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Bounds on the signed distance-k-domination number of graphs

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Abstract

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order n and edge set $E = E(G)$. A k -dominating set of G is a subset $S \subseteq V$ such that each vertex in $V \setminus S$ has at least k neighbors in S. If v is a vertex of a graph G, the open k-neighborhood of v, denoted by $N_k(v)$, is the set $N_k(v) = \{u \in V : u \neq v \text{ and } d(u, v) \leq k\}$. $N_k[v] = N_k(v) \cup \{v\}$ is the closed k-neighborhood of v. A function $f : V \longrightarrow \{-1, 1\}$ is a signed distance-k-dominating function of G, if for every vertex $v \in V$, $f(N_k[v]) = \sum_{u \in N_k[v]} f(u) \geq 1$. The signed distance-$k$-domination number, denoted by $\gamma_{k,s}(G)$, is the minimum weight of a signed distance-k-dominating function of G. In this paper, we give lower and upper bounds on $\gamma_{k,s}$ of graphs. Also, we determine the signed distance-k-domination number of graph $\gamma_{k,s}(G \vee H)$ (the graph obtained from the disjoint union $G + H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$) when $k \geq 2$.

***Keywords**:* Signed distance-k-dominating function; kth power of a graph

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order n and edge set $E = E(G)$. For a subset $S \subseteq V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$. If v is a vertex of a graph G, the open k-neighborhood of v, denoted by $N_k(v)$, is the set $N_k(v) = \{u \in V : u \neq v \text{ and } d(u, v) \leq k\}$. $N_k[v] = N_k(v) \cup \{v\}$ is the closed-k-neighborhood of v. $\delta_k(G) = min\{|N_k(v)|;\ v \in V\}$ and $\Delta_k(G) = max\{|N_k(v)|;\ v \in V\}$.

A k-*dominating set* of G is a subset $S \subseteq V$ such that every vertex in $V \setminus S$ has at least k neighbors in S. The k-domination number $\gamma_k(G)$ is the minimum cardinality among the k-*dominating sets of* G. A subset $S \subseteq V$ is a total dominating set, if for every vertex $u \in V$ there exists a vertex $v \in S$, such that u is adjacent to v. Let G be a graph with no isolated vertex. The total domination number $\gamma_t(G)$ is the minimum cardinality among the total dominating sets of G.

A function $f : V \longrightarrow \{-1, 1\}$ is a signed *distance-k-dominating* function of G, if for every vertex $v \in V$, $f(N_k[v]) = \sum_{u \in N_k[v]} f(u) \geq 1$. The signed *distance-$k$-domination* number, denoted by $\gamma_{k,s}(G)$, is the minimum weight of a signed distance-k-dominating function on G. A signed distance-1-dominating function and signed distance-1-domination number $\gamma_{1,s}(G)$ of a graph G are identified with the usual signed dominating function and signed domination number $\gamma_s(G)$ of a graph G [1].

Let $k \geq 2$ be a positive integer. A subset $S \subseteq V(G)$ is a k –packing if for every pair of vertices $u, v \in S$, $d(u,v) > k$. The k -packing number $\beta_k(G)$ is the maximum cardinality of a k-packing in G [2]. The joint of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$ [3]. Let G be a graph of order n with vertex set $\{v_1, v_2, \ldots, v_n\}$. We construct kth power G^k of a graph G, by $V(G^k) = V(G)$ and u and v are adjacent in G^k if and only if $0 < d_G(u,v) \leq k$.

2. Lower bounds on $\gamma_{k,s}(G)$

Observation 1. Let G be a graph of order n, and k be a positive integer. Then $\gamma_{k,s}(G) = \gamma_s(G^k)$.

Proof: Let f be a signed distance-k-dominating function of G. It is easy to see that for every $v \in V(G)$, $N_k[v] = N_{G^k}[v]$. Hence $f(N_{G^k}[v]) = f(N_k[v])$. Therefore f is a signed distance- k-dominating function of G if and only if f is a signed

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distance dominating set of G^k. Thus $\gamma_s(G^k) = \gamma_{k,s}(G)$.

