IJST (2012) A3 (Special issue-Mathematics): 327-330
Iranian Journal of Science & Technology

http://www.shirazu.ac.ir/en

On the theory of strips and Joachimsthal theorem in the Lorentz space $\mathbb{L}^n$, $(n > 3)$

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Abstract

In this study the theory of strips and Joachimsthal Theorem in $\mathbb{L}^3$ are generalized to Lorentz space $\mathbb{L}^n$, $(n > 3)$. Furthermore, the Joachimsthal Theorem is investigated when the strip is time-like and space-like.

Keywords: Curvature strip; semi-Euclidean space; Joachimsthal Theorem

1. Introduction

The theory of strips and Joachimsthal Theorem in $\mathbb{L}^3$ is studied in [1]. Also, in [2] the higher curvatures of a strip in E^n is studied. The behavior of curvature lines near a principal cycle common to two orthogonal surfaces, as a complement of Joachimsthal theorem, is studied in [3]. Furthermore Joachimsthal's theorems in Euclidean spaces E^n are given in [4]. Using Cartan's structure equations, Joachimsthal's theorems in semi-Euclidean spaces E_v^{n+1} are studied in [5].

In this section, some basic concepts and definitions in the Lorentz n-space $\mathbb{L}^n$ are given.

$\mathbb{R}^n$ equipped with the Lorentzian inner product

$$\langle X, Y \rangle_{\mathbb{L}} = \sum_{i=1}^{n-1} x_i y_i - x_n y_n \tag{1}$$

is called n-dimensional Lorentz space and denoted by $\mathbb{L}^n$.

In $\mathbb{L}^n$, a vector X is said to be time-like if $\langle X, X \rangle < 0$, space-like if $\langle X, X \rangle > 0$ or $X = 0$ and null if $\langle X, X \rangle = 0$ and $X \neq 0$. In addition, the norm of a vector $X \in \mathbb{L}^n$ is defined by $\|X\| = \sqrt{|\langle X, X \rangle|}$ in [6].

Let α be a curve in $\mathbb{L}^n$ and α' be the velocity vector of α, where $(\,'\,)$ denotes the derivation with respect to the parameter s.

The curve α is called
time-like if $\langle \alpha', \alpha' \rangle < 0$,
space-like if $\langle \alpha', \alpha' \rangle > 0$,
null if $\langle \alpha', \alpha' \rangle = 0$.

The bilinear function $g: V \times V \to \mathbb{R}$ is called a symmetric bilinear form on V if $g(v, w) = g(w, v)$ for every $v, w \in V$, where V is finite dimensional real vector space.

A symmetric bilinear form g on a real vector space V is
i) positive definite provided it implies $g(v, v) > 0, v \in V$,
ii) negative definite provided it implies $g(v, v) < 0, v \in V$,
iii) nondegenerate provided $g(v, w) = 0$ for all $w \in V$ implies $v = 0$ [6].

If g is a symmetric bilinear form on V, then the restriction $g\mid_{W \times W}$ for any subspace W of V, denoted by $g\mid_W$, is again symmetric and bilinear. The index ν of a symmetric bilinear form g on V is the largest integer that is the dimension of a subspace W on which $g\mid_W$ is negative definite. Obviously, $0 \leq \nu \leq \dim(V)$.

Symmetric, bilinear and nondegenerate function

$$g : \chi(M) \times \chi(M) \longrightarrow C^\infty(M, \mathbb{R})$$

is called a metric tensor on M. If g is a metric tensor with constant index on M, the pair (M, g) is called semi-Riemannian manifold. If $\dim(M) \geq 2$ and $\nu = 1$, the pair (M, g) is called a Lorentz manifold.

Let $j: M \longrightarrow \mathbb{L}^n$ be an inclusion transformation. If $j^*(g)$ is a metric tensor on M, then M is called the Lorentz submanifold. If $\dim(M) = n - 1$, the submanifold is called a hypersurface of $\mathbb{L}^n$.

