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Some new double sequence spaces in 2-normed spaces defined by two valued measure

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Abstract

In this paper, following the methods of Connor, we introduce some new generalized double difference sequence spaces using summability with respect to a two valued measure, double infinite matrix and an Orlicz function in 2-normed spaces which have unique non-linear structure and examine some of their properties.

Keywords: Convergence; μ-statistical convergence; convergence in μ-density; Orlicz function; 2-normed space; paranormed space; double sequence space

1. Introduction

The usual notion of convergence does not always capture in finite details the properties of the vast class of sequences that are not convergent. One way of including more sequences under preview is to consider those sequences that are convergent when restricted to some big set of natural numbers. By a big set one understands a set $K \subset N$ having asymptotic density equal to 1. Investigation in this line was initiated by Fast [1] and independently by Schoenberg [2] who introduced the idea of statistical convergence. Over the years and under different names statistical idea of statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence spaces point of view and linked with summability theory by Connor [3], Fridy [4], Salat [5], Tripaty [6] and many others. The notion of statistical convergence was introduced for double sequences by Mursaleen and Edely [7].

Recently Connor [8] introduced definitions of μ-statistical and μ-density convergence where μ is a [0, 1]-valued finitely additive measure defined on a field of subsets of *N*. It was shown that these two summability methods are equivalent iff the measure which generated them was "nearly" countably additive, which occurs iff the ideal associated with the measure contains no proper dense subideal containing c_0. This is a very interesting generalization of statistical convergence. In particular, Das and Bhunia recently investigated the summability of double sequences of real numbers with respect to a two valued measure and made many interesting observations (see, [9]). Quite recently Das, Savas and Bhunia [10] introduced some new generalized double difference sequence spaces using summability with respect to a two valued measure and an Orlicz function in 2-normed spaces.

The concept of 2-norm spaces was initially introduced by Gahler [11, 12] as a very interesting non-linear extension of the idea of usual normed linear spaces. Recently a lot of interesting developments have occurred in 2-normed spaces in summability theory and related topics [13-16].

The sudy of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [17] become interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having compelementary subspaces isomorphic to c_0 or $\ell_p (1 \le p < \infty)$. Subsequently Lindenstrauss and Tzafriri [18] investigated Orlicz sequence spaces in more detail and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (1 \le p < \infty)$. Recently, Parashar and Choudhary [19] have introduced and discussed some properties of the four sequence spaces defined by using Orlicz function *M*, which generalized the well known Orlicz sequence space ℓ_M and strongly summable sequence spaces. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [20]. Orlicz spaces find a number

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of useful applications in the theory of nonlinear equations.

In this article some new sequence spaces are introduced by combining a four dimension matrix transformation *A*, Orlicz functions, generalized double difference sequences and a two valued measure μ in 2-normed spaces.

2. Preliminaries

By the convergence of a double sequence we mean the convergence in Pringsheim's sense (see, [21]):
A double sequence $x = (x_{kl})$ of real numbers is said to be convergent to $\xi \in \mathrm{R}$ if for any $\varepsilon > 0$, there exists $N_\varepsilon \in N$ such that $|x_{kl} - \xi| < \varepsilon$ whenever $k, l \geq N_\varepsilon$. In this case we write $\lim_{k,l\to\infty} x_{kl} = \xi$.

A double sequence $x = (x_{kl})$ of real numbers is said to be bounded if there exists a positive real number *M* such that $|x_{kl}| < M$ for all $k, l \in N$. That is $\|x\|_{(\infty,2)} = \sup_{k,l\in N} |x_{kl}| < \infty$.

Definition 2.1. ([12]). Let *X* be a real vector space of dimension *d*, where $2 \leq d < \infty$. A 2-norm in on *X* is a function $\|.,.\| : X \times X \to \mathrm{R}$ which satisfies (i) $\|x, y\| = 0$ if and only if *x* and *y* are linearly independent; (ii) $\|x, y\| = \|y, x\|$, (iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathrm{R}$; (iv) $\|x, y + z\| = \|x, y\| + \|x, z\|$. The ordered pair $(X, \|.,.\|)$ is then called a 2-normed space.