Let G be a graph of order n, and k be a positive integer. $\delta(G^k) = \delta_k(G)$ and $\Delta_k(G^k) = \Delta_k(G)$.

Theorem 2. [4] For any graph G with $\delta \geq 2$, $\gamma_s(G) \geq n(\frac{\lceil\frac{\delta}{2}\rceil - \lfloor\frac{\Delta}{2}\rfloor + 1}{\lceil\frac{\delta}{2}\rceil + \lfloor\frac{\Delta}{2}\rfloor + 1})$.

As an immediate result from Observation 1 and Theorem 2 we have.

Corollary 3. For any graph G with $\delta_k \geq 2$, $\gamma_{k,s}(G) \geq n(\frac{\lceil\frac{\delta_k}{2}\rceil - \lfloor\frac{\Delta_k}{2}\rfloor + 1}{\lceil\frac{\delta_k}{2}\rceil + \lfloor\frac{\Delta_k}{2}\rfloor + 1})$.

Proposition 4. Let G be a graph of order n. Then $2\,\gamma_2\,(G) - n \leq \gamma_s\,(G)$.

Proof Let f be a minimum signed dominating function of G. Let $V_1 = \{u \in V : f(u) = 1\}$ and $V_{-1} = \{u \in V : f(u) = -1\}$. If $V_{-1} = \emptyset$, then the proof is clear.

If $v \in V_{-1}$ since $f(N_G[v]) \geq 1$, then v has at least two adjacent in V_1. Therefore V_1 is a 2-dominating set for G and $|V_1| \geq \gamma_2(G)$. Since $\gamma_s(G) = |V_1| - |V_{-1}|$ and $n = |V_1| + |V_{-1}|$, then $\gamma_s(G) = 2|V_1| - n$ and finally we have $\gamma_s(G) \geq 2\gamma_2(G) - n$.

Proposition 5. Let G be a graph of order n and with no isolated vertex. Then $2\,\gamma_t(G) - n \leq \gamma_s(G)$.

Proof: The proof is similar to the Proposition 4.

3. Upper bounds on $\gamma_{k,s}(G)$

Theorem 6. Let k be a positive integer. If G is a simple graph of order n and minimum degree $\delta \geq 2$ and β_{k+1} is a maximum value of $k+1$-packing sets. Then $\gamma_{k,s}(G) \leq n - 2\beta_{k+1}$, and this bound is sharp.

Proof: Let S be a $k+1$-packing set with $|S| = \beta_{k+1}$. We define $f: V \longrightarrow \{-1, 1\}$ by,

$$f(v) = \begin{cases} -1 & if\ v \in S \\ 1 & if\ v \in V - S. \end{cases}$$

It is easy to show that $f(V(G)) = n - 2\,\beta_{k+1}$. Therefore, it is sufficient to show that f is a signed distance-k-dominating function on G. Let v be a vertex in S. Since $\delta \geq 2$, then $|N_k[v]| \geq 3$ since S is a $k+1$–packing set. Hence $N_k[v] \cap S = \{v\}$, and $f(N_k[v]) \geq 1$. Now let v be a vertex in $V - S$. There are two cases.

Case 1. $N[v] \cap S \neq \emptyset$. Since S is a $k+1$ -packing set in graph G, then $|N_k[v] \cap S| = 1$ and let $N_k[v] \cap S = \{w\}$. Otherwise let u be a vertex in $N_k[v] \cap S$ different from w. This shows that $d(w, u) \leq k+1$. This is a contradiction. Since $\delta \geq 2$ therefore $f(N_k[v]) \geq 1$.

