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Exponential map and invariant form on generalized Lie groups

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Abstract

In this paper, by definition of exponential map of the Lie groups the concept of exponential map of generalized Lie groups is introduced. This has a powerful generalization to generalized Lie groups which takes each line through the origin to an order product of some one-parameter subgroup. We show that the exponential map is a C^∞- map. Also, we prove some important properties of the exponential map for generalized Lie groups. Under the identification, it is shown that the derivative of the exponential map is the identity map. One of the most powerful applications of these exponential maps is to define generalized adjoint representation of a top space, so we show that this representation is a C^∞- map. Finally, invariant forms are introduced on a generalized Lie group. We prove that every left invariant k-form are introduced on a generalized Lie group T with the finite number of identity elements is C^∞. At the end of this paper, for compact connected generalized Lie group T with the finite number of identity elements and dimension n, we show that every left invariant n-form on T is right invariant n-form.

***Keywords**:* Lie group; exponential map; differential invariant form

1. Introduction

Certain manifolds such as the torus T^2 also have in multiplication structure a generalized group structure [1, 2], moreover the generalized group operations are C^∞. A manifold such as this is called generalized Lie group or top space [3-5]. This paper is a compendium of important structures of top space. Top spaces are a particularly important class of manifolds.

In this generalized setting, several authors ([2]) studied various aspects and concepts of generalized groups and top spaces. Also, some authors consider another useful generalization of Lie groups. *Gheorghe* considered the main structures of Lie groupoid, and encountered the important instruction between groupoid and geometric theory [6, 7]. Note that this generalization is different structure from the Lie groupoid introduced by *Haefliger* [8] and has been studied by *Gheorghe* [6, 7].

The concept of top space is defined by a set of axioms. In the axioms, the (only) difference lies in comparison with notorious axioms of an ordinary Lie group [2], in the definition of unit element. Here, contrary to the situation in Lie groups [9], the unit element is not necessarily unique. Rather, it is in general, element-dependent. So, for each element t in top space T there exists unique $e(t)$ in T such that $te(t) = e(t)t = t$ plus the requirement that the map $e: T \to T$ defined by $t \longmapsto e(t)$ be homomorphism (Lie group is a special case of a top space, where unit element is unique), but there are top spaces which are not Lie groups [3]. So, top spaces are beautiful important and useful manifolds, because they have one foot in each of the two great divisions of mathematics, algebra and geometry. Their algebraic properties are derived from the generalized group axioms. Their geometric properties are derived from the identification of generalized group operations with points in a topological space. The rigidity of their structure comes from the continuity requirements of the generalized group inversion map.

In section 2 of this paper we introduce the exponential map of top spaces. This has a powerful generalization to top spaces which takes each line through the origin to an order product of some one-parameter subgroup. The derivative of the exponential map is explored. One of the most powerful applications of these exponential maps is shown i.e. generalized adjoint representation is a smooth map.

Section three is devoted to studying invariant forms on a top space. We show that the set of all left invariant k-forms on top space T is isomorphism into the space of all alternating k-tensors on the Lie algebra τ of top space T [5], by this we give an upper bound for the set of all left invariant k-forms on T. Finally, for each compact connected top space T of dimension n, we consider

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the relation between left invariant n-forms on T and right invariant n-forms on T.

Now, we recall the definition of top space and then present a characterization of some top spaces.

A top space is a nonempty d-dimensional Hausdorff, second countable manifold T admitting smooth operations

$$m: T \times T \to T \quad \text{and} \quad i: T \to T$$
$$m(t_1, t_2) = t_1 t_2 \qquad i(t) = t^{-1}$$

which satisfy the following conditions:

a) $(ts)r = t(sr)$, for all $t, s, r \in T$.

b) For each $t \in T$ there exists a unique $e(t)$ in T such that $te(t) = e(t)t = t$.

c) For each $t \in T$ there exists the inverse element $s \in T$ such that $ts = st = e(t)$.

and it holds

d) $e(t_1 t_2) = e(t_1)e(t_2)$, for all $t_1, t_2 \in T$.

It is easy to show the inverse element s is unique and by t^{-1} we mean the inverse of t.

Let T and S be top spaces, a C^∞- map $f: T \to S$ is called top space homomorphism, if

$$f(t_1 t_2) = f(t_1)f(t_2),$$

for all $t_1, t_2 \in T$. [10]

Note that each Lie group is top space, but there are top spaces which are not Lie groups.

