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Generalized fuzzy filters in ordered semigroups

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Abstract

Fuzzy semigroup theory concentrates on theoretical aspects, but also includes applications in the areas of fuzzy coding theory, fuzzy finite state machines, and fuzzy languages. In this paper, we introduce the concept of an (α,β)-fuzzy filter of an ordered semigroup S, where $\alpha,\beta \in \{\in, q, \in\vee q, \in\wedge q\}$ with $\alpha \neq \in\wedge q$. Since the concept of $(\in,\in\vee q)$-fuzzy filter is an important and useful generalization of ordinary fuzzy filter, we discuss some fundamental aspects of $(\in,\in\vee q)$-fuzzy filters. An $(\in,\in\vee q)$-fuzzy filter is a generalization of the existing concept of a fuzzy filter. The concept of $(\overline{\in},\overline{\in}\vee \overline{q})$-fuzzy filters is also introduced and some related properties are investigated. The relationships among ordinary fuzzy filters, $(\in,\in\vee q)$-fuzzy filters and $(\overline{\in},\overline{\in}\vee \overline{q})$-fuzzy filters are discussed. We discuss the concept of $(\in,\in\vee q)$-fuzzy left (right and bi)-filters and provide some characterization theorems. Finally, we extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy left (right and bi)-filter with thresholds of S.

Keywords*:* Semigroup; ordered semigroup; fuzzy set; filter; generalized fuzzy filter

1. Introduction

Mordeson et. al. in [1] presented an up-to-date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. The book concentrates on theoretical aspects, but also includes applications in the areas of fuzzy coding theory, fuzzy finite state machines, and fuzzy languages. Basic results on fuzzy subsets, semigroups, codes, finite state machines, and languages are reviewed and introduced, as well as certain fuzzy ideals of a semigroup and advanced characterizations and properties of fuzzy semigroups. The idea of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [2, 3], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das (see [2]) gave the concepts of (α,β)-fuzzy subgroups by using the " belongs to" relation ($\in$) and " quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in,\in\vee q)$-fuzzy subgroup. In particular, $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup [4]. It is now natural to investigate a similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in

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mind, Kuroki [5] introduced the notion of fuzzy bi-ideals in semigroups. Jun and Song [6] discussed general forms of fuzzy interior ideals in semigroups [7, 8]. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [9] and gave some properties of fuzzy bi-ideals in terms of $(\in,\in\vee q)$-fuzzy bi-ideals. Jun et al. [10] gave the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterized regular ordered semigroups in terms of this notion. Davvaz et al. used the idea of generalized fuzzy sets in several algebraic structures and introduced different generalized fuzzy subsystems [11-17]. In [18], Ma et al. introduced the concept of a generalized fuzzy filter of R_0-algebra and provided some properties in terms of this notion, [19]. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra (see references). The concept of a fuzzy filter in ordered semigroups was first introduced by Kehayopulu and Tsingelis in [20], where some basic properties of fuzzy filters and prime fuzzy ideals were discussed. In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, design of fast adders and error-correcting codes. A theory of fuzzy generalized sets on ordered semigroups can be developed. Using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, the concept of (α,β)-fuzzy filters in an ordered semigroup is introduced. Our aim in this paper, is to introduce and study the new sort of fuzzy filters, called (α,β)-fuzzy filters and to study $(\in,\in\vee q)$-fuzzy filters, to provide different characterizations of filters of ordered semigroups in terms of $(\in,\in\vee q)$-fuzzy filters, to extend our study in $(\in,\in\vee q)$-fuzzy left (right) and $(\in,\in\vee q)$-fuzzy bi-filters and to investigate different characterizations of left (right) and bi-filters of ordered semigroups in terms of $(\in,\in\vee q)$-fuzzy left (right) and $(\in,\in\vee q)$-fuzzy bi-filters. Finally, we introduce the concepts of fuzzy left (right and bi) -filters with thresholds.

2. Preliminaries

By an *ordered semigroup* (or *po-semigroup*) we mean a *structure* $(S,\cdot,\leq)$ in which the following are satisfied:
(OS1) $(S,\cdot)$ is a semigroup,
(OS2) $(S,\leq)$ is a poset,
(OS3) $a\leq b\rightarrow ax\leq bx$ and $xa\leq xb$ for all $a,b,x\in S$.

For a non-empty subset $A\subseteq S$, we denote $(A]:=\{t\in S | t\leq h$ for some $h\in A\}$. If $A=\{a\}$, then we write $(a]$ instead of $(\{a\}]$. For non-empty subsets $A,B\subseteq S$, we denote,

$$AB\coloneqq\{ab \mid a\in A, b\in B\}.$$

Let $(S,\cdot,\leq)$ be an ordered semigroup. A nonempty subset A of S is called a *subsemigroup* of S if $A^2\subseteq A$.

Definition 2.1. A non-empty subset F of an ordered semigroup S is called a *filter* of S if it satisfies
(i) $(\forall b\in S)(\forall a\in F)(a\leq b\rightarrow b\in F)$,
(ii) $(\forall a,b\in S)(a,b\in F\rightarrow ab\in F)$,
(iii) $(\forall a,b\in S)(ab\in F\rightarrow a,b\in F)$.
F is called a *left* (resp. *right*) *filter* of S if it satisfies conditions (i), (ii) of Definition 2.1, and
(iv) $(\forall a,b\in S)(ab\in F\rightarrow a\in F($resp. $b\in F))$.
F is called a *bi-filter* of S if it satisfies condition (i), (ii) of Definition 2.1, and
(v) $(\forall a,b\in S)(aba\in F\rightarrow a\in F)$.
Now, we recall some fuzzy logic concepts.

A *fuzzy subset* μ from a universe X is a function from X into the unit closed interval $[0,1]$ of real numbers, i.e., $\mu: X\rightarrow 0,1]$.

Definition 2.2. [20]. A fuzzy subset μ of an ordered semigroup $(S,\cdot,\leq)$ is called a *fuzzy filter* of S if it satisfies
(i) $(\forall x,y\in S)(x\leq y\rightarrow\mu(x)\leq\mu(y))$,
(ii) $(\forall x,y\in S)(\mu(xy)\geq\min\{\mu(x),\mu(y)\})$,
(iii) $(\forall x,y\in S)(\min\{\mu(x),\mu(y)\}\geq\mu(xy))$.
μ is called a *fuzzy left* (resp. *right*) *filter* of S if it satisfies condition (i) of Definition 2.2 and
(iv) $(\forall x,y\in S)(\mu(xy)\geq\mu(x)($resp. $\mu(xy)\geq\mu(y))$.
μ is called a *fuzzy bi-filter* of S if it satisfies conditions (i), (ii) of Definition 2.2 and
(v) $(\forall x,y\in S)(\mu(xyx)\geq\mu(x))$.

Let S be an ordered semigroup and $\emptyset\neq F\subseteq S$. Then the characteristic function χ_F of F is defined by

$$\chi_F: S\rightarrow 0,1],\qquad x\mapsto\begin{cases}1 & \text{if } x\in F\\ 0 & \text{if } x\notin F.\end{cases}$$

Clearly, a non-empty subset F of S is a filter if and only if the characteristic function χ_F of F is a fuzzy filter of S.

3. (α,β)-fuzzy filters

In what follows let S denote an ordered semigroup, and α and β denote any one of $\in$, q, $\in\vee q$, or $\in\wedge q$ unless otherwise specified. A fuzzy subset μ in a set S of the form

$$\mu: S\rightarrow 0,1],\quad y\mapsto\begin{cases}t & \text{if } y=x\\ 0 & \text{if } y\neq x\end{cases}$$

is said to be a *fuzzy point with support* x and *value* t and is denoted by x_t. For a fuzzy point x_t and a fuzzy subset μ of a set S, Pu and Liu [21] gave meaning to the symbol $x_t\alpha\mu$, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$. To say that $x_t\in\mu$ (resp. $x_tq\mu$) means that $\mu(x)\geq t$ (resp. $\mu(x)+t>1$), and in this case, x_t is said to belong to (resp. be *quasi-coincident with*) a fuzzy subset μ. To say that $x_t\in\vee q\mu$ (resp. $x_t\in\wedge q\mu$) means that $x_t\in\mu$ or $x_tq\mu$ (resp. $x_t\in\mu$ and $x_tq\mu$). To say that $x_t\overline{\alpha}\mu$ means that $x_t\alpha\mu$ does not hold. For a fuzzy subset μ of S and $t\in(0,1]$, the *crisp set* $U(\mu;t):=\{x\in S|\mu(x)\geq t\}$ is called the *level subset* of μ.

