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On the convergence of the VHPM for the Zakharove-Kuznetsov equations

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Abstract

In this paper, the variational homotopy perturbation method (VHPM) and its convergence is adopted for the Zakharove-Kuznetsov equations (ZK-equations). The aim of this paper is to present an efficient and reliable treatment of the VHPM for the nonlinear partial differential equations and show that this method is convergent. The convergence of the applied method is approved using the method of majorants from Cauchy-Kowalevskaya theorem of differential equations with analytical vector field.

***Keywords**:* Variational homotopy perturbation method; convergence; Zakharove-Kuznetsov equation

1. Introduction

Traveling waves are very important because various phenomena in nature such as vibration and self-reinforcing solitary waves are described by them. So, investigation of traveling wave solution plays an important role in nonlinear science. Rosenau and Hyman [1] introduced a class of partial differential equations (PDEs)

$$K(m,n):\ u_t + a(u^m)_x + (u^n)_{xxx} = 0,\ m > 0, 1 < n \le 3, \quad (1)$$

which is a generalization of the Korteweg-de Vries (KdV) equation such that the theory of water waves in shallow channels is described. For more information refer to [2]. In Eq.(1) $u_t = \frac{\partial u}{\partial t}, u_x^m = \frac{\partial u^m}{\partial x}$ and $u_{xxx}^n = \frac{\partial^3 u^n}{\partial x^3}$. For $m = n$ these are Solitary waves or so-called Compactons. Recently, Wazwaz [3] has given the new solitary patterns for the nonlinear dispersive *K(m,n)* equations:

$$u_t - a(u^m)_x + (u^n)_{xxx} = 0,\ m > 0, n > 0. \quad (2)$$

The new solitary wave special solutions with compact support for the nonlinear dispersive *K(m,n)* equations:

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0,\ m > 0, n > 0, \quad (3)$$

are presented by Ismail and Taha [4] and Wazwaz [5]. They use a finite difference method and a finite element method to investigate the approximate solutions of K(2,2) and K(3,3) in Eq.(1). The ZK-equation (shortly called ZK(m,n,k)) of the form

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)_{yyx} = 0,\ mnk \ne 0, \quad (4)$$

where m, n, k are integers and a, b, c are arbitrary constants. Eq.(4) governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [6]. Eq.(4) is solved by a different method. For instance in [7] ZK-equation was solved by the sine-cosine and the hyperbolic tangent (tanh)-function methods. In this paper, the variational Homotopy Perturbation method using He's polynomials is applied to solve ZK-equation and convergence of the considered technique is approved.

2. Methodology

To introduce the VHPM, it is necessary to know VIM and HPM.

2.1. Variational Iteration and Homotopy Perturbation Method

To illustrate the basic concepts of the VIM and HPM, first consider the following nonlinear differential equation

$$Lu + Nu = g(x), \quad (5)$$

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where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. According to the VIM [8,9,10] we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(\tau)\{Lu_n + N\tilde{u}_n - g(\tau)\}d\tau, \quad (6)$$

where $\lambda(\tau)$ is a general Lagrange multiplier. $\lambda(\tau)$ can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation and $\tilde{u}_n$ is considered as a restricted variation i.e. $\delta\tilde{u}_n = 0$. The essential idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the system of equations is in sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of ``deformation", the solution of each stage is ``close" to that at the previous stage of ``deformation". Eventually at $p = 1$, the system takes the original form of equation and the final stage of ``deformation" gives the desired solution. To illustrate the basic concept of homotopy perturbation method, consider the following nonlinear system of differential equations

$$A(U) = f(r), \qquad r \in \Omega \quad (7)$$

with boundary conditions

$$B\left(U, \frac{\partial U}{\partial n}\right) = 0, \quad r \in \Gamma$$

where A is a differential operator, B is a boundary operator, $f(r)$ is a known analytical function, and Γ is the boundary of the domain Ω. Generally speaking the operator A can be divided into two parts L and N, where L is a linear, and N is a nonlinear operator. Therefore, Eq.(7) can be rewritten as follows:

$$L(U) + N(U) - f(r) = 0.$$

We construct a homotopy $V(r, p)$: $\Omega \times [0,1] \to R^n$, which satisfies

$$H(V, p) = (1 - p)[L(V) - L(U_0)] + p[A(V) - f(r)] = 0, \quad p \in [0,1], r \in \Omega,$$

or equivalently,

$$H(V, p) = L(V) - L(U_0) + pL(U_0) + p[N(V) - f(r)] = 0. \quad (8)$$

where U_0 is an initial approximation of Eq.(7). In this method, using the homotopy parameter p, we have the following power series presentation for V,

$$V = V_0 + pV_1 + p^2V_2 + \ldots.$$

The approximate solution can be obtained by setting $p = 1$, i.e.

