On generalized conformally recurrent Kaehlerian Weyl spaces

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Abstract

In this study, 2n-dimensional (n > 2) generalized conformally recurrent Kaehlerian Weyl spaces and generalized conharmonically recurrent Kaehlerian Weyl spaces are defined. It is proved that a Kaehlerian Weyl space is generalized conformally recurrent if and only if it is generalized recurrent. Also, it is shown that a Kaehlerian Weyl space will be generalized recurrent if and only if it is generalized conharmonically recurrent.

Keywords: Kaehlerian Weyl space, recurrent Weyl space, generalized recurrency, conformally recurrency, conharmonic recurrency

1. Introduction

In the review paper "Weyl Geometry in Late 20th Century Physics" by E. Scholz, the physical concepts in Weylian geometry have been widely reviewed (see [1] and references therein). H. Weyl first introduced a gauge invariant theory to unify gravity and electromagnetic theories in 1918. This theory is not acceptable as a unified theory since the electromangetic potential does not couple to spinor which is essential for the electromagnetic theory. This does not mean that Weyl geometry has a physical meaning as well in different theories and, in the second part of the 20th century, the Weyl geometry has been studied in different research fields of physics such as: quantum mechanics [2], particle physics [3], gravity [4] and scale invariant cosmology.

In [4], spaces with complex structures are considered in Weyl geometry and geodesics of the spacetime are studied. They consider scalar-flat Kaehler metrics and hypercomplex structures and obtain shear-free congruences. Presence of this type congruence means that a pair of coupled monopole like solution is equivalent to Einstein-Weyl equation. In quantum mechanics, discovery of phenomenon such as Berry phase (geometric phase), adiabatic transition probability in two level quantum system caused physicists to consider the geometry of quantum mechanics more than Riemannian structure of Hilbert space and therefore, they consider that physical state for the Berry phase is isomorphic to Kahlerian space and physical state for adiabatic transition may be related to Kaehlerian-Weyl space [5]. It may also be worthwhile to study Dirac equation in the background of these type of spaces to understand the effects of torsion on spinor fields and to study gravitational monopoles as an application of Kaehlerian-Weyl spaces.

Despite the unified theory of Weyl not being acceptable as physical theory, it introduced a beautiful theory in differential geometry. The mathematics of the theory is a generalization of the Riemannian geometry and the connection is an instructive example of non-metric connections.

In this work, some generalized Keehlerian Weyl spaces are considered and some structures on these spaces are examined.

A differentiable manifold of dimension n having a conformal metric tensor g and a symmetric connection \( \nabla \) satisfying the compatibility condition

\[
\nabla g = 2(T \otimes g) \tag{1}
\]

where \( T \) is a 1-form (complementary covector field) is called a Weyl space which is denoted by \( W_n(g,T) \). Under the renormalization

\[
\tilde{g} = \lambda^2 g \tag{2}
\]

of the metric tensor \( g \), \( T \) is transformed by the law

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\[ \bar{T} = T + d \ln \lambda, \]  
(3)

where \( \lambda \) is a scalar function defined on \( W_n \).

An object \( A \) defined on \( W_n(g,T) \) is called a satellite of \( g \) of weight \( p \) if it admits a transformation of the form

\[ \hat{A} = \lambda^p A \]  
(4)

under the renormalization (2) of \( g \). [6-9].

The prolonged covariant derivatives of the satellite \( A \) of the tensor \( g_{ij} \), with weight \( p \) is defined in [6]

\[ \hat{\nabla}_k A = \nabla_k A - pT_k A. \]  
(5)

Writing (1) in local coordinates we find

\[ \partial_k g_{ij} - g_{ij} \Gamma^b_{ik} - g_{jk} \Gamma^n_{ji} - 2T_k g_{ij} = 0, \quad \partial_k = \frac{\partial}{\partial x^k}, \]

and using (5) we obtain

\[ \hat{\nabla}_k g_{ij} = 0 \]

where \( \Gamma^i_{kl} \) are the coefficients of the Weyl connection given by

\[ \Gamma^i_{kl} = \left\{ \frac{i}{[kl]} \right\} g^{im} \left( g_{mk} T_j + g_{ml} T_k - g_{kl} T_m \right) \]  
(6)

and \( \left\{ \frac{i}{[kl]} \right\} \) being the coefficients of the metric connection defined by

\[ \left\{ \frac{i}{[kl]} \right\} = \frac{1}{2} g^{im} \left( \partial_k g_{ml} + \partial_l g_{km} - \partial_m g_{lk} \right). \]  
(7)

We note that the prolonged covariant differentiation preserves the weights of the satellites.