The proof of the following theorem is easy and so is omitted.

Theorem 3.1. A fuzzy subset μ of S is a fuzzy filter of S if and only if each non-empty level subset $U(\mu;t)$, for all $t\in(0,1]$ is a filter of S, respectively.

Example 3.2. Let $S=\{a,b,c,d,e,f\}$ be a set with the following multiplication table and order relation "$\leq$":

$\cdot$	a	b	c	d	e	f
a	a	b	b	d	e	f
b	b	b	b	b	b	b
c	b	b	b	b	b	b
d	d	b	b	d	e	f
e	e	f	f	e	e	f
f	f	f	f	f	f	f

and

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),$$

$$(a,d),(a,e),(d,e),(b,f),(c,f),(c,e),(f,e)\}$$

Then $(S,\cdot,\leq)$ is an ordered semigroup (see [22]). Filters of S are $\{a,d,e\}$ and S. Define a fuzzy subset $\mu: S \to 0,1]$ by

$$\mu(e) = 0.8, \quad \mu(d) = 0.7, \quad \mu(a) = 0.6$$

$$\mu(b) = 0.4, \quad \mu(c) = 0.3, \quad \mu(f) = 0.5.$$

Then

$$U(\mu;t) := \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{a,d,e\} & \text{if } 0.5 < t \leq 0.6 \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

By Theorem 3.1, μ is a fuzzy filter of S.

Theorem 3.3. Let μ be a fuzzy subset of S. Then $U(\mu;t)$ is a filter of S for all $t \in (0.5,1]$ if and only if μ satisfies the following conditions:

(i) $(\forall x,y \in S)(\max\{\mu(y),0.5\} \geq \mu(x)$ *with* $x \leq y)$,

(ii) $(\forall x,y \in S)(\max\{\mu(xy),0.5\} \geq \min\{\mu(x),\mu(y)\})$,

(iii) $(\forall x,y \in S)(\max\{\mu(x),\mu(y),0.5\} \geq \mu(xy))$.

Proof: Assume that $U(\mu;t)$ is a filter of S for all $t \in (0.5,1]$. If there exist $x,y \in S$ with $x \leq y$ such that the condition (i) is not valid, that is,

$$(\exists x,y \in S, x \leq y)(\max\{\mu(y),0.5\} < \mu(x) = r),$$

then $r \in (0.5,1]$, $x \in U(\mu;r)$. But $\mu(y) < r$ implies $y \notin U(\mu;r)$, a contradiction. Hence condition (i) is valid. If there exist $x,y \in S$ such that $\max\{\mu(xy),0.5\} < \min\{\mu(x),\mu(y),0.5\} = s$, then $s \in (0.5,1]$, $x,y \in U(\mu;s)$. But $\mu(xy) < s$ implies $xy \notin U(\mu;s)$, a contradiction. Hence condition (ii) is valid. Also, if there exist $x,y \in S$ such that $\max\{\mu(x),\mu(y),0.5\} < \mu(xy) = t$, then $t \in (0.5,1]$, $xy \in U(;t)$. But $\mu(x) < t$ and $\mu(y) < t$ imply $x \notin U(\mu;t)$ and $y \notin U(\mu;t)$, a contradiction.

Conversely, suppose that μ satisfies conditions (i), (ii) and (iii). Let $x,y \in S$, $x \leq y$ be such that $x \in U(\mu;t)$ for some $t \in (0.5,1]$, then $\mu(x) \geq t$. Since $x \leq y$ so it follows by condition (i)

$$\max\{\mu(y),0.5\} \geq \mu(x) \geq t > 0.5$$

so that $\mu(y) \geq t$, that is, $y \in U(\mu;t)$. For $x,y \in U(\mu;t)$, we get

$$\max\{\mu(xy),0.5\} \geq \min\{\mu(x),\mu(y)\} \geq t > 0.5,$$

and so $\mu(xy) \geq t$. It follows that $xy \in U(\mu;t)$. For $xy \in U(\mu;t)$, we have $\max\{\mu(x),\mu(y),0.5\} \geq \mu(xy) \geq t > 0.5$ and hence $\mu(x) \geq t, \mu(y) \geq t$, it follows that $x,y \in U(\mu;t)$. Thus $U(\mu;t)$ is a filter of S for all $t \in (0.5,1]$.

The conditions (i)-(iii) of Definition 2.2, are equivalent to the following definition.

Definition 3.4. A fuzzy subset μ of an ordered semigroup S is called an *$(\in,\in)$-fuzzy filter* of S if it satisfies:

(i) $(\forall x,y \in S)(\forall t \in (0,1])(x \leq y, x_t \in \mu \to y_t \in \mu)$,

(ii) $(\forall x,y \in S)(\forall t,r \in (0,1])(x_t, y_r \in \mu \to (xy)_{\min\{t,r\}} \in \mu)$,

(iii) $(\forall x,y \in S)(\forall t \in (0,1])((xy)_t \in \mu \to x_t \in \mu, y_t \in \mu)$.

From Definition 3.4, we have the following theorem:

Theorem 3.5. A fuzzy subset μ of S is a fuzzy filter of S if and only if it satisfies the conditions (i), (ii) and (iii) of Definition 3.4.

Proof: It is straightforward.

Note that if μ is a fuzzy subset S, defined by $\mu(x) \leq 0.5$ for all $x \in S$, then the set $\{\mu_t | \mu_t \in\wedge q\mu\}$ is empty. Therefore the case when $\alpha = \in\wedge q$ is omitted in Definition 3.6.

Definition 3.6. A fuzzy subset μ of S is called an *(α,β)-fuzzy filter* of S, where $\alpha \neq \in\wedge q$, if it satisfies

(i) $(\forall x,y \in S)(\forall t \in (0,1])(x \leq y, x_t\alpha\mu \to y_t\beta\mu)$,

(ii) $(\forall x,y \in S)(\forall t,r \in (0,1])(x_t\alpha\mu, y_r\alpha\mu \to (xy)_{\min\{t,r\}}\beta\mu)$,

(iii) $(\forall x,y \in S)(\forall t \in (0,1])((xy)_t\alpha\mu \to x_t\beta\mu, y_t\beta\mu)$.

Theorem 3.7. Let μ be a non-zero *(α,β)*-fuzzy filter of S. Then the set $\mu_0 := \{x \in S \mid \mu(x) > 0\}$ is a filter of S.

Proof: Let $x,y \in S$, $x \leq y$ and $x \in \mu_0$. Then $\mu(x) > 0$. Assume that $\mu(y) = 0$. If $\alpha \in \{\in, \in\vee q\}$, then $x_{\mu(x)}\alpha\mu$ but $y_{\mu(y)}\overline{\beta}\mu$ for every $\beta \in \{\in, q \in\vee q, \in\wedge q\}$, a contradiction. Note that $x_1 q\mu$, but $y_1\overline{\beta}\mu$ for every $\beta \in \{\in, q \in\vee q, \in\wedge q\}$, a contradiction. Hence $\mu(y) > 0$, that is $y \in \mu_0$. Now let $x,y \in \mu_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Assume that $\mu(xy) = 0$ and let $\alpha \in \{\in, \in\vee q\}$, then $x_{\mu(x)}\alpha\mu$ and $y_{\mu(y)}\alpha\mu$ but $(xy)_{\min\{\mu(x),\mu(y)\}}\overline{\beta}\mu$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Note that $x_1 q\mu$ and $y_1 q\mu$ but $(xy)_{\min\{1,1\}} = (xy)_1\overline{\beta}\mu$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Hence $\mu(xy) > 0$, it follows that, $xy \in \mu_0$. Let $xy \in \mu_0$.

Then $\mu(xy) > 0$ and assume that $\mu(x) = 0$ or $\mu(y) = 0$. Let $\alpha \in \{\in, \in\vee q\}$, then $(xy)_{\mu(xy)}\alpha\mu$ but $(x)_{\mu(x)}\overline{\beta}\mu$ or $(y)_{\mu(y)}\overline{\beta}\mu$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Note that $(xy)_1 q\mu$, but $(x)_1\overline{\beta}\mu$ or $(y)_1\overline{\beta}\mu$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Hence $\mu(x) > 0$ and $\mu(y) > 0$ so $x \in \mu_0$ and $y \in \mu_0$. Therefore μ_0 is a filter of S.

4. $(\in, \in\vee q)$-fuzzy filters

Now, we introduce the concept of $(\in, \in\vee q)$-fuzzy filters in ordered semigroups and we characterize filters of ordered semigroups in terms of $(\in, \in\vee q)$-fuzzy filters.