3. Results

As an example we may take $X = R^2$ being equipped with the 2-norm $\|x, y\|$=the area of the parallelogram spanned by the vectors *x* and *y*, which may be given explicitly by the formula $\|x, y\| = |x_1 y_2 - x_2 y_1|$, $x = (x_1, x_2), y = (y_1, y_2)$. Recall that $(X, \|.,.\|)$ is a 2-Banach space if every Cauchy sequence in *X* is convergent to some x in *X*.

Recall in [20] that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, convex and non decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Natural density was generalized by Freeman and Sember in [22] by replacing C_1 with a nonnegative regular summability matrix $A = (a_{n,k})$. Thus, if *K* is a subset of *N* then the A-density of *K* is given by $\delta_A(K) = \lim_n \sum_{k\in K} a_{n,k}$ if the limit exists. Let $K \subset N \times N$ be a two-dimensional set to positive integers and let $K(m,n)$ be the said numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. The two-dimensional analogues of natural density can be defined as follows: The lower asymptotic density of a set $K \subset N \times N$ is defined as

$$\delta^2(K) = \liminf_{m,n} \frac{K(m,n)}{mn}.$$

In case the double sequence $\frac{K(m,n)}{mn}$ has a limit in the Prinhsheim sense then we say that *K* has a double natural density as

$$P - \lim_{m,n} \frac{K(m,n)}{mn} = \delta^2(K).$$

Let $K \subset N \times N$ be a two-dimensional set of positive integers, then the A-density of *K* is given by

$$\delta^2{}_A(K) = P - \lim_{m,n} \sum_{(k,l)\in K} a_{m,n,k,l}$$

provided that the limit exists. Then the notion of double asymptotic density for double sequence was presented by Mursaleen and Edely in [7] as follows.

Definition 3.1. ([7]). A double sequence $x = (x_k)$ of real numbers is said to be statistically convergent to $\xi \in \mathrm{R}$ if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(k,l) \in N \times N : |x_{kl} - \xi| \geq \varepsilon\}$.

Throughout the paper μ will denote a complete {0,1} valued finite additive measure defined on an algebra Γ of subsets of $N \times N$ that contains all subsets of $N \times N$ that are contained in the union of a finite number of rows and columns of $N \times N$ and $\mu(A) = 0$ if *A* is contained in the union of a finite number of rows and columns of $N \times N$, (see, [9]).

Definition 3.2. [9] A double sequence $x = (x_{kl})$ in a 2-normed space $(X, \|.,.\|)$ is said to be convergent to ξ in $(X,\|.,.\|)$ if for each $\varepsilon > 0$ and each $z \in X$ there exists $n_\varepsilon \in N$ such that $\|x_{kl} - \xi, z\| < \varepsilon$ for all $k,l \geq n_\varepsilon$.

Definition 3.3. [9]. Let μ be a two valued measure on $N \times N$. A double sequence $x = (x_{kl})$ in a 2-normed space $(X,\|.,.\|)$ is said to be μ-statistically convergent to some point x in X if for each pre-assigned $\varepsilon > 0$ and for each $z \in X$,

$\mu(A(z,\varepsilon)) = 0$
where $A(z,\varepsilon) = \{(k,l) \in N \times N : \|x_{kl} - x, z\| \geq \varepsilon\}$.

If a double sequence $x = (x_{kl})$ is μ-statistically convergent to a point x in a 2-normed space $(X,\|.,.\|)$ then we write

$$\mu - \lim_{k,l\to\infty} \|x_{kl} - x, z\| = 0$$

or

$$\mu - \lim_{k,l\to\infty} \|x_{kl}, z\| = \|x, z\|.$$

Here, x is called μ-statistical limit of the sequence (x_{kl}).

Definition 3.4. [10]. Let μ be a two valued measure on $N \times N$. A double sequence $x = (x_{kl})$ of the points in a 2-normed space $(X,\|.,.\|)$ is said to be $\xi \in X$ in μ-density if there exists a set $M \in \Gamma$ with $\mu(M) = 1$ such that (x_{kl}) is convergent to ξ in $(X,\|.,.\|)$.