Example 1. The Euclidean subspace $R^* = R - o$ with the product $ab = a|b|$ is a top space with the identity elements $e(T) = \{+1, -1\}$.

Example 2. The n-dimensional Euclidean space R^n with the product:

$$\big((a_1, \dots, a_n), (b_1, \dots, b_n)\big) \mapsto \Big(\frac{na_1 + \sum b_i}{n}, \dots, \frac{na_n + \sum b_i}{n}\Big)$$

is a top space which is not a Lie group.

Note that in this paper, by the following symbol we mean disjoint union.

$$\mathring{\bigcup}$$

Theorem 3. [11] If T is a top space with the finite number of identity elements, then

$$T = \mathring{\bigcup_{t \in T}} \big(e^{-1}(e(t))\big)$$

where $(e^{-1}(e(t)))'$ s are diffeomorphic Lie groups and are defined by:

$$\big(e^{-1}(e(t))\big) = \{s \in T : e(s) = e(t)\},$$

for all $s \in T$.

In the paper [11], some low dimensional top spaces are characterized. Also, in the paper [12], the properties of top spaces in the special cases are considered.

2. Exponential Map on Top Spaces

Let T be a top space, τ a Lie algebra of T, [10]. Let $Y \in \tau$ and $\gamma^{t_0 Y}$ be the integral curve of Y starting at the identity element $e(t_0) \in e(T)$, where $e(T)$ show the set of all identity elements of T. Then t_0-exponential map $exp_{t_0}: \tau \to T$ is the map which assigns $\gamma^{t_0 Y}(1)$ to Y, we write $exp_{t_0}(Y) = \gamma^{t_0 Y}(1)$.

Lemma 4. Let T be a top space. Then $\gamma^{t_0(rY)}(1) = \gamma^{t_0 Y}(r)$, where $\gamma^{t_0(rY)}$ and $\gamma^{t_0 Y}$ are the integral curves of rY and Y respectively, starting at the identity element $e(t_0)$, where $t_0 \in T$, $r \in R$ and $Y \in \tau$.

Proof: Obviously

$$\gamma^{t_0(rY)}(0) = \gamma^{t_0 Y}(0) = e(t_0)$$

also we have:

$$\frac{d}{dr}\Big|_{r=0}\big(\gamma^{t_0 Y}(ar)\big) = a\left(\frac{d}{dr}(\gamma^{t_0 Y})\right)(0)$$
$$= aY\big(e(t_0)\big).$$

Since $\gamma^{t_0(aY)}(r)$ is the integral curve of vector field aY starting at the identity element $e(t_0)$, then

$$\frac{d}{dr}\Big|_{r=0}\left(\gamma^{t_0(aY)}(r)\right) = aY\big(e(t_0)\big).$$

By existence and uniqueness theorem of the solution of the differential equation for manifolds, we deduce

$$\gamma^{t_0 Y}(ar) = \gamma^{t_0(aY)}(r).$$

Now, by replacing "a" by "r" we have $\gamma^{t_0 Y}(ar) = \gamma^{t_0(rY)}(a)$. Let $a = 1$, then $\gamma^{t_0(rY)}(1) = \gamma^{t_0 Y}(r)$.

Theorem 5. Let T_1 and T_2 be top spaces with the finite number of identity elements. Let $\varphi: T_1 \to T_2$ be a top space homomorphism and $e^{-1}\big(e(t_0)\big)$, $e^{-1}\big(e(\varphi(t_0))\big)$ be simply connected topological subspace of T_1 and T_2, respectively. Then $exp_{\varphi(t_0)} o d\varphi = \varphi o exp_{t_0}$.

Proof: At first, by Theorem 3, we know that for top space T we have:

$$T = \overset{\circ}{\bigcup_{e(t)\in T}} (e^{-1}(e(t)))$$

also, $e^{-1}(e(t))' s$ are diffeomorphic Lie groups for every $t \in T$.

Let $\psi_1: R \to T_1$ and $\psi_2: R \to T_2$ are defined by:

$$\psi_1(r) = \varphi\left(exp_{t_0}(rY)\right)$$

and

$$\psi_2(s) = exp_{\varphi(t_0)}(sd\varphi(Y)),$$

respectively, where $Y \in \tau_1$(the Lie algebra of T_1).