A Kaehlerian Weyl space denoted by \( KW_n \) is an \( n \)-dimensional \( (n = 2m) \) space with an almost complex structure \( F^i_j \) satisfying

\[ F^i_j F^j_k = -\delta^i_k, \]  
(8)

\[ g_{ij} F^i_j F^j_k = g_{jk}, \]  
(9)

\[ \hat{\nabla}_k F^i_j = 0, \quad \text{(for all } i, j, k) \]  
(10)

\[ F^i_j = g_{jk} F^j_i = -F^j_i, \]  
(11)

\[ F^ij = g_{ih} F^j_i = -F^j_i. \]  
(12)

The tensors \( F^i_j \) and \( F_{ij} \) are of weight 2 and -2, respectively [9].

The mixed curvature tensor \( R^i_{jkl} \) and the covariant curvature tensor \( R^i_{jkl} \) of \( W_n(g,T) \) are given respectively,

\[ R^i_{jkl} = \frac{\partial}{\partial x^i} \Gamma^j_{kl} - \frac{\partial}{\partial x^i} \Gamma^{i}_{jl} + \Gamma^j_{ki} \Gamma^{i}_{lj} - \Gamma^j_{lj} \Gamma^{i}_{ki}, \]  
(13)

and \( R^i_{jkl} = g_{ik} R^i_{jkl} \) where \( \Gamma^i_{jk} \)'s are defined in (6).

The Ricci tensor and the scalar curvature of \( W_n(g,T) \) are defined by

\[ R_{ij} = R^i_{ji}, \quad \text{and} \quad R = g^{ij} R_{ij}. \]  
(14)

Also, it can be seen that the anti-symmetric part of the Ricci tensor satisfies [6]

\[ R_{[ij]} = n \nabla_i T_{j}. \]  
(15)

Pure and hybrid tensors in Kaehlerian Weyl spaces are defined similar to the definitions in Riemannian spaces. Let \( T \) be a tensor field of type \((0,2)\) on \( KW_n \). If \( T \) satisfies \( T(JX,Y) = T(X,Y) \) for any vector fields \( X \) and \( Y \) and any almost complex structure \( J \) on \( KW_n \) (in local coordinates \( T_{sr} J^r J^s = T_{ji} \)) then \( T \) is said to be a hybrid tensor with respect to \( j \) and \( i \) and if \( T(JX,Y) = -T(X,Y) \) (in local coordinates \( T_{sr} J^r J^s = -T_{ji} \)), then \( T \) is said to be a pure tensor with respect to \( j \) and \( i \) in a Kaehlerian Weyl space [10]. Let

\[ H_{ij} = \frac{1}{2} R^k_{ijkl} F^i_j, \quad M_{ij} = g_{ik} R^k_j, \]  
(16)

where, \( R^k_{ijkl} = g_{ik} R^h_{jkl}, \quad R^k_j = R^h_{jkl} g^{kl}. \) Then the following relations hold [10]:

\[ a) \quad M_{ij} = \left( \frac{n-2}{n} \right) R_{ij} + 2 R_{ij} = R_{ij} + 2n(R_{ij} - R_g) \]  
(17)

\[ b) \quad H_{ij} = -M_{ij} F^i_j = M_{ij} F^i_j. \]  
(18)
2. Generalized conformally recurrent Kaehlerian Weyl spaces

An n-dimensional Weyl space is called a generalized recurrent Weyl space if its curvature tensor $\hat{R}_{ijkl}$ satisfies the condition

$$\hat{R}_{ijkl} = A R_{ijkl} + B_r (g_{ij} g_{kl} - g_{ik} g_{lj})$$

where $A$ and $B$ are non-zero two 1-forms of weights 0 and -2, respectively, \[11\]. By putting $G_{ijkl} = g_{ij} g_{kl} - g_{ik} g_{lj}$, (23) becomes

$$\hat{V}_r R_{ijkl} = A R_{ijkl} + B_r G_{ijkl}.$$  

(24) 

If the 1-form $B$ is zero, then the Weyl space reduces to recurrent Weyl space [8].

Now, we introduce the following definitions on recurrent Kaehlerian Weyl spaces.

**Definition 2.1.** An $n$-dimensional ($n = 2m$, $m > 2$) Kaehlerian Weyl space is said to be generalized recurrent if its curvature tensor $R_{ijkl}$ of weight 2 satisfies the condition

$$\hat{V}_r R_{ijkl} = A R_{ijkl} + B_r G_{ijkl},$$

(25) 

where $A$ and $B$ are two 1-forms of weights 0 and -2, respectively. Such a space is denoted by $GRKW_n$.