If $u = \{u_1, u_2, u_3, ..., u_d\}$ is the basis of the 2-normed space $(X,\|.,.\|)$, then we can have the following result.

Lemma 3.1. [10]. μ is a two valued measure. A double sequence (x_{kl}) is μ-statistically convergent to $x \in X$ if and only if $\mu - \lim_{k,l\to\infty} \|x_{kl} - x, u_i\| = 0$ for every $i = 1,2,3....,d$.

Definition 3.5. Let μ be a two valued measure on $N \times N$. A double sequence $x = (x_{kl})$ in a 2-normed space $(X,\|.,.\|)$ is said to be $\mu_A(\Delta)$-statistically convergent to L if for every positive ε
$\mu_A\left(\{(k,l) \in N \times N : \|\Delta^m x_{kl} - x, z\| \geq \varepsilon\}\right) = 0.$

In this case we write $x_{k,l} \to L(\mu_A(\Delta))$ *or* $\mu_A(\Delta) - \lim x = L$, *and*
$\mu_A(\Delta) = \{x : \exists\, L \in \mathbb{R}, \mu_A - \lim x = L\}$.

4. New double sequence spaces

Recall that a mapping $g : X \to \mathbb{R}$ is called a paranorm on X if it satisfies the following conditions: (i) $g(\theta) = 0$ where θ is the zero element of the space; (ii) $g(x)=g(-x)$; (iii) $g(x+y) \leq g(x) + g(y)$; (iv) $\lambda^n \to \lambda$ $(n \to \infty)$ and $g(x^n - x) \to 0$ $(n \to \infty)$ imply $g(\lambda^n x^n - \lambda x) \to 0$ $(n \to \infty)$ for all $x, y \in X$ (see, [23]). The ordered a pair (X, g) is called a paranormed space with respect to the paranorm g.

We now state an inequality which will be used throughout in our study: If (p_{kl}) be a bounded double sequence of non-negative real numbers and $\sup_{k,l\in N} p_{kl} = H$ and $D = max\{1, 2^{H-1}\}$, then

$$|a_{kl} + b_{kl}|^{p_{kl}} \leq D\left\{ |a_{kl}|^{p_{kl}} + |b_{kl}|^{p_{kl}} \right\}$$

for all k,l and $a_{kl}, b_{kl} \in C$, the set of all complex numbers. Also,

$$|a|^{p_{kl}} \leq max\{1, |a|^H\}$$

for all $a \in C$.

Throughout this paper we shall examine our sequence spaces using the following transformation:

Definition 4.1. Let $A = (a_{m,n,k,l})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequences Ax where the mn-th term to Ax is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

Such transformation is said to be non-negative if $a_{m,n,k,l}$ is nonnegative for all m, n, k and l.

Let $(X, \|.,.\|)$ be any 2-normed space and $S''(2-X)$ denotes X-valued sequence spaces. Clearly $S''(2-X)$ is a linear space under addition and scalar multiplication.

Definition 4.2. Suppose that, as before, μ is a two valued measure on $N \times N$ and M be an Orlicz function and $(X, \|.,.\|)$ be a 2-normed space. Further, let $p = (p_{kl})$ be a bounded sequence of positive real numbers. Now we introduce the following different types of sequence spaces, for all $\varepsilon > 0$

$$W^{\mu}(A, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \mu\left((m,n) \in N \times N : \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} \left[M\left(\left\| \frac{\Delta^m x_{kl} - L}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \varepsilon \right) = 0, \text{ for some } \rho > 0 \text{ and } L \in X \text{ and each } z \in X \right\},$$

$$W_0^{\mu}(A, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \mu\left((m,n) \in N \times N : \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} \left[M\left(\left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \varepsilon \right) = 0, \text{ for some } \rho > 0 \text{ and each } z \in X \right\},$$

$$W_{\infty}^{\mu}(A, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \exists k > 0, \mu\left((m,n) \in N \times N : \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} \left[M\left(\left\| \frac{\Delta^m x_{st}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq k \right) = 0, \text{ for some } \rho > 0 \text{ and each } z \in X \right\}$$

where

$$\Delta^m x_{kl} = \Delta^{m-1} x_{kl} - \Delta^{m-1} x_{k+1,l} - \Delta^{m-1} x_{k,l+1} + \Delta^{m-1} x_{k+1,l+1}$$

Let us consider a few special cases of the above sets.

