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Numerics of stochastic parabolic differential equations with stable finite difference schemes

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Abstract

In the present article, we focus on the numerical approximation of stochastic partial differential equations of Itˆo type with space-time white noise process, in particular, parabolic equations. For each case of additive and multiplicative noise, the numerical solution of stochastic diffusion equations is approximated using two stochastic finite difference schemes and the stability and consistency conditions of the considered methods are analyzed. Numerical results are given to demonstrate the computational efficiency of the stochastic methods.

Keywords: Stochastic partial differential equations of Itˆo type; finite difference methods; multiplicative noise; additive noise; Saul'yev method; Liu method; convergence; consistency; stability

1. Introduction and preliminaries

Many natural phenomena and physical applications are modeled by partial differential equations and the efficiency of the computed solutions are analyzed and tested. Practically, a great number of uncertainties are involved in determining these partial differential equations. So, in many areas of applicable sciences such as financial mathematics, mechanic engineering and many complex phenomena such as wave propagation, phase transition and climate change, a stochastic model for describing these uncertainties is employed. Hence, the extensive application of random effects in describing practical sciences has developed the theory of stochastic partial differential equations, or SPDEs.

Thus, providing applicable numerical techniques and high accuracy computational methods is of great importance for approximating the solution of stochastic problems.

Many effective researches for solving stochastic differential equations as well as their strong and weak approximation have been implemented by Kloeden and Platen [1], Komori [2], Milstein [3], Röβler [4] and Higham [5].

In recent years, some of the main numerical methods for solving SPDEs like finite difference and finite element schemes [6-11], and some practical techniques like the method of lines [12, 11] for boundary value problems, have been applied to the linear stochastic partial differential equations, and the results of these approaches have been experimented numerically. In [13], we have considered the approximation of stochastic parabolic equations with real valued Brownian motion using two various finite difference methods, and their numerical results are investigated. The main aim of the current work is to verify the main properties of unconditional stable finite difference schemes when they develop to the stochastic case for approximating the solutions of stochastic parabolic equations based on the two-dimensional white noise process. In other words, we illustrate how stochastic term with space-time white noise process affects the stability conditions of unconditional stable Saul'yev and Crank-Nicolson techniques when they are reformulated for approximation of stochastic diffusion equations. It can be shown that these unconditional stable finite difference methods retain their stability conditions when they apply to stochastic diffusion equations driven by one-dimensional white noise process. In general, for a given physical system many different perturbations may be considered. Basically, noise may enter the physical system either as temporal fluctuations of internal variables or as external random parameters. Internal randomness is often

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considered as additive noise terms, while external fluctuations are modeled as multiplicative noise terms. This paper is concerned with the numerical approximation of the stochastic partial differential equation of the form

$$\frac{\partial u}{\partial t}(x,t)-\gamma\frac{\partial^2 u}{\partial t^2}(x,\ t)+\lambda\sigma(u(x,\ t))\dot{W}(x,t),\quad 0\le t\le T \qquad (1)$$

$u(x,0)=u_0(x),\quad 0\le x\le 1,$

where u is a real valued function of $x\in R_+$ and $x\in R_+^d$,with initial value $u_0(x)\in C_0([0,\ 1])$ and $\dot{W}(x,t)$ denotes the space-time white noise process. The parameter γ is the viscosity term and assumed to be a positive constant. We consider the numerical solutions of SPDE (1) driven by additive noise $\sigma(u(x,\ t))=1$ using implicit stochastic Crank-Nicolson scheme, and stochastic explicit Saul'yev method with multiplicative noise $\sigma(u(x,\ t))=u(x,\ t)$, and the qualification of these stochastic difference schemes will be verified. The white noise process defined in SPDE (1) is related to the two parameter Brownian motions or Brownian sheet $W(x,\ t)$ by the following differential equation:

$$\dot{W}(x,t)=\frac{\partial^2 W}{\partial x\partial t}(x,t),0\le t\le T,\ 0\le x\le 1$$

where $\dfrac{\partial^2 W}{\partial x\ \partial t}(x,\ t)$ denotes the mixed derivative of Brownian sheet. It should be noted that this is not a derivative in the ordinary sense, since the Brownian sheet is nowhere differentiable. There are some important properties of the standard Brownian sheet that should be mentioned. Firstly, if χ_S is the characteristic function on the rectangle S, then for

$$S\subset(0,\ T)\times(a,\ b)$$

$$\int_0^T\int_a^b \chi_S dW(x,\ t)=W(S),$$

$$\int_c^d\int^b \chi_S dW(x,t)=W(b,d)-W(a,d)-W(b,c)-W(a,c).$$

Secondly, if $E(\int_0^T\int_a^b f^2(x,t)dx\ dt)<\infty$ then

$$E(\int_0^T\int_a^b f(x,t)dW\ (x,t))^2=E(\int_0^T\int_a^b f^2(x,t)dx\ dt).$$

Allen et. al. [6] have suggested the following approximation for one-dimensional white noise process $\dot{W}(x)$, for computing the approximated solution of stochastic partial differential equations. The partition $0=x_1<x_2<...<x_{N+1}=1$ is defined on the interval [0,1], where $x_i=(i-1)\Delta x$ and $\Delta x=\dfrac{1}{N}$ .Then, the following approximation is defined for the white noise process $\dot{W}(x)$ on this partition