Definition 4.1. A fuzzy subset μ of S is called an *$(\in, \in\vee q)$-fuzzy filter* of S if it satisfies

(i) $(\forall x, y \in S)(\forall t \in (0,1])(x \leq y, x_t \in \mu \rightarrow y_t \in\vee q\mu)$,

(ii) $(\forall x, y \in S)(\forall t, r \in (0,1])(x_t \in \mu, y_r \in \mu \rightarrow (xy)_{\min\{t,r\}} \in\vee q\mu)$,

(iii) $(\forall x, y \in S)(\forall t \in (0,1])((xy)_t \in \mu \rightarrow x_t \in\vee q\mu, y_t \in\vee q\mu)$.

Example 4.2. Consider the ordered semigroup as given in Example 3.2, and define a fuzzy subset μ by

$$\mu(e) = 0.8, \quad \mu(d) = 0.7, \quad \mu(a) = 0.6$$

$$\mu(c) = 0.4, \quad \mu(b) = 0.3, \quad \mu(f) = 0.45.$$

Then μ is an $(\in, \in\vee q)$-fuzzy filter of S. But

(i) μ is not an $(\in, \in)$-fuzzy filter of S, since $d_{0.68} \in \mu$ and $c_{0.38} \in \mu$ but $(dc)_{\min\{0.68,0.38\}} = b_{0.38} \notin \mu$.

(ii) μ is not a $(q, \in)$-fuzzy filter of S, since $d_{0.68}q\mu$ and $f_{0.78}q\mu$ but

$$(df)_{\min\{0.68,0.78\}} = f_{0.68} \notin \mu.$$

(iii) μ is not an $(\in, q)$-fuzzy filter of S, since $a_{0.58} \in \mu$ and $f_{0.38} \in \mu$ but

$$(af)_{\min\{0.58,0.38\}} = f_{0.38}\overline{q}\mu.$$

Theorem 4.3. A fuzzy subset μ of S is an $(\in, \in\vee q)$-fuzzy filter of S if and only if it satisfies the following conditions

(i) $(\forall x, y \in S)(x \leq y, \mu(y) \geq \min\{\mu(x), 0.5\})$,

(ii) $(\forall x, y \in S)(\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\})$,

(iii) $(\forall x, y \in S)(\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\})$.

Proof: Let μ be an $(\in, \in\vee q)$-fuzzy filter and let $x, y \in S, x \leq y$. If $\mu(x) = 0$, then $\mu(y) \geq \min\{\mu(x), 0.5\}$. Let $\mu(x) \neq 0$ and assume, on the contrary, that $\mu(y) < \min\{\mu(x), 0.5\}$. Choose $t \in (0,1]$ such that $\mu(y) < t \leq \min\{\mu(x), 0.5\}$. If $\mu(x) < 0.5$, then $\mu(y) < t \leq \mu(x)$ and so $x_t \in \mu$ but $y_t \overline{\in} \mu$, a contradiction. If $\mu(x) \geq 0.5$ then $\mu(y) < 0.5$ and so $x_{0.5} \in \mu$ but $y_{0.5} \overline{\in} \mu$, again a contradiction. Hence $\mu(y) \geq \min\{\mu(x), 0.5\}$ for all x, y with $x \leq y$. Let $x, y \in S$ and if $\mu(x) = 0$ or $\mu(y) = 0$, then $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$. Let $\mu(x) \neq 0$ and $\mu(y) \neq 0$ and assume, on the contrary, that $\mu(xy) < \min\{\mu(x), \mu(y), 0.5\}$. Choose $s \in (0,1]$ such that $\mu(xy) < s \leq \min\{\mu(x), \mu(y), 0.5\}$ if $\min\{\mu(x), \mu(y)\} < 0.5$ then $\mu(xy) < s \leq \min\{\mu(x), \mu(y)\}$ and $x_s, y_s \in \mu$ but $(xy)_s \overline{\in} \mu$, a contradiction. Let $\min\{\mu(x), \mu(y)\} \geq 0.5$, then $\mu(xy) < 0.5$ and $x_{0.5}, y_{0.5} \in \mu$ but $(xy)_{0.5} \overline{\in} \mu$, again a contradiction. Hence $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in S$. For $x, y \in S$, if $\mu(xy) = 0$, then $\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\}$. Let $\mu(xy) \neq 0$ and assume, on the contrary, that $\min\{\mu(x), \mu(y)\} < \min\{\mu(xy), 0.5\}$. Choose $r \in (0,1]$ such that $\min\{\mu(x), \mu(y)\} < r \leq \min\{\mu(xy), 0.5\}$. If $\mu(xy) < 0.5$, then $\min\{\mu(x), \mu(y)\} < r \leq \mu(xy)$ and so $(xy)_r \in \mu$ but $x_r \overline{\in} \mu$ and $y_r \overline{\in} \mu$, a contradiction. If $\mu(xy) \geq 0.5$ then $\min\{\mu(x), \mu(y)\} < 0.5$ and so $(xy)_{0.5} \in \mu$ but $x_{0.5} \overline{\in} \mu$ and $y_{0.5} \overline{\in} \mu$, again a contradiction. Hence $\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\}$ for all $x, y \in S$.

Conversely, let $x, y \in S$, $x \leq y$ and $x_t \in \mu$ for some $t \in (0,1]$, then $\mu(x) \geq t$. By hypothesis, $\mu(y) \geq \mu(x) \geq t$ and $\mu(y) \geq t$, i.e., $y_t \in\vee q\mu$. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $x_t, y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$ and so $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, r, 0.5\}$. If $\min\{t, r\} \leq 0.5$ then $\mu(xy) \geq \min\{t, r\}$ and $(xy)_{\min\{t,r\}} \in \mu$. If $\min\{t, r\} > 0.5$ then $\mu(xy) + \min\{t, r\} > 0.5 + 0.5 = 1$ and so $(xy)_{\min\{t,r\}}q\mu$. Hence $(xy)_{\min\{t,r\}} \in\vee q\mu$. For $x, y \in S$, and $(xy)_t \in \mu$ for some $t \in (0,1]$. Then $\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\} \geq \min\{t, 0.5\}$. If $t \leq 0.5$ then $\min\{\mu(x), \mu(y)\} \geq t$ and so $x_t, y_t \in \mu$. If $t > 0.5$ then $\min\{\mu(x), \mu(y)\} + t > 0.5 + 0.5 = 1$ that is, $\mu(x) + t > 1$ and $\mu(y) + t > 1$, it follows that $x_t q\mu$ and $y_t q\mu$. Therefore $x_t \in\vee q\mu$ and $y_t \in\vee q\mu$. Consequently, μ is an $(\in, \in\vee q)$-fuzzy filter of S.

Remark 4.4. A fuzzy subset μ of an ordered semigroup S is an $(\in, \in\vee q)$-fuzzy filter of S if and only if it satisfies conditions (i), (ii) and (iii) of Theorem 4.3.

Remark 4.5. By the above remark every fuzzy filter of an ordered semigroup S is an $(\in, \in\vee q)$-fuzzy filter of S. However, the converse is not true, in general.

Example 4.6. Consider the ordered semigroup as given in Example 3.2, and define a fuzzy subset μ by

$$\mu(e) = 0.8, \quad \mu(d) = 0.7, \quad \mu(a) = 0.6$$

$$\mu(b) = 0.4, \quad \mu(c) = 0.3, \quad \mu(f) = 0.5.$$

Then μ is an $(\in, \in\vee q)$-fuzzy filter of S. But μ is not an (α, β)-fuzzy filter of S, where $\alpha \in \{\in, q, \in\vee q\}$ and $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, as shown in Example 4.2.

Using Theorem 4.3, we have the following characterization of fuzzy filters of ordered semigroups.

Theorem 4.7. Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq F \subseteq S$. Then F is a filter of S if and only if the characteristic function χ_F of F is an $(\in, \in\vee q)$-fuzzy filter of S.

Theorem 4.8. Let F be a filter of S and μ a fuzzy subset of S such that

$$\mu(x) := \begin{cases} \geq 0.5 & \text{if } x \in F \\ 0 & \text{if } x \in S \backslash F. \end{cases}$$

Then

(a) μ is a $(q, \in\vee q)$-fuzzy filter of S.
(b) μ is an $(\in, \in\vee q)$-fuzzy filter of S.