By the well known chain rule theorem we have:

$$\begin{aligned} d\psi_1(1) &= d(\varphi o exp_{t_0})(rY)|_{r=1} v \\ &= (d\varphi o dexp_{t_0})(rY)|_{r=1} \\ &= d\varphi(Y). \end{aligned}$$

Since $e^{-1}(e(t_0))$ is a simply connected Lie group and $\psi_1|_{r=1} = \psi_2|_{r=1}$, then by existence and uniqueness of solution of differential equation for manifolds we have:

$$exp_{\varphi(t_0)} o d\varphi = \varphi o exp_{t_0}.$$

Theorem 6. In a neighborhood of 0 in the Lie algebra τ, the map exp_{t_0} is a diffeomorphism.

Proof: By the inverse function theorem on manifolds it is enough to show that $dexp_{t_0}$ is a surjective map at 0. We have $dexp_{t_0}: \tau \to T_{e(t_0)}$, where $T_{e(t_0)}$ is the tangent space of T at $e(t_0)$. Let $s \in T_{e(t_0)}$, then there is a left invariant vector field $Y \in \tau$ such that $Y(e(t_0)) = s$.

Now, let $\gamma^{t_0 Y}$ be the integral curve of vector field Y, then

$$\frac{d\gamma^{t_0 Y}(r)}{dr} = Y(r).$$

Also,

$$exp_{t_0}(rY) = \gamma^{t_0 Y}(r),$$

where $r \in R$.

Since τ is a vector space and $Y \in \tau$. Then $rY \in \tau$ and we have:

$$dexp_{t_0}(rY)|_{r=0} = d\gamma^{t_0 Y}(r)|_{r=0} = Y(e(t_0)) = s.$$

Therefore $dexp_{t_0}$ is a surjective map.

Now, we define a powerful map Exp on the Lie algebra τ of a top space T.

Definition 7. Let T be a top space with the order finite number of identity elements $e(t_1), \dots, e(t_n)$, also $s's'' = s''s'$, for all $s' \in e^{-1}(e(t_i))$ and $s'' \in e^{-1}\left(e(t_j)\right)$, where $i, j \in \{1, \dots, n\}$ and $i \neq j$. The map Exp is defined by order product $Exp(Y) = \gamma^{t_1 Y}(1) \dots \gamma^{t_n Y}(1)$, where $\gamma^{t_i Y}(1)$ is the integral curve of Y starting at the identity element $e(t_i)$, for all $i = 1, \dots, n$.

Note that, if T is a Lie group, then the definition of exponential map on top spaces agrees to the definition of exponential map on Lie groups.

Theorem 8. Let T be a top space with the order finite number of identity elements $e(t_1), \dots, e(t_n)$. Then

$$Exp(rY) = \gamma^{t_1 Y}(r) \dots \gamma^{t_n Y}(r).$$

Proof: By Lemma 4, we know that $\gamma^{t_i(rY)}(1) = \gamma^{t_i Y}(r1)$, for every $i \in \{1, \dots, n\}$ and $r \in R$. Then

$$\begin{aligned} Exp(rY) &= \gamma^{t_1(rY)}(1) \dots \gamma^{t_n(rY)}(1) \\ &= \gamma^{t_1 Y}(r) \dots \gamma^{t_n Y}(r). \end{aligned}$$

Theorem 9. For top space T with the Lie algebra τ, the exponential map Exp is a C^∞- map.

Proof: We know that the exp_{t_i} is a C^∞- map, for all $i \in \{1, \dots, n\}$. Since T is a top space, then the product map $exp_{t_1} \dots exp_{t_n}$ is a C^∞- map.

Theorem 10. Let T be a top space with the order finite number of identity elements $e(t_1), \dots, e(t_n)$ and τ be the Lie algebra of T. Then

a) $Exp(-rY) = (Exp(rY))^{-1}$.

b) $dExp: \tau_0 \to T_{e(t_1 \dots t_n)}$ is the identity map, under the canonical identifications of both τ_0 with τ and $T_{e(t_1 \dots t_n)}$ with the Lie algebra of the Lie group $e^{-1}(e(t_1 \dots t_n))$.

Where $r \in R$ and $Y \in \tau$.