Case 2. $N[v] \cap S = \emptyset$. If $N_k[v] \cap S = \emptyset$, then $f(N_k[v]) \geq 1$. Let $N_k[v] \cap S \neq \emptyset$ and let $N_k[v] \cap S = \{s_1, s_2, \ldots, s_r\}$. Since $d(s_i, s_j) \geq k+2$, there exists a vertex v_i on the $v - s_i$ path which is distinct from the vertex v_j on the $v - s_j$ path. Thus there exist at least r distinct vertices in $N_k[v] - S$. Suppose v_i be a vertex in $N_k[v]$ such that v_i is adjacent to s_i for each $1 \leq i \leq r$.

Therefore, $f(N_k[v]) \geq \sum_{i=1}^{r} f(v - s_i) + \sum_{i=1}^{r} f(v_i) + f(v) = 1$. And f is a signed-k-dominating function on G with weight $|V - S| - |S| = n - 2\,\beta_{k+1}$. Hence $\gamma_{k,s}(G) \leq n - 2\beta_{k+1}$.

Now, we show that the bound is sharp. The desired graph G will be the union p copies of C_4. Then $\gamma_{k,s}(G) = 2p$ and $\beta_{k+1} = p$. Therefore $\gamma_{k,s}(G) = 2p = 4p - 2p = n - 2\beta_{k+1}$. This completes the proof.

Corollary 7. Let k be a positive integer. If G is a simple graph of order n and minimum degree $\delta \geq 2$ and β_{k+1} is a maximum value of $k+1$ - packing sets, then

$$\beta_{k+1} \leq \frac{n}{2}\left(1 - \frac{\lceil\frac{\delta_k}{2}\rceil - \lfloor\frac{\Delta_k}{2}\rfloor + 1}{\lceil\frac{\delta_k}{2}\rceil + \lfloor\frac{\Delta_k}{2}\rfloor + 1}\right).$$

Proof: By Theorem 6 and Corollary 3 the proof is clear.

Theorem 8. Let G be a connected graph of order n. Let L and S be the sets of vertices degree 1 (leaves) and $N_G(L)$ (support vertices) respectively. If D is a maximum 2-packing set in $G - (L \cup S)$, then $\gamma_s(G) \leq n - 2|D|$, and this bound is sharp.

Proof: We define $f: V(G) \longrightarrow \{-1, 1\}$ by, $f(v) = \begin{cases} -1 & if\ v \in D \\ 1 & if\ v \in V - D. \end{cases}$

It is easy to show that $f(V(G)) = n - 2\,|D|$. Therefore, it is sufficient to show that f is a signed distance-2-dominating function on G. For each vertex $v \in V(G)$, if v is a vertex in L then $f(N[v]) = 2 \geq 1$. Let $v \in V - (L \cap D)$, if $N[v] \cap D = \emptyset$ then obviously $f(N[v]) \geq 1$. If $N[v] \cup D \neq \emptyset$, since D is a 2-packing set in $G - (L \cup S)$ then $|N[v] \cap D| = 1$, and let $N[v] \cap D = \{w\}$. Otherwise if u be a vertex in $N[v] \cap D$ different from w then $d(u, w) \leq 2$. This is a contradiction. Since $deg(v) \geq 2$ then $f(N[v]) \geq 2$. Finally, if $v \in D$, since $deg(v) \geq 2$ and $N[v] \cap D = \{v\}$ then $f(N[v]) \geq 1$. Therefore f is a signed dominating function of G with weight $n-2|D|$. Hence $\gamma_s \leq n-2|D|$.

Now, we show that the bound is sharp. Let $G=K_{1,n-1}$ $(n \geq 2)$. Then $\gamma_s(G)=n$ and $D=\emptyset$. Thus $\gamma_s(G)=n-2|D|$.

In Theorem 6 the graph G can be a simple disconnected graph of order n and $\delta(G)=1$. Since H_1, H_2,..., H_m are components of G, then $\gamma_s(G)=\gamma_s(H_1)+\gamma_s(H_2)+\ldots+\gamma_s(H_m)$. By a similar reason we can prove $\gamma_s(G) \leq n-2|D|$, where L and S are the sets of vertices of degree 1 and $N_G(L)$ respectively.