$$U = U_0 + U_1 + U_2 + \ldots.$$

2.2. Variational Homotopy Perturbation Method using He's polynomials

To illustrate the basic idea of the VHPM, consider the following general differential equation

$$Lu + Nu = g(x) \quad (9)$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. According to the VIM, as illustrated previously we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(\tau)(Lu_n + N\tilde{u}_n - g(\tau))d\tau, \quad (10)$$

where $\lambda(\tau)$ is a general Lagrange multiplier. Applying the homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n = u_0(x) + p\int_0^t \lambda(\tau)\left\{N\left(\sum_{n=0}^{\infty} p^n \tilde{u}_n\right)\right\}d\tau - \int_0^t \lambda(\tau)g(\tau)\,d\tau, \quad (11)$$

which is the coupling of VIM and He's polynomials and is called the modified variational iteration method (MVIM). The comparison of similar powers of p gives solutions of various orders. So, u_n can be obtained as

$$u_0 = f(x, y) - \int_0^t \lambda(\tau)g(\tau)\,d\tau, \quad (12)$$

and

$$u_{n+1} = \int_0^t \lambda(\tau)H_n d\tau,$$

where

$$H_n = \frac{1}{n!}\frac{d^n}{dp^n} N\Big(\sum_{k=0}^{n} p^k u_k\Big)\Big|_{p=0} .$$

For later numerical computation, we let the expression $\phi_n = \sum_{i=0}^{n} u_i(x,y,t)$ to denote the n-term approximation to *u(x,y,t)*. For more information about the VHPM refer to [11,12].

3. The VIM and VHPM for ZK-equation

3.1. The VIM for ZK-equation

In order to solve ZK-equation

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)yyx = 0,$$

by the VIM, correction functional can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{u_{n\tau}(x,y,\tau) + a\tilde{u}^m_{n_x}(x,y,\tau) + b\tilde{u}^n_{n_{xxx}}(x,y,\tau) + c\tilde{u}^k_{n_{yyx}}(x,y,\tau)d\tau \tag{13}$$

where $\tilde{u}_n$ is considered as restricted variations, i.e. $\delta\tilde{u}_n = 0$.

To find the optimal value of $\lambda(\tau)$, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\tau)\{u_{n\tau}(x,y,\tau) + a\tilde{u}^m_{n_x}(x,y,\tau) + b\tilde{u}^n_{n_{xxx}}(x,y,\tau) + c\tilde{u}^k_{n_{yyx}}(x,y,\tau)d\tau, \tag{14}$$

or

$$\delta u_{n+1}(x,y,t) = \delta u_n(x,y,t) + \delta\int_0^t \lambda(\tau) u_{n\tau} d\tau, \tag{15}$$

which results in

$$\delta u_{n+1}(x,y,t) = \delta u_n(x,y,t) + \delta\lambda(\tau) u_n(x,y,\tau)|_{\tau=t} - \int_0^t \lambda'(\tau) u_n(x,y,\tau) d\tau = 0. \tag{16}$$

Therefore, the stationary conditions are obtained in the following form

$$\begin{cases} 1+\lambda(\tau) = 0\,|_{\tau=t}\,, \\ \lambda'(\tau) = 0\,|_{\tau=t}\,, \end{cases} \tag{17}$$

which results in $\lambda(\tau) = -1$. Substituting this value of the Lagrange multiplier into Eq.(13) gives

$$u_{n+1}(x,y,t) = u_0(x,y,t) - \int_0^t \{u_{n\tau}(x,y,\tau) + au^m{}_{n_x}(x,y,\tau) + bu^n{}_{n_{xxx}}(x,y,\tau) + cu^k{}_{n_{yyx}}(x,y,\tau)d\tau. \tag{18}$$

The iteration formula will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations.