Proof: (a) Let $x, y \in S$, $x \leq y$ and $t \in (0,1]$ be such that $x_t q\mu$. Then $x \in F$ and $y \geq x \in F$ implies $y \in F$. If $t \leq 0.5$ then $\mu(y) \geq 0.5 \geq t$ implies $\mu(y) \geq t$ and so $y_t \in \mu$. If $t > 0.5$ then $\mu(y) + t > 0.5 + 0.5 = 1$ and $y_t q\mu$. Hence $y_t \in\vee q\mu$. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $x_t q\mu$ and $y_r q\mu$. Then $x, y \in F$ and so $xy \in F$. If $\min\{t, r\} \leq 0.5$ then $\mu(xy) \geq 0.5 \geq \min\{t, r\}$ and so $\mu(xy) \geq \min\{t, r\}$ implies $(xy)_{\min\{t,r\}} \in \mu$. If $\min\{t, r\} > 0.5$ then $\mu(xy) + \min\{t, r\} > 0.5 + 0.5 = 1$ and so $(xy)_{\min\{t,r\}} q\mu$. Hence $(xy)_{\min\{t,r\}} \in\vee q\mu$. For $x, y \in S$ and $(xy)_t q\mu$ for some $t \in (0,1]$ we have $xy \in F$ then $x, y \in F$. If $t \leq 0.5$ then $\min\{\mu(x), \mu(y)\} \geq 0.5 \geq t$ and we have $\mu(x) \geq t$ and $\mu(y) \geq t$. Hence $x_t, y_t \in \mu$. If $t > 0.5$ then $\min\{\mu(x), \mu(y)\} + t > 0.5 + 0.5 = 1$ that is $\mu(x) + t > 1$ and $\mu(y) + t > 1$, it follows that $x_t q\mu$ and $y_t q\mu$. Hence $x_t \in\vee q\mu$ and $y_t \in\vee q\mu$.

(b) Let $x, y \in S$, $x \leq y$ and $t \in (0,1]$ be such that $x_t \in \mu$. Then $x \in F$ and we have $y \in F$. If $t \leq 0.5$ then $\mu(y) \geq 0.5 \geq t$ implies $\mu(y) \geq t$, and so $y_t \in \mu$. If $t > 0.5$ then $\mu(y) + t > 0.5 + 0.5 = 1$ and $y_t q\mu$. Hence $y_t \in\vee q\mu$. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$. Then $x, y \in F$ and we have $xy \in F$. If $\min\{t, r\} \leq 0.5$ then $\mu(xy) \geq 0.5 \geq \min\{t, r\}$ and so $\mu(xy) \geq \min\{t, r\}$ implies $(xy)_{\min\{t,r\}} \in \mu$. If $\min\{t, r\} > 0.5$ then $\mu(xy) + \min\{t, r\} > 0.5 + 0.5 = 1$ and so $(xy)_{\min\{t,r\}} q\mu$. Hence $(xy)_{\min\{t,r\}} \in\vee q\mu$. For $x, y \in S$ and $(xy)_t \in \mu$ for some $t \in (0,1]$ we have $xy \in F$ then $x, y \in F$. If $t \leq 0.5$ then $\min\{\mu(x), \mu(y)\} \geq 0.5 \geq t$ and we have $\mu(x) \geq t$ and $\mu(y) \geq t$. Hence $x_t, y_t \in \mu$. If $t > 0.5$ then $\min\{\mu(x), \mu(y)\} + t > 0.5 + 0.5 = 1$, that is $\mu(x) + t > 1$ and $\mu(y) + t > 1$, it follows that $x_t q\mu$ and $y_t q\mu$. Hence $x_t \in\vee q\mu$ and $y_t \in\vee q\mu$.

Note that in the above theorem, we impose a condition on the fuzzy subset μ. Without the condition

$$\mu(x) := \begin{cases} \geq 0.5 & \text{if } x \in F \\ 0 & \text{if } x \in S \backslash F.' \end{cases}$$

μ may not be a $(q, \in\vee q)$-fuzzy filter as given in Example 4.2 (ii).

In the following theorem we give a condition for an $(\in, \in\vee q)$-fuzzy filter to be an $(\in, \in)$-fuzzy filter of S.

Theorem 4.9. Let μ be an $(\in, \in\vee q)$-fuzzy filter of S such that $\mu(x) < 0.5$ for all $x \in S$. Then μ is an $(\in, \in)$-fuzzy filter of S.

Proof: Let $x, y \in S$, $x \leq y$ and $x_t \in \mu$ for some $t \in (0,1]$. Then $\mu(x) \geq t$ and we have $\mu(y) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} = t$. Hence $y_t \in \mu$. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$ and so $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, r\}$. Hence $(xy)_{\min\{t,r\}} \in \mu$. Now, let $x, y \in S$ and $t \in (0,1]$ be such that $(xy)_t \in \mu$. Then $\mu(xy) \geq t$ and so $\min\{\mu(x), \mu(y) \geq \min\{\mu(xy), 0.5\} \geq \min\{t, 0.5\} = t$. Hence $x_t \in \mu$ and $y_t \in \mu$.

Theorem 4.10. A fuzzy subset μ of S is an $(\in, \in\vee$ q)-fuzzy filter of S if and only if the set $U(\mu; t) := \{x \in S | \mu(x) \geq t\}$ is a filter of S for all $t \in (0,0.5]$.

Proof: Assume that μ is an $(\in, \in\vee q)$-fuzzy filter of S. Let $x, y \in S$, $x \leq y$ and let $t \in (0,0.5]$ be such that $x \in U(\mu; t)$. Then $\mu(x) \geq t$ and it follows from Theorem 4.3 (i) that

$$\mu(y) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} = t$$

and so $y \in U(\mu; t)$. Let $x, y \in U(\mu; t)$ for some $t \in (0,0.5]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. From Theorem 4.3 (ii), it follows that

$$\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t,$$

and so $xy \in U(\mu; t)$. Now, let $xy \in U(\mu; t)$ for some $t \in (0,0.5]$. Then $\mu(xy) \geq t$ and so, $\min\{\mu(x), \mu(y)\} \geq min\{\mu(xy), 0.5\} = t$, i.e., $\mu(x) \geq t$, and $\mu(y)\} \geq t$. It follows that $x, y \in U(\mu; t)$. Conversely, let for all $t \in (0,0.5]$ the set $U(\mu; t) := \{x \in S | \mu(x) \geq t\}$ be a filter of S. If there exist $x, y \in S$, $x \leq y$ such that $\mu(y) < \min\{\mu(x), 0.5\}$, then we can choose $t \in (0,0.5)$ such that $\mu(y) < t \leq \min\{\mu(x), 0.5\}$, then $x_t \in \mu$ but $y_t \notin \mu$, a contradiction. Hence $\mu(y) \geq \min\{\mu(x), 0.5\}$ for all $x, y \in S$ with $x \leq y$. If there exist $x, y \in S$ such that

$$\mu(xy) < min\{\mu(x), \mu(y), 0.5\}.$$

We can choose $s \in (0,0.5]$ such that $\mu(xy) < s \leq \min\{\mu(x), \mu(y), 0.5\}$. Then $x, y \in U(\mu; s)$ but $xy \notin U(\mu; s)$, a contradiction. Hence $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in S$. Finally, if there exist $x, y \in S$ such that $\min\{\mu(x), \mu(y)\} < \mu(xy)$, then $\min\{\mu(x), \mu(y)\} < r \leq \mu(xy)$ for some $r \in (0,0.5]$. Thus $xy \in U(\mu; r)$ but $x, y \notin U(\mu; r)$. Since $U(\mu; r)$ is a filter of S, and $xy \in U(\mu; r)$ then $x, y \in U(\mu; r)$, a contradiction. Hence $\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\}$ for all $x, y \in S$.

For any fuzzy subset μ of an ordered semigroup S and $t \in (0,1]$, we consider the following two subsets

$$Q(\mu; t) := \{x \in S | x_t q\mu\} \text{and} [\mu]_t := \{x \in S | x_t \in\vee q\mu\}.$$

It is clear that $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$.

Theorem 4.11. A fuzzy subset μ of S is an *(*$\in, \in\vee q$*)*-fuzzy filter of S if and only if for all $t \in (0,1]$ $[\mu]_t (\neq \emptyset)$ is a filter of S.

Proof: Assume that μ is an $(\in, \in\vee q)$-fuzzy filter of S and let $t \in (0,1]$ be such that $[\mu]_t \neq \emptyset$. Let $x, y \in S$, $x \leq y$ and $x \in \mu]_t$. Then $\mu(x) \geq t$ or $\mu(x) + t > 1$. Since μ is an $(\in, \in\vee q)$-fuzzy filter and $x \leq y$, we have $\mu(y) \geq \min\{\mu(x), 0.5\}$. We consider the cases:

(i) $\mu(x) \geq t$,

(ii) $\mu(x) + t > 1$.