Let M be a hypersurface in $\mathbb{L}^n$. It is called time-like hypersurface if normal of M is space-like (space-like hypersurface if normal of M is time-like).

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Received: 12 December 2011 / Accepted: 8 February 2012

Let M be a time-like hypersurface in $\mathbb{L}^n$ and α be a time-like curve on M. The geometric shape which is constituted by points of curve α and surface tangents at these points is called time-like surface strip along the given curve and is denoted by (α, M). The strip (α, M) is called space-like surface strip if M is space-like hypersurface and α is a space-like curve.

2. The Higher Curvatures of a Strip in $\mathbb{L}^n$

Let M be a hypersurface in $\mathbb{L}^n$ and α be a curve given with the arc-parameter s. Let $\{V_1, V_2, \dots, V_n\}$ be a Frenet n-frame at the point $\alpha(s)$, taking $Z_1 = V_1$ and $\mathcal{F}_0^1$ be a set of orthonormal frame $\{Z_1, Z_2, \dots, Z_n\}$ at the point $\alpha(s)$. Taking Z_n a unit normal vector of M at the point $\alpha(s)$, $\{Z_1, Z_2, \dots, Z_{n-1}\}$ is an orthonormal base of $T_M(\alpha(s))$.

For vector fields Z_i

$$Z_i = \sum_{j=1}^{n} g_{ij} \frac{\partial}{\partial x_j}, \qquad 1 \le i, j \le n$$

may be written, where x_i, $1 \le i \le n$, are coordinate functions in $\mathbb{L}^n$. We may write

$$Z = GE \tag{2}$$

where $[Z_1 \; Z_2 \; \dots \; Z_n]^T = Z$, $[\frac{\partial}{\partial x_1} \; \frac{\partial}{\partial x_2} \; \dots \; \frac{\partial}{\partial x_n}]^T = E$ and $G \in O_v(n)$. If Eq. (2) is derivated with respect to the arc-parameter s, then

$$\frac{dZ}{ds} = \frac{dG}{ds} E$$

is obtained. Moreover, since $G \in O_v(n)$, we may write

$$GG^{-1} = G(\varepsilon G^T \varepsilon) = I_n, \tag{3}$$

where ε is a sign matrix of G. That is

$$\varepsilon = \begin{bmatrix} \varepsilon_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \varepsilon_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \varepsilon_0 \end{bmatrix},$$

$$\varepsilon_0 = \begin{cases} 1, & if \; Z_i \; space-like \\ -1, & if \; Z_i \; time-like. \end{cases}$$

If Eq. (3) is derived with respect to s,

$$\left(\frac{dG}{ds} G^{-1}\right) + \varepsilon \left(\frac{dG}{ds} G^{-1}\right)^T \varepsilon = 0$$

is obtained. If we denote

$$\frac{dG}{ds} G^{-1} = \Omega$$

we have

$$\Omega^T = -\varepsilon \Omega \varepsilon.$$

This shows that Ω is a semi anti-symmetric matrix. Then we may write

$$\frac{dZ}{ds} = \Omega Z.$$

This expression can be written in the matrix form as follows:

$$\begin{bmatrix} \frac{dZ_1}{ds} \\ \frac{dZ_2}{ds} \\ \vdots \\ \frac{dZ_{n-1}}{ds} \\ \frac{dZ_n}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_0 t_{12} & \varepsilon_0 t_{13} & \cdots & \varepsilon_0 t_{1(n-1)} & \varepsilon_0 t_{1n} \\ -\varepsilon_0 t_{12} & 0 & \varepsilon_0 t_{23} & \cdots & \varepsilon_0 t_{2(n-1)} & \varepsilon_0 t_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\varepsilon_0 t_{1(n-1)} & -\varepsilon_0 t_{2(n-1)} & -\varepsilon_0 t_{3(n-1)} & \cdots & 0 & \varepsilon_0 t_{(n-1)n} \\ -\varepsilon_0 t_{1n} & -\varepsilon_0 t_{2n} & -\varepsilon_0 t_{3n} & \cdots & -\varepsilon_0 t_{(n-1)n} & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \\ Z_n \end{bmatrix} \tag{4}$$