3.2. The VHPM for ZK-equation

Using the value of Lagrange multiplier that was calculated in the previous section and applying the VHPM gives:

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \ldots &= f(x,y) \\ &- ap\int_0^t ((u_0 + pu_1 + p^2u_2 + \ldots)^m)_x d\tau \\ &- bp\int_0^t ((u_0 + pu_1 + p^2u_2 + \ldots)^n)_{xxx} d\tau \\ &- cp\int_0^t ((u_0 + pu_1 + p^2u_2 + \ldots)^k)_{yyx} d\tau. \end{aligned} \tag{19}$$

The comparison of similar powers of p gives solutions of various orders, and the component which constitutes $u(x,y,t)$ is written as $u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t).$

4. *Convergence of VHPM for ZK equations*

Consider the initial value problem

$$\dot{u} = L'u + Nu\,,\ n > 0 \tag{20}$$

with initial condition

$$u(0) = f(x,y),$$

where dot denotes differential in time t.

4.1. Assumption

Let $L' : X \to Y$ form a continuous semigroup $E(t) = \exp(tL')$ for $t \in \mathrm{R}_+$, $N(u) : X \to X$ be analytic near $u = f$ and X be a Banach algebra with the property

$$\|fg\|_X \le \|f\|_X \|g\|_X \qquad f, g \in X. \tag{21}$$

Note that for ZK- equation $L' = 0$, so $E(t) = I$.

By Duhamel's principle, problem (20) can be reformulated as an integral equation

$$u(t) = E(t)f + \int_0^t E(t-\tau)N(u(\tau))d\tau, \quad n > 1. \quad (22)$$

If *N(u)* is analytic near *f*, it satisfies a local Lipschitz condition in the ball $B_\delta(f)$ of a radius $\delta > 0$ centered at *f*, i.e., there is a constant $K_\delta > 0$ such that

$$\|N(u) - N(\tilde{u})\|_x \le k_\delta \|u - \tilde{u}\|_x, \quad \forall \|u - f\|_x \le \delta, \ \forall \|\tilde{u} - f\|_x \le \delta. \quad (23)$$

In the following, the convergence of the VHPM is approved for Eq.(20).

4.2. Theorem (Picard-Kato)

Let L and $N(u)$ satisfy assumption 4.1 and $f \in X$. There exists a $T > 0$ and a unique solution $u(t)$ of the initial-value problem Eq.(20) on *[0,T]*, such that

$$u(t) \in C([0,T], X) \cap C^1([0,T], X) \quad (24)$$

and $u(0) = f$. Moreover, the solution $u(t)$ depends continuously on the initial data f. See [13] for the proof of Picard-Kato theorem. Using this theorem local well-posedness of solutions of the initial-value problem (20) with Lipschitz vector field *N(u)* can be proved for small time intervals.

4.3. Theorem (Cauchy-Kowalevskaya)

Let assumption 4.1 be satisfied with $X = Y$ and $u(t)$ be a unique solution of Eq.(20) in $C^1([0,T], X)$, where $T > 0$ is the maximal existence time. Then there exist $\tau \in (0,T)$ such that $u : [0,\tau] \to X$ is also a real analytic function. For further details and the proof of Cauchy-Kowalevskaya theorem see [13].

By Cauchy estimates there exist constants $a, b > 0$ such that:

$$\frac{1}{k!}\left\|\partial_u^k N(u)\right\| \le \frac{b}{a^k} \quad k \ge 0 \quad (25)$$

where $\partial_u N(u)$ denotes operator in the sense of Frechet derivative, e.g., $\partial_u N(u) = N'(u)$ is the Jacobian operator. The Taylor series of $N(u)$ at f converges for any $\|u - f\|_x < a$, and moreover, we obtain

$$\|N(u)\|_x \le \sum_{k=0}^{\infty} \frac{1}{k!}\left\|\partial_u^k N(f)\right\|_x \left\|(u-f)^k\right\|_x \le b \sum_{k=0}^{\infty} \frac{\|u-f\|_x^k}{a^k} = \frac{ab}{a-\rho} := g(\rho), \quad (26)$$

where $\rho = \|u - f\|_x < a$. From the majorant function $g(\rho)$, it is clear that

$$\frac{1}{k!}\left\|\partial_u^k N(f)\right\|_x \le \frac{1}{k!}\partial_\rho^k g(0) \quad k \ge 0. \quad (27)$$