(1) If $M(x) = x$, then for all $x \in [0, \infty)$ the above classes of sequences are denoted by $W^{\mu}(A, \Delta^m, p, \|.,.\|)$, $W_0^{\mu}(A, \Delta^m, p, \|.,.\|)$ and $W_{\infty}^{\mu}(A, \Delta^m, p, \|.,.\|)$ respectively.

(2) If $p_{k,l} = 1$, for all $(k,l) \in N \times N$, we denote the above classes of sequences by $W^{\mu}(A, M, \Delta^m, \|.,.\|)$, $W_0^{\mu}(A, M, \Delta^m, \|.,.\|)$ and $W_{\infty}^{\mu}(A, M, \Delta^m, \|.,.\|)$ respectively.

(3) If $M(x) = x$, for all $x \in [0, \infty)$, and $p_{k,l} = 1$ for all $(k,l) \in N \times N$, then we denote the above spaces by $W^{\mu}(A, \Delta^m, \|.,.\|)$, $W_0^{\mu}(A, \Delta^m, \|.,.\|)$ and $W_{\infty}^{\mu}(A, \Delta^m, \|.,.\|)$ respectively.

(4) If we take $A = C(1, 1)$, i.e., the double Cesaro matrix, we denote the above classes of sequences by $W^{\mu}(M, \Delta^m, p, \|.,.\|)$, $W_0^{\mu}(M, \Delta^m, \|.,.\|)$ and $W_{\infty}^{\mu}(M, \Delta^m, p, \|.,.\|)$ respectively.

(5) If we take $A = (C, 1, 1)$ and $p_{k,l} = 1$ for all $(k,l) \in N \times N$, then we denote the above spaces by $W^{\mu}(M, \Delta^m, \|.,.\|)$, $W_0^{\mu}(M, \Delta^m, \|.,.\|)$ and $W_{\infty}^{\mu}(M, \Delta^m, p, \|.,.\|)$ respectively.

(6) If we take $A = (C, 1, 1)$ and $M(x) = x$ for all $x \in [0, \infty)$, and $p_{k,l} = 1$ for all $(k,l) \in N \times N$, then we denote the above spaces by $W^{\mu}(\Delta^m, \|.,.\|)$, $W_0^{\mu}(\Delta^m, \|.,.\|)$, $W_{\infty}(\Delta^m, \|.,.\|)$ and $W_{\infty}^{\mu}(\Delta^m, \|.,.\|)$ respectively.

(7) Let us consider the following notations and definitions. The double sequence $\theta_{r,s} = (k_r, l_s)$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty,$$
$$l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,$$

and let $\bar{h}_{r,s} = h_r h_s$, $\theta_{r,s}$ is determined by $I_{r,s} = \{(i,j) : k_{r-1} < i \leq k_r \text{ and } l_{s-1} < j \leq l_s\}$, (see, [24], [25]). If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\bar{h}_{r,s}} & if\ (k,l) \in I_{r,s}; \\ 0 & otherwise. \end{cases}$$

We write

$$W^{\mu}(\theta, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \mu\left((r,s)\in N\times N : \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in I_{rs}} \left[M\left(\left\| \frac{\Delta^m x_{kl} - L}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \varepsilon \right) = 0, \text{ for some } \rho>0 \text{ and } L\in X \text{ and each } z\in X \right\},$$

$$W_0^{\mu}(\theta, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \mu\left((r,s)\in N\times N : \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in I_{rs}} \left[M\left(\left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \varepsilon \right) = 0, \text{ for some } \rho>0 \text{ and each } z\in X \right\},$$

$$W_\infty^{\mu}(\theta, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \exists k>0, \mu\left(\left\{(r,s)\in N\times N : \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in I_{rs}} \left[M\left(\left\| \frac{\Delta^m x_{st}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq k \right\}\right) = 0, \text{ for some } \rho>0 \text{ and each } z\in X \right\}.$$