Here the functions

$$t_{ij} : I \longrightarrow \mathbb{R}, \qquad 1 \le i, j \le n$$

are called higher curvature functions of the strip (α, M) and the real number $t_{ij}(s) \in \mathbb{R}$ is said to be higher curvature of (α, M) at the point $\alpha(s)$ for every $s \in I$. If S is the shape operator of M and α is a curvature line, we may write

$$S(Z_1) = k_1 Z_1, \; k_1 \in \mathbb{R}.$$

Moreover, since

$$S(Z_1) = -\frac{dZ_n}{ds},$$

$$t_{2n} = t_{3n} = \cdots = t_{(n-1)n} = 0$$

is obtained from the matrix Ω. Then we can state the following theorem:

Theorem 2.1. Let (α, M) be a strip in $\mathbb{L}^n$. If α is a curvature strip, then for the higher curvature functions $t_{ij} : I \longrightarrow \mathbb{R}$, $1 \le i, j \le n$, we have

$$t_{2n} = t_{3n} = \cdots = t_{(n-1)n} = 0.$$

If $t_{2n} = t_{3n} = \cdots = t_{(n-1)n} = 0$ for the strip (α, M) in $\mathbb{L}^n$, the strip is said to be curvature strip. If we take $n = 3$, then t_{23} equals 0. This shows that (α, M) is a curvature strip in $\mathbb{L}^3$.

Theorem 2.2. Let M_1 and M_2 be two time-like hypersurfaces in $\mathbb{L}^n$ and α be a differentiable time-like curve, where $\alpha(I) \subset M_1 \cap M_2$. If time-like strips (α, M_1) and (α, M_2) are curvature strips, then the angle between (α, M_1) and (α, M_2) is constant.

Proof: Let $\{Z_1, Z_2, \dots, Z_{n-1}, Z_n\}$ and $\{X_1, X_2, \dots, X_{n-1}, X_n\}$ be vector field systems of the strips (α, M_1) and (α, M_2), respectively. Let t_{ij} and $\bar{t}_{ij}$, $1 \leq i, j \leq n$ be higher curvature functions of (α, M_1) and (α, M_2), respectively. In this case, from (4)

$$\frac{dZ_n}{ds} = -\varepsilon_0 t_{1n} Z_1 \tag{5}$$

and

$$\frac{dX_n}{ds} = -\varepsilon_0 \bar{t}_{1n} X_1 \tag{6}$$

are obtained. If θ is the angle between (α, M_1) and (α, M_2), we can write $\langle Z_n, X_n\rangle = \cos\theta$ (see [7]). If this expression is derived with respect to s,

$$\langle \frac{dZ_n}{ds}, X_n\rangle + \langle Z_n, \frac{dX_n}{ds}\rangle = -\sin\theta \frac{d\theta}{ds}$$

is obtained. If we use (5) and (6),

$$-\varepsilon_0 t_{1n}\langle Z_1, X_n\rangle - \varepsilon_0 \bar{t}_{1n}\langle Z_n, X_1\rangle = -\sin\theta \frac{d\theta}{ds} \tag{7}$$

is obtained. Since $\{V_1, V_2, \dots, V_n\}$ is Frenet n-frame at the point $\alpha(s)$, we have $Z_1 = V_1$ and $X_1 = V_1$. Therefore, from (7) $\sin\theta \frac{d\theta}{ds} = 0$ is obtained. If $\sin\theta = 0$, then $\theta = 0$ or $\theta = \pi$. This means that the time-like strips (α, M_1) and (α, M_2) are congruent. Then $\frac{d\theta}{ds} = 0$ is obtained. This shows that θ is constant.