Now consider the majorant problem

$$\begin{cases} \dfrac{d\rho}{dt} = g(\rho) & t > 0, \\ \rho(0) = 0, \end{cases} \quad (28)$$

where $\rho \in \mathrm{R}_+$. The majorant problem has an explicit solution

$$\rho(t) = a - \sqrt{a^2 - 2abt},$$

which is an analytical function of t on $(-\infty, \frac{a}{2b})$. Using these relations the convergence of the VHPM can be approved for ZK- equations.

4.4. Convergence theorem

Let the assumption 4.1 be satisfied and $u(t)$ be a unique solution of Eq.(20) in $C([0,T), X)$, where $T > 0$ is the maximal existence time. As mentioned before, $u_n(t)$ is defined as

$$u_{n+1} = \int_0^t \lambda(\tau) H_n d\tau.$$

For ZK-equations $\lambda(\tau) = -1$. So,

$$u_{n+1} = \int_0^t -H_n d\tau. \quad (29)$$

There exist a $\tau \in (0,T)$ such that the nth partial sum $S_n = \sum_{k=0}^n u_k$, when $n \to \infty$ converges to the solution $u(t)$ in $C([0,\tau], X)$.

4.5. Remark

Existence and uniqueness of the solution $u(t)$ in $C([0,T),X)$ is approved in Picard-Kato Theorem.

4.6. proof

From Picard-Kato theorem, for any $\delta > 0$ there exist a $t_0 \in (0,T)$ such that

$$sup_{t\in[0,t_0]}\|u_0 - f\|_x \le \frac{\delta}{2}. \tag{30}$$

Choosing $\delta < 2a$, where a is the radius of analyticity of $N(u)$ near f. The Cauchy estimates, Eq.(25) can be generalized as

$$\frac{1}{k!}\|\partial^k N(u_0)\|_x \le \sum_{m\ge k}\frac{m(m-1)\ldots(m-k+1)\|\partial^m N(f)\|_x\|u_0 - f\|_x^{m-k}}{m!k!}$$
$$\le b\sum_{m\ge k}\frac{m(m-1)\ldots(m-k+1)\|u_0 - f\|_x^{m-k}}{m!k!a^m}$$
$$= \frac{\partial_\rho^k g(\rho)}{k!}, \tag{31}$$

where $\rho(t)$ satisfy the majorant problem Eq.(28) for $t\in[0,\frac{a}{2b})$. Then, $\rho(t)$ and all its derivatives with respect to t are increasing functions of t and so are $g(\rho(t))$ and all its derivatives with respect to ρ. By using semigroup property and Eq.(29)

$$\|u_1\|_x \le \int_0^t\|H_0\|_x d\tau \le \int_0^t\|N(u_0(x,y)\|_x d\tau$$
$$\le \int_0^t g(\rho(\tau))d\tau \le tg(\rho(t))$$
$$= t\rho'(t),$$
$$\|u_2\|_x \le \int_0^t\|H_1\|_x d\tau \le \int_0^t\|N'(u_0(x,y))u_1(x,y,\tau)\|_x d\tau$$
$$\le \int_0^t g'((\rho(\tau)))g(\rho(\tau))d\tau$$
$$\le \frac{t^2 g'(\rho(t))g(\rho(t))}{2!}$$
$$= \frac{t^2\rho''(t)}{2!}. \tag{32}$$

By induction, we assume that

$$\|u_k\| \le \frac{t^k\partial_t^k\rho(t)}{k!}$$

and prove that the same relation remains true at $k = n+1$:

$$\|u_{n+1}\| \le \frac{t^{n+1}\partial_t^{n+1}\rho(t)}{(n+1)!} \qquad t\in[0,\frac{a}{2b}).$$