As a final illustration let

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\overline{\lambda}_{i,j}} & if\ (k,l)\in \overline{I}_{i,j}; \\ 0 & otherwise. \end{cases}$$

where we shall denote $\overline{\lambda}_{i,j}$ by $\lambda_i\mu_j$ and $(k\in I_i, l\in I_j)$ by $(k,l)\in \overline{I}_{i,j}$. Let $\lambda=(\lambda_i)$ and $\mu=(\mu_j)$ be two non-decreasing sequences of positive real numbers such that each tend to ∞ and $\lambda_{i+1}\leq \lambda_i+1,\ \lambda_1=0$ and $\mu_{j+1}\leq \mu_j+1,\ \mu_1=0$. Then our definition reduces to the following

$$W^{\mu}(\lambda, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \mu\left((i,j)\in N\times N : \frac{1}{\overline{\lambda}_{i,j}} \sum_{(k,l)\in \overline{I}_{i,j}} \left[M\left(\left\| \frac{\Delta^m x_{kl} - L}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \varepsilon \right) = 0, \text{ for some } \rho>0 \text{ and } L\in X \text{ and each } z\in X \right\},$$

$$W_0^{\mu}(\lambda, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \mu\left((i,j)\in N\times N : \frac{1}{\overline{\lambda}_{i,j}} \sum_{(k,l)\in \overline{I}_{i,j}} \left[M\left(\left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \varepsilon \right) = 0, \text{ for some } \rho>0 \text{ and each } z\in X \right\},$$

$$W_\infty^{\mu}(\lambda, M, \Delta^m, p, \|.,.\|) = \left\{ x \in S''(2-X) : \exists k>0, \mu\left(\{(i,j)\in N\times N : \frac{1}{\overline{\lambda}_{i,j}} \sum_{(k,l)\in \overline{I}_{i,j}} \left[M\left(\left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq k \}\right) = 0, \text{ for some } \rho>0 \text{ and each } z\in X \right\}.$$

We now have

Theorem 4.1. $W^{\mu}(A, M, \Delta^m, p, \|.,.\|)$, $W_0^{\mu}(A, M, \Delta^m, p, \|.,.\|)$ and $W_\infty^{\mu}(A, M, \Delta^m, p, \|.,.\|)$ are linear spaces.

Proof: We shall prove the theorem for $W_0^{\mu}(A, M, \Delta^m, p, \|.,.\|)$ and others can be proved similarly. Let $\varepsilon>0$ be given. Assume that $x,y\in W_0^{\mu}(A, M, \Delta^m, p, \|.,.\|)$ and $\alpha,\beta\in \mathrm{R}$, where $x=(x_{kl})$ and $y=(y_{kl})$. Further, let $z\in X$. Then

$$\mu\left(\left\{(m,n)\in N\times N : \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\| \frac{\Delta^m x_{kl}}{\rho_1}, z \right\| \right) \right]^{p_{kl}} \geq \varepsilon \right\}\right) = 0 \qquad (1)$$

for some $\rho_1>0$ and

$$\mu\left(\left\{(m,n)\in N\times N : \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\| \frac{\Delta^m y_{St}}{\rho_2}, z \right\| \right) \right]^{p_{st}} \geq \varepsilon \right\}\right) = 0 \qquad (2)$$

for some $\rho_1>0$.