$$\frac{d\hat{W}}{dx}(x)=\frac{1}{\Delta x}\sum_{i=1}^{N}\eta_i\sqrt{\Delta x},$$

where

$$\eta_i=\frac{1}{\sqrt{\Delta x}}\int_{x_i}^{x_{i+1}}dW(x,t)\ \ ,i=1,...,N,$$

i.e. $\eta_i\sim N(0,1)$, and

$$\chi_i(x)=\begin{cases}1 & if\ \ x_i\le x<x_{i+1}\\ 0 & otherwise.\end{cases}$$

Clearly, this estimation is similar to the discrete time approximation of continuous time white noise when the solution of stochastic differential equations is numerically simulated. (see for example Kloeden and Platen [1]). Similarly, an approximate noise process is constructed to the generalized zero mean Gaussian process. Following the approach of Allen et. al.[6], the space [0, 1] ×[0,T] is partitioned by rectangles $[x_i,x_{i+1}]\times[t_j,t_{j+1}]$, where $x_i=(i-1)\Delta x$ and $t_j=(j-1)\Delta t$ for $i=1,...,M$ and $j=1,...,N$.

The following approximation for the mixed derivative of the generalized Gaussian white noise process can then be made with respect to the partition,

$$\frac{\partial^2\hat{W}}{\partial t\partial x}(x,t)=\frac{1}{\Delta x\Delta t}\sum_{i=1}^{M}\sum_{j=1}^{N}\eta_{ij}\sqrt{\Delta x\Delta t}\chi_i(x)\chi_j(t),$$

where $\eta_{ij}\sim N(0,\ 1),\ \Delta t=T/N,\ \Delta x=1/M$ and

$$\chi_i(x)=\begin{cases}1 & if\ \ x_i\le x<x_{i+1}\\ 0 & otherwise.\end{cases}$$

defines the characteristic function for x, and $\chi_j(t)$ is defined similarly for t , and

$$\eta_{ij} = \frac{1}{\sqrt{\Delta x \Delta t}} \int_{x_i}^{x_{i+1}} \int_{t_j}^{t_{j+1}} dW(x,t).$$

The outline of the paper is as follows: In section 2, the explicit unconditional stable Saul'yev method is reformulated for the stochastic parabolic equation driven by multiplicative noise and the stability and consistency conditions are investigated for the stochastic case. In section 3, the stochastic Crank-Nicolson implicit method is applied to the stochastic diffusion equation driven by additive noise and the efficiency of the proposed method is analyzed. The numerical results are presented in section 4 to support the theoretical analysis. Finally, some concluding remarks are given.

2. Multiplicative noise

The Saul'yev method was first introduced by Saul'yev [14] for solving initial value problems based on the two approximations that are implemented for computations proceeding in alternating directions, e.g., from left to right and from right to left [15, 16]. In applying the left to right Saul'yev method to the stochastic diffusion equation, the time derivative is approximated with the usual forward-difference expression and the space derivative is approximated by

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j-1}^{n+1} - u_j^{n+1} - u_j^n + u_{j+1}^n}{\Delta x^2}.$$

So, the stochastic difference scheme (SDS) that approximate the stochastic diffusion equation with multiplicative noise (σ(u)= u) is:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \gamma \frac{u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + \lambda u_j^n \frac{\partial^2 \hat{W}}{\partial t \partial x}\Big|_{jn}$$

or

$$u_j^{n+1} = u_j^n + \gamma \frac{\Delta t}{\Delta x^2}[u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}]$$

$$+ \frac{\lambda}{\Delta x} u_j^n W_j^n. \qquad (2)$$

where $\rho = \frac{\Delta t}{\Delta x^2}$. Here, u_j^n is intended as an approximation to $u(j\Delta x, n\Delta t)$; and $W_j^n = W(R_{j,n})$, where $R_{j,n}$ is the rectangle $[j\Delta x, (j+1)\Delta x] \times [n\Delta t, (n+1)\Delta t]$.

Basically, these schemes discretize continuous space and time into an evenly distributed grid system, and the values of the state variables are evaluated at each node of the grids. Considering a uniform space grid Δx and time grid Δt in the time-space lattice, we can estimate the solution of equation at the points of this lattice. The value of the approximate solution at the point $(k\Delta x, n\Delta t)$ will be denoted by u_k^n, where n, k are integers.

We want to approximate the solution of SPDE (1) for the case of multiplicative noise by random variable u_k^n defined by stochastic difference scheme (2), which is the stochastic version of Saul'yev method. A similar formulation can be considered for the right to left Saul'yev method. For all proposed schemes, the increments of Wiener process are assumed independent of the state u_k^n.