Proof: For part ***(a)***, since T is a top space with the finite number of identity elements, let $e(T) = \{e(t_1), \dots, e(t_n)\}$ be an order set of all identities elements. Also

$$T = \overset{\circ}{\bigcup_{e(t)\in T}} (e^{-1}(e(t)))$$

we know that $e^{-1}(e(t))$ is Lie group with the identity element $e(t)$, then

$$exp(-rY) = (exp(rY))^{-1}.$$

Then

$$Exp(-rY) = exp_{t_1}(-rY) \dots exp_{t_n}(-rY)$$

$= (exp_{t_1}(rY))^{-1} \dots (exp_{t_n}(rY))^{-1} (Exp(rY))^{-1}.$

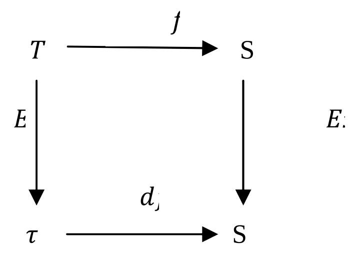
Part ***(b)*** follows immediately from the Lie group theory.

Let T be a top space with the finite number of identity elements and τ be its Lie algebra. Let $Y \in \tau$ and $H(e(t), r) = exp_t(Y)$, by theorems in differential equations on the dependence of solutions on initial conditions, for every $t \in T$ there is a positive number ε and a neighborhood V of $e(t)$ such that H is defined and C^∞- map on $\left(V \cap e^{-1}(e(t))\right) \times (-\varepsilon, \varepsilon)$. Since the real numbers used as the second variable of H are parameter value along a curve, by uniqueness on initial condition in differential equation, they satisfy an additive property, i.e. if $v \in V$, $r_1, r_2, r_1 + r_2 \in (-\varepsilon, \varepsilon)$, then $H(H(v, r_1), r_2) = H(v, r_1 + r_2)$. Also, if we are given a C^∞- map having domain of the same type as H and satisfying the additive property, we get a vector field having H, for this let ξ_v be defined by $\xi_v(r) = (v, r)$. Then at v the value of the vector field is $Y(v) = d(Ho\ \xi_v)(0)$.
So, we have the following theorem:

Theorem 11. Let T be a top space with the order finite number of identity elements $e(t_1), \dots, e(t_n)$, Y be a left invariant vector field on T and $Exp_{t_i}(Y)(r) = \gamma^{t_iY}(r)$, where $i = 1, \dots, n$. Then there is neighborhood V of $e(t_i)$ and open interval $(-\varepsilon, \varepsilon)$ such that $Exp_{t_i}(Y)(r)$ is a C^∞- map, where $t \in \left(V \cap e^{-1}(e(t_i))\right)$, $r \in (-\varepsilon, \varepsilon)$ and $i = 1, \dots, n$.

The exponential map on top spaces is also natural in the following sense:

Theorem 12. Given any two top spaces T and S, for every top space homomorphism $f: T \to S$ that preserves ordering of identities, the following diagram commutes:



where τ is the Lie algebra of T and s is the Lie algebra of S.

Proof: It is enough to show that $foExp_T(Y) = Exp_S odf(Y)$, where $Y \in \tau$.

$$\begin{aligned} foExp_T(Y) &= f\left(\gamma^{t_1Y}(1) \dots \gamma^{t_nY}(1)\right) \\ &= f(\gamma^{t_1Y}(1)) \dots f\left(\gamma^{t_nY}(1)\right). \end{aligned}$$

Since $e^{-1}(e(t_i))$ is Lie group, for every $i = 1, \dots, n$, then

$$\begin{aligned} &f\left(\gamma^{t_1Y}(1)\right) \dots f\left(\gamma^{t_nY}(1)\right) \\ &= \left(\gamma^{t_1df|_1Y}(1)odf|_1\right) \dots \left(\gamma^{t_ndf|_1Y}(1)odf|_1\right) \\ &= Exp_S odf|_1(Y). \end{aligned}$$

Then $foExp_T(Y) = Exp_S odf(Y)$.

Now, by a representation as a map of top space T into a Lie algebra of T we show that generalized representation is a C^∞- map. Let T be a top space with the finite number of identities elements $e(t_1), \dots, e(t_n)$, for every $t_0 \in T$ we have a top space homomorphism $A_{t_0}: T \to e^{-1}(e(t_0)) \subseteq T$ defined by $A_{t_0}(t) = t_0 t (t_0)^{-1}$. Moreover, the map $t_0 \longmapsto A_{t_0}$ is a homomorphism. The map: $T \longrightarrow GL(\tau)$ defined by $GAD(t_0) = dA_{t_0}$ is called generalized adjoint representation of T.

Theorem 13. GAD is a C^∞-representation of T into Lie algebra τ of T.