As $\rho(t)$ is analytic in t for all $t\in[0,\frac{a}{2b})$, for any small $p > 0$ there exists a $C^\infty -$ function $\tilde{\rho}^p(t)$ on $[0,\frac{a}{2b})$ such that

$$\rho((1+p)t) = \sum_{k=0}^n \frac{p^k t^k\partial_t^k\rho(t)}{k!} + \frac{p^{n+1}t^{n+1}\tilde{\rho}^p(t)}{(n+1)!},$$

if $u_n^p = \sum_{k=0}^n p^k u_k$ then

$$\|u_n^p\| \le \sum_{k=0}^n p^k\|u_k\|_x \le \sum_{k=0}^n \frac{p^k t^k\partial_t^k\rho(t)}{k!}$$
$$= \rho((1+p)t) - \frac{p^{n+1}t^{n+1}\tilde{\rho}^p(t)}{(n+1)!}.$$

By definition of H_n, we have

$$H_n = \frac{1}{n!}\frac{d^n}{dp^n}N(\sum_{k=0}^\infty p^k u_k)|_{p=0}$$
$$= \frac{1}{n!}\frac{d^n}{dp^n}N(u_n^p)|_{p=0} \tag{33}$$

so that

$$\|H_n\|_x \le \frac{1}{n!}\left\|\frac{d^n}{dp^n}N(u_n^p)\right\|_x|_{p=0} \le \frac{\frac{d^n}{dp^n}g(\rho((1+p)t))}{n!}|_{p=0}$$
$$\le \frac{t^n\frac{d^n}{d\mu^n}g(\rho(\mu))}{n!}|_{\mu=t} = \frac{t^n p_n(g(\rho(t))}{n!}$$
$$= \frac{t^n\partial^{n+1}\rho(t)}{n!}, \tag{34}$$

where p_n is a polynomial of g and its derivatives up to the $n-th$ order with positive coefficients (the same as in the proof of Cauchy-Kowalevskaya theorem). Using the iterative formula, Eq. (29), we obtain

$$\|u_{n+1}\|_x \le \int_0^t \|H_n\|_x d\tau \le \frac{t^{n+1}\partial^{n+1}\rho(t)}{(n+1)!}.$$

Therefore, $u(t)=\sum_{n=0}^{\infty}u_n$ is majorant in X by the power series

$$\rho(2t)=\sum_{k=0}^{\infty}\frac{t^k\partial_t^k\rho(t)}{k!}=a-\sqrt{a^2-4abt}$$

which converges for all $|t|<\frac{a}{4b}$. Recall the constraint $t_0\in[0,T]$ in bound Eq.(30). By the Weierstrass M-test $u(t)=\sum_{n=0}^{\infty}u_n$ convergence in $C([0,\tau],X)$ for any $\tau\in(0,\tau_0)$, where $\tau_0=min\{t_0,\frac{a}{4b}\}$, to the unique solution $u(t)$ of our equation.

4.7. Corollary

There exists a constant $C_. >\cdot$ such that the error of the VHPM is bounded by

$$E_n = sup\|u-S_n\|_x \le C_0(\frac{2b\tau}{a})^{n+1} \qquad n\ge 1$$

where (a,b,τ) are defined in convergence theorem.

4.8. proof

From convergence theorem we have

$$sup\|u_n\|_x \le \frac{\tau^n\partial_\tau^n\rho(\tau)}{n!}.$$

It follows from the explicit form for $\rho(t)$ that

$$\rho^n(\tau)=\frac{(2n-3)!!b^n a^n}{(a^2-2ab\tau)^{n-1/2}}.$$

As a result, we obtain

$$E_n \le \sum_{k=n+1}^{\infty} sup\|u_k\|_x \le \sum_{k=n+1}^{\infty}\frac{\tau^k\rho^k(\tau)}{k!}$$
$$\le \sqrt{a^2-2ab\tau}\sum_{k=n+1}^{\infty}\frac{(2k-3)!!}{k!}(\frac{ab\tau}{a^2-2ab\tau})^k$$
$$\le \sqrt{a^2-2ab\tau}(\frac{2ab\tau}{a^2-2ab\tau})^{n+1}$$
$$\sum_{k=0}^{\infty}\frac{(2k+2n-1)!!}{2^{k+n+1}(k+n+1)!}(\frac{2ab\tau}{a^2-2ab\tau})^k, \qquad (35)$$

as $\frac{(2k+2n-1)!!}{2^{k+n+1}(k+n+1)!}\le\frac{1}{2n+2k}\le 1, \qquad \forall n,k\ge 1,$
we obtain

$$E_n \le \sqrt{a^2-2ab\tau}(\frac{2ab\tau}{a^2-2ab\tau})^{n+1}\sum_{k=0}^{\infty}(\frac{2ab\tau}{a^2-2ab\tau})^k.$$