If the 1-form $B$ is zero, then $GRKW_n$ is recurrent Kaehlerian Weyl space.

On the other hand, the conformal curvature tensor $C_{ijkl}^h$ of $W_n$ is given by Miron \[12\]

$$C_{ijkl}^h = R_{ijkl}^h + \delta^h_{ij} L_{ij}^h - \delta^h_{ik} L_{ik}^h + \delta^h_{il} L_{il}^h - 2 \delta^h_{ij} L_{ij}^h,$$

(26) 

where

$$L_{ij}^h = -\frac{R_{ij}^h}{n-2} + \frac{2}{n(n-2)} R_{[ij]}^h + \frac{R g_{ij}}{2(n-1)(n-2)},$$

(27) 

and

$$L_{ik}^h = g_{ik}^h L_{ik}^h.$$

(28) 

Considering (17), (27) becomes

$$L_{ij} = -\frac{1}{n-2} M_{ij} + \frac{R_{ij}^h}{n(n-2)} + \frac{R g_{ij}}{2(n-1)(n-2)},$$

(29) 

and

$$L_{ij} = -\frac{1}{(n-2)} M_{ij} + \frac{1}{(n-4)(n-2)} (M_{ij} - M_{ji})$$

(30) 

Also, from (15), (17), (27) and (30) we obtain

$$L_{[ij]} = -\frac{1}{n} R_{[ij]} = -\nabla_{[T_{ij}]}$$

(31) 

**Definition 2.2.** An $n$-dimensional ($n = 2, m > 2$) Kaehlerian Weyl space is said to be generalized conformally recurrent if its conformal curvature tensor $C_{ijkl}$ of weight 2 satisfies the condition

$$\hat{V}_r C_{ijkl} = A C_{ijkl} + B_r G_{ijkl},$$

(32) 

where $A$, $B$ are two 1-forms of weights 0 and $-2$, respectively and $C_{ijkl} = C_{ijkl}^h g_{kl}$.

We can state the following theorem concerning the generalized conformally recurrent Kaehlerian Weyl space.

**Theorem 2.1.** A Kaehlerian Weyl space $KW_n$ is generalized conformally recurrent if and only if it is generalized recurrent.

**Proof:** Assume $KW_n$ is generalized conformally recurrent. Transvecting (26) by $g_{kl}$, we get

$$C_{ijkl} = R_{ijkl} + g_{kl} L_{ij} - g_{ij} L_{ik} + g_{ij} L_{ij} - 2 g_{ij} L_{[jk]}.$$  

(33) 

By taking the prolonged covariant derivative of (33) and using (32), (33) we have,
\[ \hat{\nabla}_i R_{ijk} + g_{il} \hat{\nabla}_i L_{lj} - g_{lj} \hat{\nabla}_i L_{ij} + g_{ij} \hat{\nabla}_i L_{lk} - g_{il} L_{lj} - 2 g_{lj} L_{ij} + g_{ij} L_{lk} = A_i [ R_{ijk} + g_{il} L_{lj} - g_{lj} L_{ij} + g_{ij} L_{lk} - g_{il} L_{lj} - 2 g_{lj} L_{ij} + g_{ij} L_{lk} ] + B_i G_{ijk}. \]  

(34)

Transvecting (34) by \( F^{jk} \) and using (16), (17), (18) and (30) we obtain

\[ \begin{align*}
(3) = & \frac{(n-3)}{(n-2)} \hat{\nabla}_i H_{ij} + \frac{1}{(n-2)} \hat{\nabla}_i H_{ij} \\
\ & - \frac{1}{(n-1)(n-2)} R_{ij} + \frac{1}{(n-4)} g_{ij} F^{\beta} \hat{\nabla}_j M_{jk} \\
\ & = A_i \left[ \frac{(n-3)}{(n-2)} H_{ij} + \frac{1}{(n-2)} H_{ij} \\
\ & - \frac{1}{(n-1)(n-2)} R_{ij} + \frac{1}{(n-4)} g_{ij} F^{\beta} M_{jk} \\
\ & \right] \\
\ & + \frac{1}{2} B_i G_{ijk} F^{\beta}.
\end{align*} \]

(35)

Also, transvecting (35) by \( F^{il} \) and using (20) we get

\[ \hat{\nabla}_i R = A_i R + \frac{n(1-n)}{(n-2)} B_i. \]

(36)