Since $\|.,.\|$ is a 2-norm, Δ^m is linear, therefore the following inequality holds:

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\| \frac{\Delta^m(\alpha x_{kl}+\beta y_{kl})}{|\alpha|\rho_1+|\beta|\rho_2}, z \right\| \right) \right]^{p_{st}}$$

$$\leq D \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2} M\left(\left\| \frac{\Delta^m x_{kl}}{\rho_1}, z \right\| \right) \right]^{p_{st}}$$

$$+D\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2} M\left(\left\|\frac{\Delta^m y_{kl}}{\rho_2},z\right\|\right)\right]^{p_{st}}$$

$$\leq DF\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}}{\rho_1},z\right\|\right)\right]^{p_{kl}}$$

$$+DF\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m y_{kl}}{\rho_2},z\right\|\right)\right]^{p_{kl}}$$

where

$$F = max\left\{1,\left[\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}\right]^H,\left[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}\right]^H\right\}.$$

From the above inequality we get,

$$\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m(\alpha x_{kl}+\beta y_{kl})}{|\alpha|\rho_1+|\beta|\rho_2},z\right\|\right)\right]^{p_{kl}}\geq\varepsilon\right\}$$

$$\subseteq\left\{(m,n)\in N\times N: DF\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}}{\rho_1},z\right\|\right)\right]^{p_{kl}}\geq\frac{\varepsilon}{2}\right\}$$

$$\cup\left\{(m,n)\in N\times N: DF\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m y_{st}}{\rho_2},z\right\|\right)\right]^{p_{kl}}\geq\frac{\varepsilon}{2}\right\}.$$

Hence from (1) and (2) the required result is proved.

Theorem 4.2. For any fixed $(m,n)\in N\times N$, $W_\infty^\mu(A, M, \Delta^m, p, \|.,.\|)$ is a paranormed space with respect to the paranorm $g_{mn}: X\to \mathrm{R}$, defined by

$$g_{mn}(x)=\inf_{z\in X}\left\{\rho^{\frac{p_{mn}}{H}}:\left(\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}}{\rho},z\right\|\right)\right]^{p_{kl}}\right)^{\frac{1}{H}}\leq 1,\ \forall z\in X\right\}$$

Proof: The proof follows on the same lines adopted by Savas[15]. So it was omitted.

Theorem 4.3. Let $X(\Delta^{m-1}), m\geq 1$ stands for $W^\mu(A, M, \Delta^{m-1}, p, \|.,.\|)$ or $W_0^\mu(A, M, \Delta^{m-1}, p, \|.,.\|)$ or $W_\infty^\mu(A, M, \Delta^{m-1}, p, \|.,.\|)$. Then $X(\Delta^{m-1})\underset{\neq}{\subset} X(\Delta^m)$. In general, $X(\Delta^i)\underset{\neq}{\subset} X(\Delta^m)$ for all $i=1,2,3,....,m-1$.

Proof: We shall give the prove for $W_0^\mu(A, M, \Delta^{m-1}, p, \|.,.\|)$ only. It can be proved in a similar way for $W^\mu(A, M, \Delta^{m-1}, p, \|.,.\|)$ and $W_\infty^\mu(A, M, \Delta^{m-1}, p, \|.,.\|)$.

Let $x=(x_{kl})\in W_0^\mu(A, M, \Delta^{m-1}, p, \|.,.\|)$. Also, Let $\varepsilon>0$ be given. Then

$$\mu\left(\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^{m-1}x_{kl}}{\rho},z\right\|\right)\right]^{p_{kl}}\geq\varepsilon\right\}\right)=0 \quad (3)$$

for some $\rho>0$. Since M is non-decreasing and convex it follows that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}}{4\rho},z\right\|\right)\right]^{p_{kl}} = \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^{m-1}x_{k+1,l+1}-\Delta^{m-1}x_{k+1,l}-\Delta^{m-1}x_{k,l+1}+\Delta^{m-1}x_{kl}}{4\rho},z\right\|\right)\right]^{p_{kl}}$$