Proof: GAD is evidently a top space homomorphism, so that it suffices to show that it is C^∞, and in fact, to show that it is C^∞ in canonical coordinate neighborhood. First, we note that for a fixed point $t_0 \in T$ the map $t_0 \longmapsto A_{t_0}$is C^∞. In fact, it is the composition of maps involving top space operations which are

$$t \longmapsto ((t_0)^{-1}, t, t_0) \longmapsto (t_0)^{-1} t t_0.$$

Now, by Theorem 11, for every y in canonical coordinate neighborhood we have $y = Exp_T(X)$, and by Theorem 12 we have:

$$\begin{aligned} A_{t_0}(y) &= A_{t_0}\left(Exp_T(X)\right) \\ &= Exp_T\left(d(t_0)(X)\right) \\ &= Exp_T(GAD(t_0)X). \end{aligned}$$

If we choose a basis $X_1, \dots, X_d$ of τ, then $GAD(x)$ is given in terms of matrix (b_{ij}), where

$$GAD(x)X_j = \sum b_{ij} X_i.$$

Now, for $y = \exp(tX_j)$ we get $A_{t_0}(y)$ are $tb_{ij}(x)$, $i = 1, \dots, d$ being defined for t sufficiently small. Since A_{t_0} is C^∞ in x, this means that $b_{ij}(x)$ is C^∞ in x for all i, j that is, $GAD(x)$ is C^∞.

3. Invariant Forms on Top Spaces

In this section we introduce left and right invariant differential forms on top space. Just as there are left invariant vector fields on a top space T [5], so there are also left invariant differential forms. For $t \in T$, let $l_t: T \longmapsto T$ be left multiplication by t. A k-form ω on T is said to be left invariant if $(l_t)^*(\omega) = \omega$, for all $t \in T$. This means for all $t, t_0 \in T$, $(l_t)^*(\omega_{tt_0}) = \omega_t$, where $(l_t)^*$ show the dual of left multiplication l_t. Then, we have:

Theorem 14. Let T be a top space with the finite number of identity elements $e(t_1), \dots, e(t_n)$. Then a left invariant k-form is uniquely determined by its values at the identity elements $e(t_1), \dots, e(t_n)$.

Proof: It is clear that $(l_{s^{-1}})^*(\omega_{e(t)}) = \omega_s$, where $e(t)$ is the identity element of s, i.e. $s \in e^{-1}(e(t))$. Since, by Theorem 3, the top space T can be written by disjoint union of $e^{-1}(e(t_i))$, where $i \in \{1, \dots, n\}$, the proof is completed.

Theorem 15. Let T be a top space with the finite number of identity elements . Then every left invariant k-form ω on top space T is C^∞.

Proof: It would be evident that $e^{-1}(e(t))$ is a Lie group, for all $t \in T$. Since $e^{-1}(e(t))$ is a Lie group, then ω is a C^∞ k-form on $e^{-1}(e(t))$, by the same process of the previous theorem ω is C^∞.

Similarly, a k-form ω on top space T is said to be right invariant if $(r_t)^*(\omega) = \omega$, for all $t \in T$, where $r_t: T \longmapsto T$ is defined by $r_t\ (t_0) = tt_0$, for all $t_0 \in T$.

Corollary 16. Every right invariant k-form on top space T, with the finite number of identity elements, is C^∞.

Note that in the following theorem by $\Omega^k(T)^T$ we mean the set of all left invariant k-form on T.

Theorem 17. Let τ be the Lie algebra of top space T with the finite number of identity elements. Let $\wedge^k(\tau^*)$ be the space of all alternating k-tensors on the Lie algebra τ. Then $\Omega^k(T)^T$ is isomorphic into $\wedge^k(\tau^*)$.

Proof: Suppose $\phi: \Omega^k(T)^T \to \wedge^k(\tau^*)$ is defined by $\phi(\omega) = \omega_{e(t)}$, where $t \in T$. Since
$\phi(\omega + \omega') = (\omega \wedge \omega')_{e(t)} = \omega_{e(t)} \wedge \omega'_{e(t)}$,
where $\wedge$ denote the wedge product, then ϕ is a linear map.