If $(\frac{2ab\tau}{a^2-2ab\tau})<1$ then, $\tau\in(0,\frac{a}{4b})$ and we have

$$E_n \le \sqrt{a^2-2ab\tau}(\frac{2ab\tau}{a^2-2ab\tau})^{n+1}(\frac{1}{1-\frac{2ab\tau}{a^2-2ab\tau}})$$
$$\le C_0(\frac{2ab\tau}{a^2-2ab\tau})^{n+1}$$
$$\le C_0(\frac{2b\tau}{a})^{n+1} \qquad (36)$$

where $C_0=\frac{\sqrt{a^2-2ab\tau}}{1-\frac{2ab\tau}{a^2-2ab\tau}}$. When $n\to\infty, E_n\to 0$. So, with an appropriate choice for a and b ZK-equations will be convergent for $|t|<\frac{a}{4b}$.

Now we would like to choose two special equations, namely ZK(2,2,2) and ZK(3,3,3) with specific initial conditions.

First consider the ZK(2,2,2) equation:

$$u_t+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0, \qquad (37)$$

with specific initial conditions $u(x,y,0)=f(x,y)=-\frac{4}{3}\eta cosh^2(x+y)$,
where η is an arbitrary constant.

Assume $\eta=1$ and proceeding as before, using VIM the lagrange multiplier is determined as $\lambda=-1$. Based on the VHPM we have:

$$u_0+pu_1+p^2u_2+\ldots = f(x,y)$$

$$-p\int_0^t((u_0+pu_1+p^2u_2+\ldots)^2)_x d\tau$$
$$-1/8p\int_0^t((u_0+pu_1+p^2u_2+\ldots)^2)_{xxx} d\tau$$
$$-1/8p\int_0^t((u_0+pu_1+p^2u_2+\ldots)^2)_{yyx} d\tau. \quad (38)$$

Comparing the coefficient of similar powers of p or using Eq.(29), we have:

$$p^0: u_0(x,y,t) = -\frac{4}{3}cosh^2(x+y),$$
$$p^1: u_1(x,y,t) = -\int_0^t(u_0^2)_x d\tau$$
$$-\frac{1}{8}\int_0^t(u_0^2)_{xxx} d\tau - \frac{1}{8}\int_0^t(u_0^2)_{yyx} d\tau$$
$$= (-\frac{224}{9}cosh^3(x+y)sinh(x+y)$$
$$-\frac{32}{3}cosh(x+y)sinh^3(x+y))t$$
$$p^2: u_2(x,y,t) = -\int_0^t(2u_0u_1)_x d\tau$$
$$-\frac{1}{8}\int_0^t(2u_0u_1)_{xxx} d\tau - \frac{1}{8}\int_0^t(2u_0u_1)_{yyx} d\tau$$
$$= (-\frac{5056}{27}cosh^6(x+y) - \frac{46784}{27}sinh^2(x+y)cosh^4(x+y)$$
$$-\frac{8128}{9}sinh^4(x+y)cosh^2(x+y) - \frac{64}{3}sinh^6(x+y))t^2$$
$$p^3: u_3(x,y,t) = -\int_0^t(2u_0u_2+u_1^2)_x d\tau$$
$$-\frac{1}{8}\int_0^t(2u_0u_2+u_1^2)_{xxx} d\tau$$
$$-\frac{1}{8}\int_0^t(2u_0u_2+u_1^2)_{yyx} d\tau$$
$$= (-\frac{12869632}{243}sinh(x+y)cosh^7(x+y)$$
$$-\frac{6025216}{27}sinh^3(x+y)cosh^5(x+y)$$
$$-\frac{9551872}{81}sinh^5(x+y)cosh^3(x+y)$$
$$-\frac{192512}{21}sinh^7(x+y)cosh(x+y))t^3$$

and $u(x,y,t) = \sum_{i=0}^{\infty}u_i(x,y,t)$. By substituting ϕ_4 in Eq.(37) and studying the result of that in Fig. 1, it is clear that our approximation satisfies the equation with a high level of accuracy. So, $\sum_{n=0}^{\infty}u_n$ will be convergent to the exact solution. To Continue, consider the ZK(3,3,3) equation