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left(\left[\frac{1}{4}M\left(\left\|\frac{\Delta^{m-1}x_{s+1,t+1}}{\rho},z\right\|\right)\right]^{p_{kl}}+\left[\frac{1}{4}M\left(\left\|\frac{\Delta^{m-1}x_{s+1,t}}{\rho},z\right\|\right)\right]^{p_{kl}}\right.$$

$$\left.+\left[\frac{1}{4}M\left(\left\|\frac{\Delta^{m-1}x_{s,t+1}}{\rho},z\right\|\right)\right]^{p_{kl}}+\left[\frac{1}{4}M\left(\left\|\frac{\Delta^{m-1}x_{s,t}}{\rho},z\right\|\right)\right]^{p_{kl}}\right)$$

$$D^2G\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left(\left[M\left(\left\|\frac{\Delta^{m-1}x_{k+1,l+1}}{\rho},z\right\|\right)\right]^{p_{kl}}+\left[M\left(\left\|\frac{\Delta^{m-1}x_{k+1,l}}{\rho},z\right\|\right)\right]^{p_{kl}}\right.$$

$$\left.+\left[M\left(\left\|\frac{\Delta^{m-1}x_{k,l+1}}{\rho},z\right\|\right)\right]^{p_{kl}}+\left[M\left(\left\|\frac{\Delta^{m-1}x_{kl}}{\rho},z\right\|\right)\right]^{p_{kl}}\right)$$

where $G = max\left\{1, \left(\frac{1}{4}\right)^H\right\}$. Hence we have

$$\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}}{4\rho},z\right\|\right)\right]^{p_{kl}}\geq\varepsilon\right\}$$

$$\subseteq\left\{(m,n)\in N\times N: D^2G\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^{m-1} x_{k+1,l+1}}{\rho},z\right\|\right)\right]^{p_{kl}}\geq\frac{\varepsilon}{4}\right\}$$

$$\cup\left\{(m,n)\in N\times N: D^2G\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^{m-1} x_{k+1,l}}{\rho},z\right\|\right)\right]^{p_{kl}}\geq\frac{\varepsilon}{4}\right\}$$

$$\cup\left\{(m,n)\in N\times N: D^2G\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^{m-1} x_{k,l+1}}{\rho},z\right\|\right)\right]^{p_{kl}}\geq\frac{\varepsilon}{4}\right\}$$

$$\cup\left\{(m,n)\in N\times N: D^2G\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^{m-1} x_{k,l}}{\rho},z\right\|\right)\right]^{p_{kl}}\geq\frac{\varepsilon}{4}\right\}.$$

Using (3) we get

$$\mu\left(\left\{m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}}{4\rho},z\right\|\right)\right]^{p_{kl}}\geq\varepsilon\right\}\right)=0$$

Therefore, $x=(x_{kl})\in W_0^{\mu}(A, M, \Delta^m, p, \|.,.\|)$. This completes the proof.

Theorem 4.4. (*i*) Let $0<\inf p_{kl}\leq p_{kl}\leq 1$. Then $W^{\mu}(A, M, \Delta^m, p, \|.,.\|)\subset W^{\mu}(A, M, \Delta^m, \|.,.\|)$.
(*ii*) $1<p_{kl}\leq\sup p_{kl}\leq\infty$. Then

$W^{\mu}(A, M, \Delta^m, \|.,.\|)\subset W^{\mu}(A, M, \Delta^m, p, \|.,.\|)$.

Proof: (*i*) Let $x=(x_{kl})\in W^{\mu}(A, M, \Delta^m, p, \|.,.\|)$. Let also $\varepsilon>0$ be given. Then

$$\mu\left(\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^{m-1} x_{kl}}{\rho},z\right\|\right)\right]^{p_{kl}}\geq\varepsilon\right\}\right)=0 \quad (4)$$

for some $\rho>0$. Since $0<\inf p_{kl}\leq p_{kl}\leq 1$, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]$$

$$\leq\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]^{p_{kl}}.$$

So

$$\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]\geq\varepsilon\right\}$$

$$\subseteq\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]^{p_{kl}}\geq\varepsilon\right\}$$

Finally, from (4) we get

$$\mu\left(\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z\right\|\right)\right]\geq\varepsilon\right\}\right)=0.$$