Now, let $\phi(\omega) = \phi(\omega')$, then $\omega_{e(t)} = \omega'_{e(t)}$. Then for every $t_0 \in e^{-1}(e(t))$, we have:

$$(l_{t_0})^*(\omega_{e(t)}) = (l_{t_0})^*(\omega'_{e(t)}),$$

thus $\omega_t = \omega'_{t_0}$.
Let $t_1 \in T - e^{-1}(e(t))$, then $(r_{t_1})^*\ (l_{t_1})^*(\omega_{t_0}) = (r_{t_1})^*(l_{t_1})^*(\omega'_{t_0})$, then $\omega_{t_1 t_0 t_1} = \omega'_{t_1 t_0 t_1}$, and so $\omega_{e(t_1 t_0 t_1)} = \omega'_{e(t_1 t_0 t_1)}$, since $e(t_1 t_0 t_1) = e(t_1)$, then $\omega_{e(t_1)} = \omega'_{e(t_1)}$. Thus $\omega_s = \omega'_s$, where $s \in e^{-1}(e(t_1))$. By the same process we can show that $\omega_r = \omega'_r$, for all $r \in T$. Therefore $\omega = \omega'$, on T.

Then ϕ is an injective map, and this completes the proof of Theorem 17.

So, for a top space T with the finite number of identity elements we have:

Corollary 18. $dim\Omega^k(T)^T \leq dim \wedge^k(\tau^*)$.

Lemma 19. Let T be a top space of dimension n, with the finite number of identity elements and the Lie algebra τ. Then for each $t_0 \in T$, the differential at the identity $e(t_0)$ of the map $A_{t_0} = l_{t_0} o r_{t_0^{-1}}: T \longrightarrow T$ is a linear transformation $dA_{t_0}: T_{e(t_0)} \longrightarrow T_{e(t_0)}$. Also, the map $GAD: T \to GL(T_{e(t_0)}) \subseteq GL(\tau)$ defined by $GAD(t_0) = dA_{t_0}$ is a top space homomorphism.

Proof: By chain rule theorem we have $dA_{t_0} = dl_{t_0} o\ dr_{t_0^{-1}}$, and then $dA_{t_0}$is a linear map. Since the Lie algebra τ of T is a vector space, and dl_{t_0} and $dr_{t_0^{-1}}$ are injective and surjective linear transformation on Lie group $e^{-1}(e(t_0))$, then dA_{t_0} is an isomorphism of vector spaces. Now, let $(U, y^1, \dots, y^n)$ be a chart about $e(t_0)$ in T. Relative to this chart, the map dA_{t_0} at $e(t_0)$ is represented by the Jacobian matrix. Since $A_{t_0}(t) = t_0 t t_0^{-1}$ is a C^∞- map, all entires of Jacobian matrix are C^∞ and then $GAD(t_0)$ is a C^∞- map of t_0.

Theorem 20. Suppose T is a compact connected top space of dimension n, with a finite number of identity elements. Then every left invariant n -form on T is right invariant n -form.

Proof: Let ω be a left invariant n -form on T. For any $t_0 \in T$, it is easy to show that $(r_{t_0})^*\ \omega$ is also left invariant. Since T is a manifold of dimension n, then $dim\Omega^k(T)^T = dim\wedge^k(\tau^*) = 1$, $(r_{t_0})^*\ \omega = g(t_0)\omega$, for some nonzero real constant $g(t_0)$ depending on t_0.

Now, we show that $g: T \to R - 0$ is a homomorphism of top space T into Lie group $R - 0$ with the the product of nonzero real numbers (Note that each Lie group is top space).
g is a homomorphism because:
$g(t_0)g(t_0')\omega = (r_{t_0})^*\ (r_{t_0'})^*\omega = g(t_0 t_0')\omega$, then $g(t_0)g(t_0')\omega = g(t_0 t_0')\omega$, since ω is a n -form and $dim\wedge^k(\tau^*) = 1$, then $g(t_0)g(t_0') = g(t_0 t_0')$.

To show that g is a C^∞- map, we have:

$$g(t_0)\omega_{e(t_0)} = ((r_{t_0})^*\ \omega)_{e(t_0)} = (r_{t_0})^*\ \omega_{e(t_0)}$$

$$= (r_{t_0})^* (l_{t_0^{-1}})^* (\omega_{e(t_0)}),$$

thus $g(t_0)$ is induced by map $dA_{t_0}: T \to T$, then we have $g(t_0) = det\, (dA_{t_0}(t_0))$, since dA_{t_0} is a C^∞- map of t_0 (Lemma 19), then g is C^∞.

As the continuous image of a compact connected set T, the set $g(T) \subseteq R - 0$ is compact connected, and then $g(T) = 1$.

Hence $(r_{t_0})^* \omega = \omega$, for all $t_0 \in T$, and the proof is completed.

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