(i) If $t > 0.5$, then $\mu(y) \geq \min\{\mu(x), 0.5\} = 0.5$ and $\mu(y) + t > 0.5 + 0.5 = 1$. Hence $y_t q\mu$. If $t \leq 0.5$ then $\mu(y) \geq \min\{\mu(x), 0.5\} \geq t$, and so $y_t \in \mu$, i.e., $y \in U(\mu; t) \subseteq \mu]_t$. It follows that $y \in \mu]_t$.

(ii) $\mu(x) + t > 1$. If $t > 0.5$, then $\mu(y) \geq min\{\mu(x), 0.5\} \geq min\{1 - t, 0.5\} = 1 - t$, that is, $\mu(y) + t > 1$ and hence $y_t q\mu$, i.e., $y \in Q(\mu; t) \subseteq \mu]_t$. If $t \leq 0.5$ then $\mu(y) \geq \min\{\mu(x), 0.5\} \geq \min\{1 - t, 0.5\} = 0.5 \geq t$ and so $y_t \in \mu$, hence $y \in \mu]_t$. Thus in both cases, we have $y \in \mu]_t$. Let $x, y \in \mu]_t$. Then $\mu(x) \geq$ t or $\mu(x) + t > 1$, and $\mu(y) \geq t$ or $\mu(y) + t > 1$. We consider the cases:

(i) $\mu(x) \geq t$ and $\mu(y) \geq t$,

(ii) $\mu(x) \geq t$ and $\mu(y) + t > 1$,

(iii) $\mu(x) + t > 1$ and $\mu(y) \geq t$,

(iv) $\mu(x) + t > 1$ and $\mu(y) + t > 1$.

For case (i), from Theorem 4.3 (ii), we have

$$\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = \begin{cases} 0.5 & \text{if } t > 0.5, \\ t & \text{if } t \leq 0.5, \end{cases}$$

and so $\mu(xy) + t > 0.5 + 0.5 = 1$, i.e., $(xy)_t q\mu$ or $(xy)_t \in \mu$. Therefore, $xy \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$. For the case (ii), assume that $t > 0.5$. Then $1 - t < 0.5$. If $\min\{\mu(y), 0.5\} \leq \mu(x)$, then $\mu(xy) \geq \min\{\mu(y), 0.5\} > 1 - t$, and if $\min\{\mu(y), 0.5\} > \mu(x)$, then $\mu(xy) \geq \mu(x) \geq t$. Hence $xy \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$ for $t > 0.5$. Suppose that $t \leq 0.5$. Then $1 - t \geq 0.5$. If $\min\{\mu(x), 0.5\} \leq \mu(y)$, then

$$\mu(xy) \geq \min\{\mu(x), 0.5\} \geq t,$$

and if $\min\{\mu(x), 0.5\} > \mu(y)$, then $\mu(xy) \geq \mu(y) > 1 - t$. Thus $xy \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$ for $t \leq 0.5$. For the case (iii), we have the same discussion as in case (ii). For case (iv), if $t > 0.5$ then $1 - t < 0.5$. Hence

$$\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\} = \begin{cases} 0.5 > 1 - t & \text{if } \min\{\mu(x), \mu(y)\} \geq 0.5, \\ \min\{\mu(x), \mu(y)\} > 1 - t & \text{if } \min\{\mu(x), \mu(y)\} < 0.5, \end{cases}$$

and so $xy \in Q(\mu; t) \subseteq \mu]_t$. If $t \leq 0.5$, then $1 - t \geq 0.5$. Thus

$$\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\} = \begin{cases} 0.5 \geq t & \text{if } \min\{\mu(x), \mu(y)\} \geq 0.5, \\ \min\{\mu(x), \mu(y)\} > 1 - t & \text{if } \min\{\mu(x), \mu(y)\} < 0.5, \end{cases}$$

which implies that $xy \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$. Let $xy \in \mu]_t$. Then $\mu(xy) \geq t$ or $\mu(xy) + t > 1$. Assume that $\mu(xy) \geq t$. Then

$$\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\} \geq \min\{t, 0.5\} = \begin{cases} t & \text{if } t \leq 0.5, \\ 0.5 > 1 - t & \text{if } t > 0.5, \end{cases}$$

so that $x, y \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$. Suppose that $\mu(xy) + t > 1$. If $t > 0.5$, then

$$\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\} = \begin{cases} 0.5 > 1 - t & \text{if } \mu(xy) \geq 0.5, \\ \mu(xy) > 1 - t & \text{if } \mu(xy) < 0.5, \end{cases}$$

and thus $x, y \in Q(\mu; t) \subseteq \mu]_t$. If $t \leq 0.5$, then

$$\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\} = 0.5 \geq t$$

and so $x, y \in U(\mu; t) \subseteq \mu]_t$. Consequently, $[\mu]_t$ is a filter of S.

Conversely, suppose that for all $t \in (0,1]$, the set $[\mu]_t (\neq \emptyset)$ is a filter of S. If there exist $x_0, y_0 \in S$ with $x_0 \leq y_0$ such that $\mu(y_0) < \min\{\mu(x_0), 0.5\}$, then $\mu(y_0) < t_0 \leq \min\{\mu(x_0), 0.5\}$ for some $t_0 \in (0,0.5]$. It follows that $x_0 \in U(\mu; t_0) \subseteq \mu]_{t_0}$, and so $x_0 \in \mu]_{t_0}$. Thus $\mu(x_0) \geq t_0$ or $\mu(x_0) + t_0 > 1$. Since $y_0 \geq x_0 \in \mu]_{t_0}$ and for $t_0 \in (0.0.5]$, $[\mu]_{t_0}$ is a filter, we have $y_0 \in \mu]_{t_0}$. Hence $\mu(y_0) \geq t_0$ or $\mu(y_0) + t_0 > 1$, a contradiction. Thus $\mu(y) \geq min\{\mu(x), 0.5\}$ for all x, y with $x \leq y$. If there exist $a, b \in S$ such that $\mu(ab) < \min\{\mu(a), \mu(b), 0.5\}$. Then $\mu(ab) < t_1 \leq \min\{\mu(a), \mu(b), 0.5\}$ for some $t_1 \in (0,0.5]$. Hence $a, b \in U(\mu; t_1) \subseteq \mu]_{t_1}$ and so

$ab \in \mu]_{t_1}$. Thus $\mu(ab) \geq t_1$ or $\mu(ab) + t_1 > 1$, a contradiction. So $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in S$. Finally, if there exist $c, d \in S$ such that $\min\{\mu(c), \mu(d)\} < \{\mu(cd), 0.5\}$, then $\min\{\mu(c), \mu(d)\} < t_2 \leq \{\mu(cd), 0.5\}$ for some $t_2 \in (0,0.5]$. Hence $cd \in U(\mu; t_2) \subseteq \mu]_{t_2}$. It follows that $cd \in \mu]_{t_2}$ and hence $c, d \in \mu]_{t_2}$. Thus $\mu(c) \geq t_2$ or $\mu(c) + t_2 > 1$ and $\mu(d) \geq t_2$ or $\mu(d) + t_2 > 1$, a contradiction. Thus $\min\{\mu(x), \mu(y)\} \geq \min\{\mu(xy), 0.5\}$ for all $x, y \in S$. Thus μ is a fuzzy filter of S.

Let μ be a fuzzy subset of S and $J = \{t \mid t \in (0,1]$ and $\mu_t (\neq \emptyset)$ is a filter of $S\}$. When $J = (0,1]$, μ is an ordinary filter of the ordered semigroup S (Theorem 3.1). When $J = (0,0.5]$, then μ is an $(\in, \in\vee q)$-fuzzy filter of S (Theorem 4.10).

Consider $J = \{t | t \in (0,1]$ and $\mu_t (\neq \emptyset)$ is a filter of $S\}$. In the following, we give the answer of the questions:

(1) If $J = (0.5,1]$, what kind of fuzzy filters of S will be μ?
(2) If $J = (r, s]$, $(r, s \in (0,1])$, will μ be a kind of fuzzy filter of S or not?