(ii) Let $p_{kl}\geq 1$ for each *k,l*, and $\sup p_{k,l}$. Let $(x_{k,l})\in W^{\mu}(A, M, \Delta^m, \|.,.\|)$. Then for each $0<\varepsilon<1$ there exists a positive integer *N* such that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]\le\varepsilon<1$$

for all $n,m\ge N$. This implies that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]^{p_{kl}}$$
$$\le\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z\right\|\right)\right].$$

So we have

$$\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]^{p_{kl}}\ge\varepsilon\right\}$$
$$\subseteq\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}-L}{\rho},z\right\|\right)\right]\ge\varepsilon\right\}.$$

As above we have

$$\mu\left(\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z\right\|\right)\right]^{p_{kl}}\ge\varepsilon\right\}\right)=0.$$

This completes the proof.

The following corollary follows immediately from the above theorem.

Corollary 4.1. Let $A=(C,\ 1,\ 1)$ double Cesaro matrix and let M be an Orlicz function.

(a) If $0<\inf p_{kl}\le p_{kl}<1$, then $W^{\mu}(M,\ \Delta^m,p,\ \|.,.\|)\subset W^{\mu}(M,\ \Delta^m,\ \|.,.\|)$.

(b) $1<p_{kl}\le\sup p_{kl}<\infty$, then $W^{\mu}(M,\ \Delta^m,\ \|.,.\|)\subset W^{\mu}(M,\ \Delta^m,\ p,\ \|.,.\|)$.

In the next theorem we establish a connection between $W^{\mu}\left(A,M,\ \Delta^m,\ p,\ \|.,.\|\right)$ and $\mu_A(\Delta)$. We now have

Theorem 4.5. If M is an Orlicz function and $0<h=\inf_{k,l}p_{kl}\le p_{kl}\le\sup_{k,l}p_{kl}=H<\infty$, then $W^{\mu}\left(A,M,\Delta^m,p,\|.,.\|\right)\subset\mu_A(\Delta)$.

Proof: If $x\in W^{\mu}\left(A,M,\Delta^m,p,\|.,.\|\right)$, then there exists $\rho>0$ such that

$$\mu\left(\left\{(m,n)\in N\times N:\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{kl}}{\rho},z\right\|\right)\right]^{p_{kl}}\ge\varepsilon\right\}\right)=0. \quad (5)$$

Then given $\varepsilon>0$ and let $\varepsilon_1=\dfrac{\varepsilon}{\rho}$ we obtain the following:

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z\right\|\right)\right]^{p_{kl}}$$
$$=\sum_{k,l=0,0;\|\Delta^m x_{k,l}-L,z\|\ge\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z\right\|\right)\right]^{p_{k,l}}$$
$$+\sum_{k,l=0,0;\|\Delta^m x_{k,l}-L,z\|<\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z\right\|\right)\right]^{p_{k,l}}$$
$$\ge\sum_{k,l=0,0;\|\Delta^m x_{k,l}-L,z\|\ge\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\left\|\frac{\Delta^m x_{k,l}-L}{\rho},z\right\|\right)\right]^{p_{k,l}}$$
$$\ge\sum_{k,l=0,0;\|\Delta^m x_{k,l}-L,z\|\ge\varepsilon}^{\infty,\infty} a_{m,n,k,l}\left(\min\left\{\left[M(\varepsilon_1)\right]^h,\left[M(\varepsilon_1)\right]^H\right\}\right)$$
$$\ge\left(\min\left\{\left[M(\varepsilon_1)\right]^h,\left[M(\varepsilon_1)\right]^H\right\}\right)\sum_{k,l=0,0;\|\Delta^m x_{k,l}-L,z\|\ge\varepsilon}^{\infty,\infty} a_{m,n,k,l}$$
$$\ge\left(\min\left\{\left[M(\varepsilon_1)\right]^h,\left[M(\varepsilon_1)\right]^H\right\}\right)\mu_A\left(\left\{(k,l)\in N\times N:\|\Delta^m x_{k,l}-L,z\|\ge\varepsilon\right\}\right).$$

Finally, from (5) we have $x\in\mu_A(\Delta)$.

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