But there exists a natural question here. What would happen if $k>1$? We are going to answer this question by concept of kth G^k of the graph G. Firstly, we have the following lemma.

Lemma 9. Let G be a simple graph of order n and G^k be the k^{th} power of the graph G. Then $D \subseteq V(G)$ is a maximum set of tk-packing vertices if and only if $D \subseteq V(G^k)$ is a maximum set of t-packing vertices.

Proof: Since every edge in G^k is equal to a path with length $l \leq k$ we have u and v, two vertices in $V(G^k)$ such that there is no path between them with length $l \leq t$ if and only if u and v are two vertices in $V(G)$ such that there is no path between them with length $l \leq tk$. This shows that $D \subseteq V(G^k)$ is a set of t-packing vertices if and only if $D \subseteq V(G)$ is a set of tk-packing vertices. Also, it is easy to see that $D \subseteq V(G^k)$ is maximum if and only if $D \subseteq V(G)$ is maximum. This completes the proof.

Theorem 10. Let $k \geq 2$ be a positive integer. If G is a simple graph and each component is order $n \geq 3$, with minimum degree $\delta=1$ and S is a maximum $2k$-packing set, then $\gamma_{k,s}(G) \leq n-2\,\beta_{2k}$, where $\beta_{2k}=|S|$, and this bound is sharp.

Proof: Let G^k be the kth power of the graph G. By Observation 1 we have $\gamma_{k,s}(G)=\gamma_s(G^k)$. Since $n \geq 3$ then $\delta(G^k) \geq 2$. Therefore, by Theorem 6 we have $\gamma_{k,s}(G)=\gamma_s(G^k) \leq n-\beta_2(G^k)$. Finally by Lemma 9 we have $\gamma_{k,s}(G) \leq n-2\,\beta_{2k}\,(G)$.

Now we show that the bound is sharp. The desired graph G will be the union t copies of star $K_{1,2}$. Then $\gamma_{k,s}(G)=t$, $n=3t$, and $\beta_{2k}=t$. Therefore $\gamma_{k,s}(G)=t=3t-2t=n-2\beta_{2k}$. This completes the proof.

Observation 11. Let G and H be two simple graphs. If $k \geq 2$ then

$$\gamma_{k,s}(G \vee H) = \begin{cases} 1 & if \quad |V(G)|+|V(H)| \quad is \quad odd \\ 2 & if \quad |V(G)|+|V(H)| \quad is \quad even. \end{cases}$$

Now we show that for any integer k we can find a simple graph G such that $\gamma_s\,(G)=k$.

Theorem 12. For any integer k, there exists a connected graph G with $\gamma_s\,(G)=k$.

Proof: We consider four cases.

Case 1. Let $k<0$. We consider the star $K_{1,2|k|+2}$ with vertices v_1, v_2, ..., $v_{2|k|+2}$ and central vertex v. We add vertices u_i $(1 \leq i \leq 2|k|+2)$ be adjacent to v_i and v_{i+1} in modulo $2|k|+2$. Then we add edges $v_iv_{i+1} (1 \leq i \leq 2|k|+2)$ in modulo $2|k|+2$. Finally, we add vertices w_i $(1 \leq i \leq |k|+1)$ adjacent to $v_{2\mathrm{i}-1}$ and $v_{2\mathrm{i}}$ (when $k=-3$, G is illustrated in Figure 1).

We define $f\colon V(G) \longrightarrow \{1,-1\}$ by,

$$f(u)=\begin{cases} 1 & if \quad u \in \{v_1, v_2, \ldots, v_{2|k|+2}\} \\ -1 & if \in \{u_1, u_2, \ldots, u_{2|k|+2}\} \cup \{w_1, w_2, \ldots, w_{|k|+1}\}. \end{cases}$$

In the following, we prove that f is a signed dominating function of G. By symmetry it is sufficient to show that $f(N[u]) \geq 1$ for $u \in \{v, v_1, u_1, w_1\}$. $f(N[v])=f(v)+\sum_{i=1}^{2|k|+2} f(v_i)=2|k|+3 \geq 1$.