On the other hand, multiplying (35) by \( g^{il} \) and using (18) we obtain

\[ F^{jk} \hat{\nabla}_j M_{jk} = A_i F^{jk} M_{jk} + \frac{1}{2} B_i G_{ijk} F^{\beta} g^{ij}. \]

(37)

Since \( G_{ijk} F^{jk} g^{ij} = 0 \)

\[ F^{jk} \hat{\nabla}_j M_{jk} = A_i F^{jk} M_{jk}. \]

(38)

Substituting (36) and (38) in (35) we get

\[ \hat{\nabla}_i H_{ij} = A_i H_{ij} + \frac{n-1}{n-2} B_i F_{ij}. \]

(39)

By multiplying (39) by \( F^{ji} \) we obtain,

\[ \hat{\nabla}_i M_{ij} = A_i M_{ij} - \frac{n-1}{n-2} B_i g_{ij}. \]

(40)

Substituting (36) and (40) into (30) gives us,

\[ \hat{\nabla}_i L_{ij} = A_i L_{ij} + \frac{1}{2(n-2)} B_i g_{ij}. \]

(41)

Using (41), (34) reduces to

\[ \hat{\nabla}_i R_{ijk} = A_i R_{ijk} + \frac{n-1}{n-2} B_i G_{ijk}. \]

(42)

Hence, the necessary part of the theorem is proved.

Conversely, assume that \( KW_n \) is generalized recurrent with 1-forms \( A \) and \( B \), i.e:

\[ \hat{\nabla}_i R_{ijk} = A_i R_{ijk} + B_i G_{ijk}. \]

(43)

Multiplying (43) by \( F^{ij} \) and using (16) we get

\[ \hat{\nabla}_i H_{ij} = A_i H_{ij} + B_i F_{ij}. \]

(44)

Transvecting (44) by \( F^{ij} \) we obtain

\[ \hat{\nabla}_i R = A_i R - n B_i. \]

(45)

Also, since \( H_{ij} = M_{ij} F_{ij} \) and (44) we get

\[ \hat{\nabla}_i M_{ij} = A_i M_{ij} - B_i g_{ij}. \]

(46)

Hence from (30), (44), (45), we get

\[ \hat{\nabla}_i L_{ij} = A_i L_{ij} + \frac{1}{2(n-1)} B_i g_{ij}, \]

(47)

and

\[ \hat{\nabla}_i L_{ij} = A_i L_{ij}. \]

(48)

Taking prolonged covariant derivative of (33) and using (47) and (48) we get,

\[ \hat{\nabla}_i C_{ijk} = A_i C_{ijk} + \frac{(n-2)}{(n-1)} B_i G_{ijk}, \]

(49)

which implies that sufficiency part of the theorem is proved.

Remark: If, in particular, \( B_i \) is zero we obtain the results in [10].

3. Generalized conharmonically recurrent Kaehlerian Weyl spaces

Let \( W_n(g_{ij}, T_k) \) and \( \tilde{W}_n(g_{ij}, \tilde{T}_k) \) be two Weyl spaces with connections \( \nabla_k \) and \( \tilde{\nabla}_k \), respectively and let the map \( \tau: W_n \to \tilde{W}_n \) be a conformal mapping. As a special case, let the transformed expressions of the fundamental metric tensor \( g_{ij} \)
and the coefficients of Weyl connection $\Gamma^i_{kl}$ be in the following forms

$$ g^{i} = g_{j}, \quad g^{ij} = g^{ji}, $$

$$ \tilde{\Gamma}^i_{kl} = \Gamma^i_{kl} + \delta^i_l P_k + \delta^i_k P_l - g_{kl} g^{im} P_m $$

where the vector $P_k$ is called conformal mapping vector such that

$$ P_k = T_k - \tilde{T}_k. $$

Let $A$ be the differentiable harmonic function with weight $\{p\}$ defined by

$$ \overrightarrow{A} = e^{\frac{1}{2} \int P_d u} A, \quad c = \frac{2(1 - p) - n}{2} $$

and the vector field $P_k$ satisfies

$$ g^{ik} \nabla_k P_i + \frac{1}{2} (n - 2) P^k P_k = 0. $$

If a conformal transformation with $P_k$ satisfying (53) transforms a harmonic function into a harmonic function then it is called conharmonic transformation.