Now we express and prove the Theorem 2.2 in the case of the strips (α, M_1) and (α, M_2) being space-like.

Theorem 2.3. Let M_1 and M_2 be two space-like hypersurfaces in $\mathbb{L}^n$ and α be a differentiable space-like curve, where $\alpha(I) \subset M_1 \cap M_2$. If the space-like strips (α, M_1) and (α, M_2) are curvature strips, then the angle between (α, M_1) and (α, M_2) is constant.

Proof: Let $\{Z_1, Z_2, \dots, Z_{n-1}, Z_n\}$ and $\{X_1, X_2, \dots, X_{n-1}, X_n\}$ be vector field systems of the strips (α, M_1) and (α, M_2), respectively. Let t_{ij} and $\bar{t}_{ij}$, $1 \leq i, j \leq n$, be higher curvature functions of (α, M_1) and (α, M_2), respectively. In this case, from (4)

$$\left.\begin{aligned} \frac{dZ_n}{ds} &= -\varepsilon_0 t_{1n} Z_1 \\ \frac{dX_n}{ds} &= -\varepsilon_0 \bar{t}_{1n} X_1 \end{aligned}\right\} \tag{8}$$

or

$$\left.\begin{aligned} \frac{dZ_n}{ds} &= -t_{1n} Z_1 \\ \frac{dX_n}{ds} &= -\bar{t}_{1n} X_1 \end{aligned}\right\} \tag{9}$$

are obtained. If θ is the angle between (α, M_1) and (α, M_2), then we may write

$$\langle Z_n, X_n\rangle = ch\theta \text{ (see [7])},$$

since Z_n and X_n are unit time-like vectors. If this expression is derivated with respect to s,

$$\langle \frac{dZ_n}{ds}, X_n\rangle + \langle Z_n, \frac{dX_n}{ds}\rangle = sh\theta \frac{d\theta}{ds}$$

is obtained. If we use equation (9), then

$$-t_{1n}\langle Z_1, X_n\rangle - \bar{t}_{1n}\langle Z_n, X_1\rangle = sh\theta \frac{d\theta}{ds} \tag{10}$$

is obtained. Since $\{V_1, V_2, \dots, V_n\}$ is Frenet n-frame at the point $\alpha(s)$, we have $Z_1 = V_1$ and $X_1 = V_1$. Therefore, from (10)

$$\frac{d\theta}{ds} sh\theta = 0. \tag{11}$$

If $sh\theta = 0$, then $\theta = 0$. That means the space-like strips (α, M_1) and (α, M_2) are congruent. So $sh\theta \neq 0$. Hence $\frac{d\theta}{ds} = 0$. It is seen that θ is constant.

Theorem 2.4. Let M_1 and M_2 be two hypersurfaces in $\mathbb{L}^n$. Let α be a nonplanar time-like curve on M_1 and β be any time-like curve on M_2. Let P be a hypersurface which is rolling along the curves α and β on M_1 and M_2. If the time-like strips (α, M_1) and (β, M_2) are curvature strips, then the distance between the corresponding points is constant.

Proof: Suppose that α and β are two curves with the arc-parameter s_1 and s_2, respectively. Let $\{Z_1, Z_2, \dots, Z_{n-1}, Z_n\}$ and $\{X_1, X_2, \dots, X_{n-1}, X_n\}$ be strip vector field systems at the point $\alpha(s_1)$ and $\beta(s_2)$, respectively. Since the points $\alpha(s_1)$ and $\beta(s_2)$ are at the common tangent space of M_1 and M_2, we may say $V(s_1) \in T_M(\alpha(s_1))$, where $V(s_1)$ is a unit vector on the line combining the points $\alpha(s_1)$ and $\beta(s_2)$. Therefore, we may write

$$V(s_1) = \sum_{i=1}^{n-1} h_i Z_i \quad , \quad h_i(s_1) \in \mathbb{R}. \tag{12}$$

