl. Math. Letters*, *24*, 1554-1560.

[6] Mursaleen, M., Karakaya, V., Polat, H. & Şimşek, N. (2011). Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means. *Comput. Math. Appl*. *62*, 814-820.

[7] Mursaleen, M. & Noman, A. K. (2012). Compactness of matrix operators on some new difference sequence spaces. *Linear Algebra Appl. 436*(1), 41-52.

[8] Mursaleen, M. & Noman, A. K. (2011). The Hausdorff measure of noncompactness of matrix operators on some BK spaces. *Operators and Matrices 5*(3), 473-486.

[9] Başar, F. & Malkowsky, E. (2011). The characterization of compact operators on spaces of strongly summable and bounded sequences. *Appl. Math. Comput.*, *217*, 5199-5207.

[10] Başarır, M. & Kayıkçı, M. (2009). On the generalized B^m-Riesz difference sequence space and β-property. *J. Inequal. Appl.* Volume 2009, Article ID 385029, 18 pages, doi:10.1155/2009/385029.

[11] Mursaleen, M. (2011). Application of measure of noncompactness to infinite system of differential equations. *Canadian Math. Bull*. doi:10.4153/CMB-2011-170-7.

[12] Malkowsky, E. & Mursaleen, M. (2011). Compact matrix operators between the spaces $m(\phi)$, $n(\phi)$ and ℓ_p. *Bull. Korean Math. Soc*., *48*(5), 1093-1103.

[13] Başarır, M. & Kara, E. E. (2011). On some difference sequence spaces of weighted means and compact operators. *Ann. Funct. Anal. 2*(2), 116-131.

[14] Kara, E. E. & Başarır, M. (2011). On some Euler B^m-difference sequence spaces and compact operators. *J. Math. Anal. Appl. 379*, 499-511.

[15] Mursaleen, M. & Mohiuddine, S. A. (2012). Applications measures of noncompactness to the infinite system of differential equations in ℓ_p spaces. *Nonlinear Analysis, 75*, 2111-2115.

[16] Mursaleen, M. & Noman, A. K. (2011). On generalized means and some related sequence spaces, *Comput. Math. Appl. 61*, 988-999.

[17] Başarır, M. & Kara, E. E. (2012). On the B-difference sequence space derived by generalized weighted mean and compact operators. *J. Math. Anal. Appl*. *391*, 67-81.

[18] Başar, F. (2011). *Summability Theory and Its Applications*. Bentham Science Publishers, e-books, Monographs, Istanbul-2012, ISBN: 978-1-60805-252-3, to appear.

[19] Wilansky, A. (1986). *Summability Through Functional Analysis*. *Vol. 85*, Amsterdam, North-Holland Math. Studies.

[20] Malkowsky, E. & Rakočević, V. (2000). An introduction into the theory of sequence spaces and measures of noncompactness, *Zb. Rad. (Beogr.)*, *9*(17) 143-234.

[21] Başarır, M. & Öztürk, M. (2011), On some generalized B^m-difference Riesz sequence spaces and uniform Opial property. *J. Inequal. Appl.* Volume 2011, Article ID 485730, 17 pages, doi:10.1155/2011/485730.

[22] Mursaleen, M. & Noman, A. K. (2010). On the spaces of λ-convergent and bounded sequences. *Thai J. Math. 8*(2), 311-329.

[23] Mursaleen, M. & Noman, A. K. (2011). On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ I. *Filomat*, *25*(2), 33-51.

[24] Mursaleen, M. & Noman, A. K. (2011). On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ II. *Math. Commun*. *16*, 383-398.

[25] Mursaleen, M. & Noman, A. K. (2010), On some new difference sequence spaces of non-absolute type. *Math. Comput. Modelling*, *52*, 603-617.