Definition 4.12. A fuzzy subset μ of S is called an $(\overline{\in}, \overline{\in}\vee \overline{q})$-*fuzzy filter* of S if it satisfies the following assertions:

(Fi) $(\forall x, y \in S)(\forall t \in (0,1])(y_t \overline{\in} \mu \to x_t \overline{\in}\vee \overline{q}\mu$ with $x \leq y)$,

(Fii) $(\forall x, y \in S)(\forall t, r \in (0,1])((xy)_{\min\{t,r\}} \overline{\in} \mu \to x_t \overline{\in}\vee \overline{q}\mu$ or $y_r \overline{\in}\vee \overline{q}\mu)$,

(Fiii) $(\forall x, y \in S)(\forall t \in (0,1])(x_t \overline{\in} \mu, y_t \overline{\in} \mu \to (xy)_t \overline{\in}\vee \overline{q}\mu)$.

Example 4.13. Consider the ordered semigroup as given in Example 3.2 and define a fuzzy subset μ as follows:

$$\mu(e) = 0.6, \quad \mu(d) = 0.5, \quad \mu(a) = 0.4,$$

$$\mu(b) = 0.2, \quad \mu(c) = 0.3, \quad \mu(f) = 0.35.$$

Then

$$U(\mu; t) = \begin{cases} S & \text{if } 0 < t \leq 0.2 \\ \{a, d, e\} & \text{if } 0.35 < t \leq 0.4 \\ \emptyset & \text{if } 0.6 < t. \end{cases}$$

Then μ is an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy filter of S.

Theorem 4.14. A fuzzy subset μ of S is an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy filter of S if and only if it satisfies the following conditions:

(Fiv) $(\forall x, y \in S)$ $(\max\{\mu(y), 0.5\} \geq \mu(x))$,
(Fv) $(\forall x, y \in S)$ $(\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\})$,
(Fvi) $(\forall x, y \in S)$ $(\max\{\mu(x), \mu(y), 0.5\} \geq \mu(xy))$.

Proof: (Fi)→(Fiv). Let $x, y \in S$, $x \leq y$ be such that $\max\{\mu(y), 0.5\} < \mu(x) = t$. Then $t \in (0.5,1]$, $y_t \overline{\in} \mu$ but $x_t \in \mu$. By (Fi), we have $x_t \overline{q}\mu$. Then $t \leq \mu(x)$ and $t + \mu(x) \leq 1$, which implies $t \leq 0.5$, contradiction. Hence (Fiv) is valid.

(Fiv)→ (Fi). Let $y_t \overline{\in} \mu$ then $\mu(y) < t$. (a) If $\mu(y) \geq \mu(x)$, then $\mu(x) < t$, and so $x_t \overline{\in} \mu$. Thus $x_t \overline{\in}\vee \overline{q}\mu$. (b) If $\mu(y) < \mu(x)$, then by (Fiv) we have $0.5 \geq \mu(x)$. Putting $x_t \in \mu$, then $t \leq \mu(x) \leq 0.5$. It follows that $x_t \overline{\in}\vee \overline{q}\mu$.

(Fii)→(Fv). If there exist $x, y \in S$ such that

$$\max\{\mu(xy), 0.5\} < min\{\mu(x), \mu(y)\} = s,$$

then $s \in (0.5,1]$, $(xy)_s \overline{\in} \mu$ but $x_s \in \mu, y_s \in \mu$. By (Fii), we have $x_s\overline{q}\mu$ or $y_s\overline{q}\mu$. Then ($s \leq \mu(x)$ and $s + \mu(x) \leq 1$) or ($s \leq \mu(y)$ and $s + \mu(y) \leq 1$), which implies $s \leq 0.5$, contradiction. Hence (Fv) is valid.

(Fv)→(Fii). Let $(xy)_{\min\{t,r\}} \overline{\in} \mu$ then $\mu(xy) < \min\{t, r\}$. (a) If $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, then $\min\{\mu(x), \mu(y)\} < \min\{t, r\}$ and so $\mu(x) < t$ or $\mu(y) < r$. It follows that $x_t \overline{\in} \mu$ or $y_r \overline{\in} \mu$. Thus $x_t \overline{\in}\vee \overline{q}\mu$ or $y_r \overline{\in}\vee \overline{q}\mu$. (b) If $\mu(xy) < min\{\mu(x), \mu(y)\}$ then by (Fv), $0.5 \geq \min\{\mu(x), \mu(y)\}$. Let $x_t \in \mu$ or $y_r \in \mu$, then $t \leq \mu(x) \leq 0.5$ or $r \leq \mu(y) \leq 0.5$. It follows that $x_t\overline{q}\mu$ or $y_r\overline{q}\mu$. Thus $x_t \overline{\in}\vee \overline{q}\mu$ or $y_r \overline{\in}\vee \overline{q}\mu$.

(Fiii)→(Fvi). If there exist $x, y \in S$ such that

$$\max\{\mu(x), \mu(y), 0.5\} < \mu(xy) = r,$$

then $r \in (0.5,1]$, $x_r \overline{\in} \mu$, $y_r \overline{\in} \mu$ but $(xy)_r \in \mu$. By (Fiii), we have $(xy)_r\overline{q}\mu$. Then $r \leq \mu(xy)$ and $r + \mu(xy) \leq 1$, which implies $r \leq 0.5$, contradiction. Hence (Fvi) is valid.

(Fvi)→(Fiii). Let $x_t \overline{\in} \mu$ and $y_t \overline{\in} \mu$, then $\mu(x) < t$ and $\mu(y) < t$, that is $\min\{\mu(x), \mu(y)\} < t$. (a) If $\min\{\mu(x), \mu(y)\} \geq \mu(xy)$, then $\mu(xy) < t$ and so $(xy)_t \overline{\in} \mu$. It follows that $(xy)_t \overline{\in}\vee \overline{q}\mu$. (b) If $\min\{\mu(x), \mu(y)\} < \mu(xy)$, then by (Fvi), $0.5 \geq \mu(xy)$. Let $(xy)_t \in \mu$, then $t \leq \mu(xy) \leq 0.5$. It follows that $(xy)_t\overline{q}\mu$. Thus $(xy)_t \overline{\in}\vee \overline{q}\mu$.

Lemma 4.15. Let μ be fuzzy subset of S. Then $U(\mu; t)(\neq \emptyset)$ is a filter of S if and only if it satisfies the conditions (Fiv)-(Fvi) of Theorem 4.14.

Theorem 4.16. A fuzzy subset μ of S is an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy filter of S if and only if $U(\mu; t)(\neq \emptyset)$ is a filter of S for all $t \in (0.5,1]$.

Proof: Follows from Theorem 4.14 and Lemma 4.15.

In [23], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds, which is a generalization of Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup. Based on [23], we can extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy filter with thresholds as follows:

Definition 4.17. Let $r, s \in 0,1]$ and $r < s$. Let μ be a fuzzy subset of an ordered semigroup S. Then μ is called a *fuzzy filter with thresholds* (r, s) of S if it satisfies the following assertions:
(i) $(\forall x, y \in S)(x \leq y, \max\{\mu(y), r\} \geq \min\{\mu(x), s\})$,
(ii) $(\forall x, y \in S)(\max\{\mu(xy), r\} \geq \min\{\mu(x), \mu(y), s\})$,
(iii) $(\forall x, y \in S)(\max\{\mu(x), \mu(y), r\} \geq \min\{\mu(xy), s\})$.

If μ is a fuzzy filter of S with thresholds of S, then we can conclude that μ is an ordinary fuzzy filter when $r = 0, s = 1$ and μ is an $(\in, \in\vee q)$-fuzzy filter when $r = 0, s = 0.5$. Now, we characterize fuzzy filters with thresholds $(r, s]$ of S, by their level filters.

Theorem 4.18. A fuzzy subset μ of an ordered semigroup S is a fuzzy filter with thresholds (r, s) of S if and only if $\mu_t(\neq \emptyset)$ is a filter of S for all $t \in (r, s]$.