Furthermore, we may write

$$\beta(s_2) = \alpha(s_1) + \lambda(s_1)\mathrm{V}(s_1) \tag{13}$$

Since (α, M_1) and (β, M_2) are curvature strips, from (4) we may write

$$\left.\begin{array}{l}\frac{dZ_n}{ds_1} = -\varepsilon_0 t_{1n} Z_1 \\ \frac{dX_n}{ds_2} = -\varepsilon_0 \bar{t}_{1n} X_1\end{array}\right\}. \tag{14}$$

Since time-like hypersurfaces M_1 and M_2 have the common tangent space along α and β, the unit normal vector fields of M_1 and M_2 are the same. In this case, from (14)

$$t_{1n} Z_1 ds_1 = \bar{t}_{1n} X_1 ds_2.$$

can be written. If the norm of the two sides of the above equation is taken, since X_1 and Z_1 are unit time-like vector fields

$$\frac{ds_1}{ds_2} = \frac{|\bar{t}_{1n}|}{|t_{1n}|} \tag{15}$$

is obtained. Let us denote

$$\frac{|\bar{t}_{1n}|}{|t_{1n}|} = k. \tag{16}$$

If Eq. (13) is derivated with respect to s_1, and Eqs. (15) and (16) are kept in mind, then

$$X_1 = kZ_1 + k\frac{d\lambda}{ds_1} V(s_1) + k\lambda(s_1)\frac{dV}{ds_1} \tag{17}$$

is obtained. If Eq. (12) is replaced in the last equation and we consider $\langle X_1, X_n \rangle = 0$ and $Z_n = X_n$, then

$$\sum_{i=1}^{n-1} \varepsilon_0 h_i t_{in} = 0$$

is obtained. Since (α, M_1) is curvature strip, we have $t_{2n} = t_{3n} = \cdots = t_{(n-1)n} = 0$. In this case, since $t_{1n} \neq 0$, from the last equation we have $h_1(s_1) = 0$. If we consider Eq. (13), then

$$\langle Z_1, V(s_1) \rangle = 0 \tag{18}$$

is obtained. Moreover, since $Z_n = X_n$, Z_1 and X_1 are linear dependent. Therefore we have

$$\langle X_1, V(s_1) \rangle = 0.$$

If Eq. (17) is replaced in the last equation, then we obtain

$$k\frac{d\lambda}{ds_1} = 0.$$

Since the curve α isn't planar, $k \neq 0$. This shows that λ is constant.

Example 2.1. Let's take the Lorentz sphere

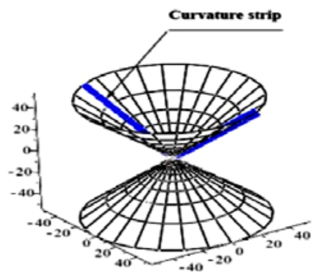
$$S = \{\alpha(u,v) = (rchu\cos v,\ rchu\sin v, rshu): 0 \leq v \leq 2\pi, u \in \mathbb{R}\}$$

in $\mathbb{L}^3$. If Lorentz sphere S is derived with respect to parameter u, we obtain

$$\alpha_u = (rshu\cos v,\ rshu\sin v, rchu).$$

If we compute geodesic curvature k_g of the parameter curve α_u, we find $k_g = 0$. Here $k_g = t_{12}$. Then we can say that the α_u is a geodesic curve on S. In this case the geodesic torsion τ_g of α_u is equal to the torsion τ of α_u and we compute $\tau = 0$. Since $\tau_g = \tau$, we obtain $\tau_g = t_{23} = 0$. This shows that the pair of (α_u, S) is a curvature strip.

Now let us show this with a Fig.



Acknowledgments

The authors thank the referee for the useful suggestions and remarks for the revised version. This study is supported by the University of Ondokuz Mayıs Project no PYO.FEN.1904.11.007.

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