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0, \quad (39)$$

with specific initial conditions $u(x,y,0) = f(x,y) = 1.5\eta sinh((x+y)/6)$, as mentioned in the previous example we assume $\eta = 1$. By using VIM the lagrange multiplier is determined as $\lambda = -1$. Applying the VHPM gives:

$$p^0: u_0(x,y,t) = 1.5sinh((x+y)/6),$$
$$p^1: u_1(x,y,t) = -\int_0^t(u_0^3)_x d\tau$$
$$-2\int_0^t(u_0^3)_{xxx} d\tau - 2\int_0^t(u_0^3)_{yyx} d\tau$$
$$= (-3sinh^2((x+y)/6)cosh((x+y)/6)$$
$$-\frac{3}{8}cosh^3((x+y)/6))t$$
$$p^2: u_2(x,y,t) = -\int_0^t(3u_0^2u_1)_x d\tau$$
$$-2\int_0^t(3u_0^2u_1)_{xxx} d\tau - 2\int_0^t(3u_0^2u_1)_{yyx} d\tau$$
$$= (\frac{273}{64}sinh^5((x+y)/6) + \frac{1641}{64}sinh^3((x+y)/6)cosh^2((x+y)/6)$$
$$+\frac{381}{64}sinh((x+y)/6)cosh^4((x+y)/6))t^2$$
$$p^3: u_3(x,y,t) = -\int_0^t(3u_0u_1^2+3u_0^2u_2)_x d\tau$$
$$-2\int_0^t(3u_0u_1^2+3u_0^2u_2)_{xxx} d\tau$$
$$-2\int_0^t(3u_0u_1^2+3u_0^2u_2)_{yyx} d\tau$$
$$= (-\frac{39851}{256}sinh^6((x+y)/6)cosh((x+y)/6)$$
$$-\frac{114915}{256}sinh^4((x+y)/6)cosh^3((x+y)/6)$$
$$-\frac{16455}{128}sinh^2((x+y)/6)cosh^5((x+y)/6)$$
$$-\frac{505}{256}cosh^7((x+y)/6))t^3 \quad (40)$$

and proceeding as before $u(x,y,t) = \sum_{i=0}^{\infty}u_i(x,y,t)$. Substituting ϕ_4 in Eq.(39) and studying Fig. 2 shows that the VHPM gives the solution of Eq. (39) which has an excellent agreement with the exact one. So, our applied method is convergent for this example too.

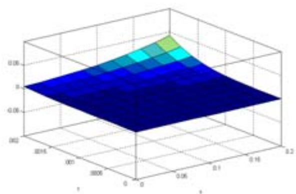

Fig. 1. The result of substituting ϕ_4 in Eq.(37) at $y = .1$

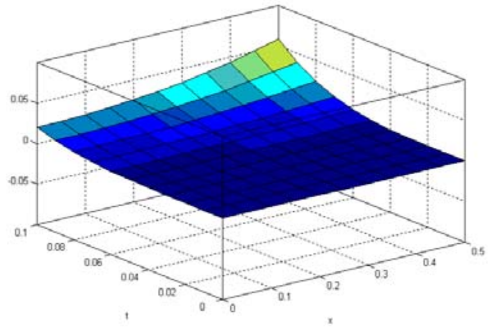

Fig. 2. The result of substituting ϕ_4 in Eq.(39) at $y = .1$

5. Conclusion

In this work, the variational homotopy perturbation method using He's polynomials is used to solve Zakharov-Kuznetsove equations. As shown, this method is an effective and straightforward technique. One important object of our research is the examination of the convergence of the variational homotopy perturbation method using He's polynomials. Convergence theorems are given in general for partial differential equations and the result is examined on Zakharov-Kuznetsove equations as special cases. The results show that the VHPM is a convergent method and can be used to solve other linear and nonlinear equations.

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