Proof: Let μ be a fuzzy filter with thresholds of S and $t \in (r, s]$. Let $x, y \in S$ with $x \leq y$. Let $x \in \mu_t$, then $\mu(x) \geq t$ and from (i) of Definition 4.17, it follows that

$$\max\{\mu(y), r\} \geq \min\{\mu(x), s\} \geq \min\{t, s\} \geq t > r,$$

so $\mu(y) \geq t$ and hence, $y \in \mu_t$. Let $x, y \in \mu_t$, then $\mu(x) \geq t, \mu(y) \geq t$ and by (ii) of Definition 4.17, we have

$$\max\{\mu(xy), r\} \geq \min\{\mu(x), \mu(y), s\} \geq \min\{t, s\} \geq t,$$

and hence $\mu(xy) \geq t$ implies that $xy \in \mu_t$. For $xy \in \mu_t$, we have $\mu(xy) \geq t$ and by (iii) of Definition 4.17, it follows that

$$\max\{\mu(x), \mu(y), r\} \geq \min\{\mu(xy), s\} \geq \min\{t, s\} \geq t > r,$$

and so $\mu(x) \geq t, \mu(y) \geq t$, that is, $x, y \in \mu_t$. Conversely, let μ be a fuzzy subset of S such that $\mu_t(\neq \emptyset)$ is a filter of S for all $t \in (r, s]$. If there exist $x, y \in S$ with $x \leq y$ such that

$$\max\{\mu(y), r\} < min\{\mu(x), s\} = t,$$

then $t \in (r, s]$, $\mu(y) < t$ and $x \in \mu_t$. Since $y \geq x \in \mu_t$ and μ_t is a filter of S, we have $y \in \mu_t$. Then $\mu(y) \geq t$, a contradiction. Hence $\max\{\mu(y), r\} \geq \min\{\mu(x), s\}$ for all $x, y \in S$ with $x \leq y$. If there exist $x, y \in S$ such that

$$\max\{\mu(xy), r\} < min\{\mu(x), \mu(y), s\} = t,$$

then $t \in (r, s]$, $\mu(xy) < t$ and $x_t, y_t \in \mu$. Since μ_t is a filter of S and $x_t, y_t \in \mu$, we have $(xy)_{\min\{t,t\}} = (xy)_t \in \mu$, then $\mu(xy) \geq t$, a contradiction. Hence $\max\{\mu(xy), r\} \geq \min\{\mu(x), \mu(y), s\}$ for all $x, y \in S$. In a similar way we can prove that $\max\{\mu(x), \mu(y), r\} \geq \min\{\mu(xy), s\}$ for all $x, y \in S$. Therefore μ is a fuzzy filter of S.

Remark 4.19. (1) By Definition 4.17, we have the following conclusion: if μ is a fuzzy filter with thresholds $(r, s]$ of S, then we have:
(i) μ is an ordinary fuzzy filter when $r = 0, s = 1$;
(ii) μ is an $(\in, \in\vee q)$-fuzzy filter when $r = 0, s = 0.5$;
(iii) μ *is an* $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy filter when $r = 0.5, s = 1$.
(2) By Definition 4.17, we can define other kinds of fuzzy filters of S, such as fuzzy filter with thresholds (0.5,0.6] and (0.4,0.8] of S, etc.
(3) However, the fuzzy filters with thresholds of S may not be an ordinary fuzzy filter, may not be an $(\in, \in\vee q)$-fuzzy filter, and may not be $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy filter, respectively, as shown in the following example:

Example 4.20. Consider the ordered semigroup as given in Example 3.2 and define a fuzzy subset μ as follows:

$$\mu(e) = 0.8, \quad \mu(d) = 0.7, \quad \mu(a) = 0.6$$
$$\mu(b) = 0.4, \quad \mu(c) = 0.3, \quad \mu(f) = 0.5.$$

Then

$$U(\mu; t) = \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{a, d, e\} & \text{if } 0.5 < t \leq 0.6 \\ \{a, d, e, f\} & \text{if } 0.4 < t \leq 0.5 \\ \{a, d, e, f, b\} & \text{if } 0.3 < t \leq 0.4 \\ \emptyset & \text{if } 0.8 < t. \end{cases}$$

Thus, μ is a fuzzy filter with thresholds (0.5,0.6] of S. But μ are neither a fuzzy filter and $(\in, \in\vee q)$-fuzzy filter nor an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy filter of S.

5. $(\in, \in\vee q)$-fuzzy left(right) filters

Definition 5.1. A fuzzy subset of S is called an *$(\in, \in\vee q)$-fuzzy left* (resp. *right*) *filter* of S if it satisfies
(i) $(\forall x, y \in S)(\forall t \in (0,1])(x \leq y, x_t \in \mu \rightarrow y_t \in \vee q\mu)$,
(ii) $(\forall x, y \in S)(\forall t, r \in (0,1])(x_t \in \mu, y_r \in \mu \rightarrow (xy)_{\min\{t,r\}} \in\vee q\mu)$,
(iii) $(\forall x, y \in S)(\forall t \in (0,1])(x_t \in \mu \rightarrow (xy)_t \in\vee q\mu$(resp. $(yx)_t \in\vee q\mu$)).

μ is called an *$(\in, \in\vee q)$-fuzzy bi-filter* of S if it satisfies conditions (i) and (ii) of Definition 5.1, and
(i) $(\forall x, y \in S)(\forall t \in (0,1])(x_t \in \mu \rightarrow (xyx)_t \in\vee q\mu)$.

Example 5.2. Consider a set $S = \{a, b, c, d, e, f\}$ with the following multiplication table and order relation "$\leq$"

.	a	b	c	d	e	f
a	b	c	d	d	d	d
b	c	d	d	d	d	d
c	d	d	d	d	d	d
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	f	f	f	f	f	f

and

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(a,d),(b,d),(c,d),(d,e),(d,f),(a,e),(b,e),(c,e),(a,f),(b,f),(c,f)\}.$$

Then $(S,\cdot,\leq)$ is an ordered semigroup (see [17]). Left filters of S are $\{e,f\}$ and S. Define a fuzzy subset $\mu: S \to 0,1]$ by:

$$\mu(a) = \mu(b) = \mu(c) = \mu(d) = 0.3$$
$$\mu(e) = 0.7,$$
$$\mu(f) = 0.5$$

Then

$$U(\mu;t) := \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{e,f\} & \text{if } 0.3 < t \leq 0.5 \\ \emptyset & \text{if } 0.7 < t \leq 1. \end{cases}$$

Then μ is an $(\in, \in\vee q)$-fuzzy left filter of S.

Theorem 5.3. For a fuzzy subset μ of S, the conditions (i)-(iii) respectively, of Definition 5.1, are equivalent to the following conditions:
(i) $(\forall x,y \in S)(\mu(y) \geq \min\{\mu(x), 0.5\})$,
(ii) $(\forall x,y \in S)(\mu(xy) \geq \min\{\mu(x),\mu(y), 0.5\})$,
(iii) $(\forall x,y \in S)(\mu(xy) \geq \min\{\mu(x), 0.5\}$(*resp.* $\mu(xy) \geq \min\{\mu(y), 0.5\})$).

Proof: The proof follows from Theorem 4.3.

Theorem 5.4. For a fuzzy subset μ of S, the conditions (i)-(ii), and (iv) respectively, of Definition 5.1, are equivalent to the following conditions:
(L1) $(\forall x,y \in S)(\mu(y) \geq \min\{\mu(x), 0.5\})$,
(L2) $(\forall x,y \in S)(\mu(xy) \geq \min\{\mu(x),\mu(y), 0.5\})$,
(L3) $(\forall x,y \in S)(\mu(xyx) \geq \min\{\mu(x), 0.5\})$.

Proof: The proof of (L1) $\leftrightarrow$ (i) and (L2) $\leftrightarrow$ (ii) follows from Theorem 4.3.

(L3) $\to$ (iii). Let $x,y \in S$. If $\mu(x) = 0$, then $\mu(xyx) \geq \min\{\mu(x), 0.5\}$. Let $\mu(x) \neq 0$ and assume on the contrary that $\mu(xyx) < \min\{\mu(x), 0.5\}$. Choose $t \in (0,1]$ such that $\mu(xyx) < t \leq \min\{\mu(x), 0.5\}$. If $\mu(x) < 0.5$, then $\mu(xyx) < t \leq \mu(x)$ and so $x_t \in \mu$, but $(xyx)_t \overline{\in} \mu$, a contradiction. If $\mu(x) \geq 0.5$ then $\mu(xyx) < 0.5$ and so $x_{0.5} \in \mu$ but $(xyx)_{0.5} \overline{\in} \mu$, again a contradiction. Hence $\mu(xyx) \geq \min\{\mu(x), 0.5\}$ for all $x,y \in S$.
(iii)$\to$ (L3). Let $x,y \in S$ and $t \in (0,1]$ be such that $x_t \in \mu$. Then $\mu(x) \geq t$, and by (iii) we have $\mu(xyx) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\}$. If $t \leq 0.5$, then $\mu(xyx) \geq t$, i.e., $(xyx)_t \in \mu$. If $t > 0.5$, then $\mu(xyx) + t > 0.5 + 0.5 = 1$, i.e., $\mu(xyx) + t > 1$ and so $(xyx)_t q\mu$. Therefore, $(xyx)_t \in\vee q\mu$.