It is shown that the conharmonic curvature tensor $K_{ijk}$ of $W_n(g_{ij}, T_k)$ can be given by [13]

$$ K_{ijk} = C_{ijk} + \frac{R}{(n - 2)(n - 1)} G_{ijk}, \quad n > 2 $$

where $C_{ijk}$ is the conformal curvature tensor of Weyl space and $G_{ijk} = g_{ij} g_{ik} - g_{ik} g_{ij}.$

**Definition 3.1.** An $n$-dimensional $(n = 2m)$ Kaehlerian Weyl space is said to be generalized conharmonically recurrent if its conharmonic curvature tensor $K_{ijk}$ of weight 2 satisfies the condition

$$ \nabla_i K_{ijk} = A_i K_{ijk} + B_i G_{ijk}, $$

where $A$ and $B$ are non-zero two 1-forms of weights 0 and $-2$, respectively.

**Theorem 3.1.** A Kaehlerian Weyl space is generalized recurrent if and only if it is generalized conharmonically recurrent.

**Proof:** Assume $KW_n$ is generalized recurrent, then

$$ \nabla_i K_{ijk} = A_i K_{ijk} + B_i G_{ijk} $$

From Theorem 2.1 and (49) we have

$$ \nabla_i C_{ijk} = A_i C_{ijk} + \frac{n - 2}{(n - 1)} B_i G_{ijk}. $$

By taking the prolonged covariant derivative of (54) we get

$$ \nabla_i K_{ijk} = \nabla_i C_{ijk} + \frac{1}{(n - 1)(n - 2)} B_i G_{ijk} \nabla_i R. $$

Using (57) in (58) we obtain,

$$ \nabla_i K_{ijk} = A_i C_{ijk} + \frac{n - 2}{(n - 1)} B_i G_{ijk} + \frac{1}{(n - 1)(n - 2)} G_{ijk} \nabla_i R. $$

Considering (45), (59) becomes

$$ \nabla_i K_{ijk} = A_i K_{ijk} + \frac{1}{(n - 1)(n - 2)} B_i G_{ijk} (A_i R - nB_i). $$

Therefore, from (54) we get

$$ \nabla_i K_{ijk} = A_i K_{ijk} + \frac{1}{(n - 1)(n - 2)} B_i G_{ijk}. $$

Hence the necessary part of the theorem is proved.

Conversely, assume that

$$ \nabla_i K_{ijk} = A_i K_{ijk} + B_i G_{ijk}, $$

then, (58) becomes

$$ A_i K_{ijk} + B_i G_{ijk} = \nabla_i C_{ijk} + \frac{1}{(n - 1)(n - 2)} G_{ijk} \nabla_i R. $$

Using (54), we have

$$ A_i \left[ C_{ijk} + \frac{R}{(n - 1)(n - 2)} G_{ijk} \right] + B_i G_{ijk} = \nabla_i C_{ijk} + \frac{1}{(n - 1)(n - 2)} G_{ijk} \nabla_i R. $$

Multiplying both sides of (64) by $F^{jk}$ and using (16), we get
\[ \begin{align*}
2 & \left( \frac{n-3}{n-2} H_y + \frac{1}{(n-2)(n-2)} R F_y \right) \\
& + \left( \frac{1}{(n-4)} g_y F^{\beta} M_{\beta} \right) \\
& + \left( \frac{1}{(n-1)(n-2)} g_y F^{\beta} \hat{M}_{\beta} \right) \\
& = 2 \frac{(n-3)}{(n-2)} \hat{V}_y H_y + \frac{1}{(n-2)} \hat{V}_y H_y - \frac{1}{(n-1)(n-2)} F_y \hat{V}_y R \\
& + \frac{1}{(n-4)} g_y F^{\beta} \hat{V}_y M_{\beta} \\
& + \frac{1}{(n-1)(n-2)} g_y F^{\beta} \hat{V}_y \hat{M}_{\beta} \\
& = 2 \hat{V}_y R = A_y R - \frac{n(n-2)}{(n-4)} B_y, \quad (n > 4). \\
& \text{(66)}
\end{align*} \]

Since \( g_{ijk} F^{ij} F^{jk} = 2n \), by transvecting (20) with \( F^{ij} \) and using (20) we obtain
\[ \hat{V}_y R = A_y R - \frac{n(n-2)}{(n-4)} B_y, \quad (n > 4). \]

Hence, by using (55) and (58) we get
\[ \hat{V}_y C_{ijk} = A_y C_{ijk} + \frac{n(n-2)}{(n-1)(n-4)} B_y G_{ijk}, \quad (n > 4). \]

From Theorem 2.1.
\[ \hat{V}_y R_{ijk} = A_y R_{ijk} + \frac{(n-1)}{(n-4)} B_y G_{ijk}, \quad (n > 4). \]

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