$f(N[v_1])=f(v)+f(v_1)+f(v_2)+f(v_{2|k|+2})+f(u_1)+f(u_{2|k|+2})+f(w_1)=1 \geq 1$.

$f(N[u_1])=f(u_1)+f(v_1)+f(v_2)=1 \geq 1$. $f(N[w_1])=f(v_1)+f(v_2)+f(w_1)=1 \geq 1$.

Therefore f is a signed dominating function of G with weight

$f(V(G))=1+2|k|+2-|k|-1-2|k|-2=-|k|=k$. Hence $\gamma_s(G) \geq k$.

On the other hand, let g be a minimum signed dominating function on G such that $\gamma_s(G)=g(V(G))$, we have, $g\big(V(G)\big)=\sum_{u \in V(G)} g(u)=\sum_{i=1}^{2|k|+2} g(N[u_i]) + \sum_{i=1}^{|k|+1} g(w_i)+g(v) \geq \sum_{i=1}^{2|k|+2}(1) + \sum_{i=1}^{|k|+1}(-1) -1= |k| \geq k$. Therefore $\gamma_s\,(G)=k$.

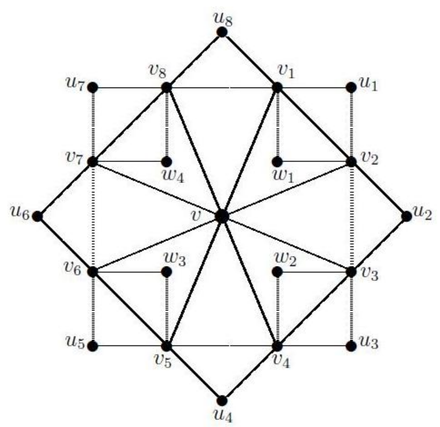

Fig. 1. example of Theorem 12 for $k=-3$

Case 2. If $k = 0$. We consider the Hajos graph G_H (Fig. 2).
We define $f: V(G) \longrightarrow \{1, -1\}$ by,

$$f(u) = \begin{cases} 1 & if \quad v \in \{u_1, u_2, u_3\} \\ -1 & if \quad v \in \{v_1, v_2, v_3\}. \end{cases}$$

It is easy to see that f is a signed dominating function of G_H, with weight 0. Therefore $\gamma_s(G_H) \leq 0$. On the other hand, let g be a minimum signed dominating function of G_H such that $\gamma_s(G_H) = g(V(G_H))$. We have, γ_s (G_H)= $g(V(G_H)) = \sum_{u \in V(G_H)} g(u) = g(N[u_1]) + g(v_3) \geq 1 - 1 = 0$. Therefore, γ_s (G_H)≥ 0. Hence γ_s $(G_H) = 0$.

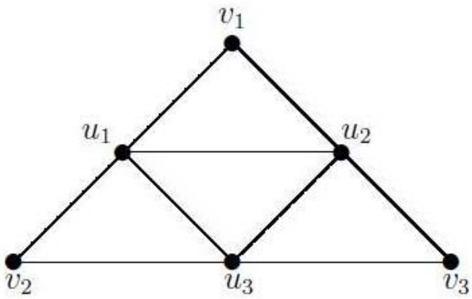

Fig. 2. Hajous graph

Case 3. If $k = 1$. Obviously for the complete graph K_{2n+1} we have γ_s (K_{2n+1})=1.

Case 4. If $k \geq 2$. We consider the star $K_{1,k-1}$. It is easy to see that γ_s $(K_{1,k-1}) = k$. This completes the proof.

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