Theorem 5.5. A fuzzy subset μ of an ordered semigroup S is an $(\in, \in\vee q)$-fuzzy bi-filter of S if and only if it satisfies conditions (L1)-(L3) of Theorem 5.4.

Theorem 5.6. Let F be a bi-filter of S and μ be a fuzzy subset of S such that

$$\mu(x) := \begin{cases} \geq 0.5 & \text{if } x \in F \\ 0 & \text{if } x \in S \backslash F. \end{cases}$$

Then
(a) μ is a $(q, \in\vee q)$-fuzzy bi-filter of S.
(b) μ is an $(\in, \in\vee q)$-fuzzy bi-filter of S.

Proof: The proof follows from Theorem 4.8.

Theorem 5.7. Let μ be an $(\in, \in\vee q)$-fuzzy bi-filter of S such that $\mu(x) < 0.5$ for all $x \in S$. Then μ is an $(\in, \in)$-fuzzy bi-filter of S.

Proof: It is straightforward.

Theorem 5.8. A fuzzy subset μ of S is an $(\in, \in\vee$ q)-fuzzy bi-filter of S if and only if the set $U(\mu;t) (\neq \emptyset)$ is a bi-filter of S for all $t \in (0, 0.5]$.

Proof. It is straightforward.

Theorem 5.9. For a fuzzy subset μ of S, the following assertions are equivalent:
(i) μ is an $(\in, \in\vee q)$-fuzzy bi-filter of S.
(ii) $(\forall t \in (0,1])([\mu]_t \neq \emptyset \to \mu]_t$ is a bi-filter of $S)$.

Proof: The proof follows from Theorem 4.11.

Definition 5.10. Let $r, s \in 0,1]$ and $r < s$. Let μ be a fuzzy subset of an ordered semigroup S. Then μ is called a *fuzzy left(right) filter with thresholds* (r,s) of S if it satisfies the following assertions:
(i) $(\forall x,y \in S)(x \leq y, \max\{\mu(y), r\} \geq \min\{\mu(x), s\})$,
(ii) $(\forall x,y \in S)(\max\{\mu(xy), r\} \geq \min\{\mu(x),\mu(y), s\})$,
(iii) $(\forall x,y \in S)(\max\{\mu(xy), r\} \geq \min\{\mu(x), s\}$(resp. $\max\{\mu(xy), r\} \geq \min\{\mu(y), s\}$)).

If μ is a fuzzy filter with thresholds (r,s) of S, then we can conclude that μ is an ordinary fuzzy bi-filter when $r = 0, s = 1$ and μ is an $(\in, \in\vee q)$-fuzzy left (right) filter when $r = 0, s = 0.5$. Now, we characterize fuzzy left (right) filters with thresholds by their level left (right) filters.

Theorem 5.11. A fuzzy subset μ of an ordered semigroup S is a fuzzy left(right) filter with thresholds (r, s) of S if and only if $\mu_t(\neq \emptyset)$ is a left (right) filter of S for all $t \in (r, s]$.

Proof: The proof follows from Theorem 4.18.

Definition 5.12. Let $r, s \in 0,1]$ and $r < s$. Let μ be a fuzzy subset of an ordered semigroup S. Then μ is called a *fuzzy bi-filter with thresholds* (r, s) of S if it satisfies the following assertions:

(i) $(\forall x, y \in S)(x \leq y, \max\{\mu(y), r\} \geq \min\{\mu(x), s\})$,
(ii) $(\forall x, y \in S)(\max\{\mu(xy), r\} \geq \min\{\mu(x), \mu(y), s\})$,
(iii) $(\forall x, y \in S)(\max\{\mu(xyx), r\} \geq \min\{\mu(x), s\})$.

If μ is a fuzzy filter with thresholds (r, s) of S, then we can conclude that μ is an ordinary fuzzy filter when $r = 0, s = 1$ and μ is an $(\in, \in\vee q)$-fuzzy bi-filter when $r = 0, s = 0.5$. Now, we characterize fuzzy bi-filters with thresholds by their level bi-filters.

Theorem 5.13. A fuzzy subset μ of an ordered semigroup S is a fuzzy bi-filter with thresholds (r, s) of S if and only if $\mu_t(\neq \emptyset)$ is a bi-filter of S for all $t \in (r, s]$.

Proof: The proof follows from Theorem 4.18.

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References

[1] Mordeson, J. N., Malik, D. S. & Kuroki, N. (2003). Fuzzy Semigroups, Studies in Fuzziness and Soft Computing *Vol. 131*, Berlin, Springer-Verlag.

[2] Bhakat, S. K. & Das, P. (1996). $(\in, \in\vee q)$-fuzzy subgroups. *Fuzzy Sets and Systems*, *80*, 359-368.

[3] Bhakat, S. K. & Das, P. (1996). Fuzzy subrings and ideals redefined. *Fuzzy Sets and Systems*, *81*, 383-393.

[4] Rosenfeld, A. (1971). Fuzzy groups. *J. Math. Anal. Appl.*, *35*, 512-517.

[5] Kuroki, N. (1981). On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets and Systems*, *5*, 203-215.

[6] Jun, Y. B. & Song, S. Z. (2006). Generalized fuzzy interior ideals in semigroups, *Information Sciences*. *176*, 3079-3093.

[7] Khan, A. & Shabir, M. (2009). *(α,β)*-fuzzy interior ideals in ordered semigroups. *Lobachevskii Journal of Mathematics*, *30*, 30-39.

[8] Shabir, M. & Khan, A. (2008). Characterizations of ordered semigroups by the properties of their fuzzy generalized bi-ideals. *New Mathematics and Natural Computation*, *4*, 237-250.

[9] Kazanci, O. & Yamak, S. (2008). Generalized fuzzy bi-ideals of semigroup. *Soft Computing*, *12*, 1119-1124.

[10] Jun, Y. B., Khan, A. & Shabir, M. (2009). Ordered semigroups characterized by their $(\in, \in\vee q)$-fuzzy bi-ideals. *Bull. Malays. Math. Sci. Soc.*, *32*(2),, 391-408.

[11] Davvaz, B., Kazanci, O. & Yamak, S. (2009). Generalized fuzzy n-ary subpolygroups endowed with interval valued membership functions. *Journal Intelligent and Fuzzy Systems*, *20*, 159-168.

[12] Davvaz, B. & Mozafar, Z. (2009). $(\in, \in\vee q)$-fuzzy Lie subalgebra and ideals. *International Journal of Fuzzy Systems*, *11*(2) (2009), 123-129.

[13] Davvaz, B. & Corsini, P. (2008). On (α, β)-fuzzy H_v-ideals of H_v-rings, *Iran. J. Fuzzy Syst.*, *5*, 35-47.

[14] Davvaz, B., Zhan, J. & Shum, K. P. (2008). Generalized fuzzy polygroups endowed with interval valued membership functions, *Journal of Intelligent and Fuzzy Systems*, *19*, 181-188.

[15] Davvaz, B. (2008). Fuzzy *R*-subgroups with thresholds of near-rings and implication operators, *Soft Computing*, *12*, 875-879.

[16] Davvaz, B. (2006). $(\in, \in\vee q)$-fuzzy subnear-rings and ideals. *Soft Computing*, *10*, 206-211.

[17] Kazanci, O. & Davvaz, B. (2009). Fuzzy *n*-ary polygroups related to fuzzy points, *Computers & Mathematics with Applications*, *58*, 1466-1474.

[18] Ma, X., Zhan, J. & Jun, Y. B. (2009). On $(\in, \in\vee q)$-fuzzy filters of R_0-algebras, *Math. Log. Quart.*, *55*, 493-508.

[19] Liu, L. & Li, K. (2005). Fuzzy implicative and Boolean filters of R_0-algebras, *Information Sciences*, *171*, 61-71.

[20] Kehayopulu, N. & Tsingelis, M. (2002). Fuzzy sets in ordered groupoids, *Semigroup Forum*, *65*, 128-132.

[21]. Pu, P. M & Liu, Y. M. (1980) Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence., *J. Math. Anal. Appl.*, *76*, 571-599.

[22] Lee, S. K. & Park, K. Y. (2003). On right (left) duo po-semigroups, *Kangweon-Kyungki Math. Jour.*, *11*, 147-153.

[23] Yuan, X., Zhang, C. & Ren, Y. (2003). Generalized fuzzy groups and many-valued implications. *Fuzzy Sets and Systems*, *138*, 205-211.

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