2.1. Stability analysis

Stability is probably the most important problem in any algorithm since it is a necessary rather than sufficient condition for accuracy. Applied to parabolic equations, Saul'yev's technique is unconditionally stable and, because it is explicit, it is not necessary to solve a large system of simultaneous equations at each time step in the algorithm like implicit unconditional stable methods [16]. Consequently, we are concerned with studying the stability analysis of the Saul'yev SDS for approximating the stochastic diffusion equation with space-time process based on multiplicative noise.

Von Neumann introduced a method to prove stability using Fourier analysis so that it can give necessary and sufficient condition for the stability of deterministic finite difference schemes [17, 18]. If $u \in l_2$ and $\hat{u}^{n+1}$ are the Fourier transformation of u^{n+1} then

$$u_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{\Delta\pi}}^{\frac{\pi}{\Delta\pi}} e^{im\Delta x\xi} \hat{u}^{n+1}(\xi) d\xi, \qquad (3)$$

or in the discrete form:

$$\hat{u}^{n+1} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{m=\infty} e^{-im\Delta x\xi} u_m^{n+1} \Delta x, \qquad (4)$$

where ξ is a real variable. Substituting in a stochastic difference scheme, we have

$$\hat{u}^{n+1}(\xi) = g(\Delta x\xi, \Delta t, \Delta x)\hat{u}^n(\xi). \qquad (5)$$

that $g(\Delta x\xi, \Delta t, \Delta x)$ is the amplification factor of the stochastic difference scheme. The decision whether a scheme is stable or not can be simplified by the aid of amplification factor.

Like the deterministic case, we get the following necessary and sufficient condition for a scheme's stability via its amplification factor, see Roth [19]

$$E\,|\,g(\Delta x\xi,\Delta t,\Delta x)\,|^2 \le 1+K\Delta t. \qquad (6)$$

Theorem 1. The stochastic Saul'yeu scheme is stable for:

$$r\,\Delta x \ge \frac{\lambda^2}{e^{1+e\lambda^2}}$$

according to the Fourier-Transformation analysis for the stochastic diffusion equation (1) with multiplicative noise.

Proof: According to the Fourier-inversion-formula u_m^n has the following transformation:

$$u_m^n = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{im\Delta x\xi}\hat{u}^n(\xi)d\xi,$$

substituting in the stochastic Saul'yev Scheme we have:

$$(1+\gamma\rho)\hat{u}^{n+1}(\xi)-(\gamma\rho)e^{-i\Delta x\xi}\hat{u}^{n+1}(\xi)=$$
$$(\gamma\rho)e^{i\Delta x\xi}\hat{u}^n(\xi)+(1-\gamma\rho)\hat{u}^n(\xi)+\frac{\lambda}{\Delta x}\hat{u}^n(\xi)W_j^n,$$

or

$$[(1+\gamma\rho)-(\gamma\rho)e^{-i\Delta x\xi}]\hat{u}^{n+1}(\xi)=$$
$$[(\gamma\rho)e^{i\Delta x\xi}+(1-\gamma\rho)]\hat{u}^n(\xi)+\frac{\lambda}{\Delta x}\hat{u}^n(\xi)W_j^n,$$

and then

$$\hat{u}^{n+1}(\xi)=\{\frac{1-\gamma\rho+\gamma\rho e^{i\Delta x\xi}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}+$$
$$\frac{\lambda}{\Delta x(1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi})}W_j^n\}\hat{u}^n(\xi). \qquad (7)$$

So, the amplification factor of the stochastic Saul'yev scheme is:

$$g(\Delta x\xi,\Delta t,\Delta x):=\{\frac{1-\gamma\rho+\gamma\rho e^{i\Delta x\xi}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}+$$
$$\frac{\frac{\lambda}{\Delta x}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}W_j^n\}.$$

$$E\,|\,g(\Delta x\xi,\Delta t,\Delta x)\,|^2=E\,|\,\frac{1-\gamma\rho+\gamma\rho e^{i\Delta x\xi}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}$$
$$+\frac{\frac{\lambda}{\Delta x}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}W_j^n\,|^2$$

$$=E\,|\,\frac{1-\gamma\rho+\gamma\rho e^{i\Delta x\xi}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}\,|^2+E\,|\,\frac{\frac{\lambda}{\Delta x}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}W_j^n\,|^2$$
$$+2E\,|\,\frac{1-\gamma\rho+\gamma\rho e^{i\Delta x\xi}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}\times\frac{\frac{\lambda}{\Delta x}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}W_j^n\,|.$$

because of the independence of the Wiener process, we have:

$$E\,|\,g(\Delta x\xi,\Delta t,\Delta x)\,|^2=(\frac{1-\gamma\rho+\gamma\rho e^{i\Delta x\xi}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}})^2$$
$$+(\frac{\frac{\lambda}{\Delta x}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}})^2\,\Delta t\Delta x. \qquad (8)$$

Since for every γ, ρ and Δx we have:

$$|\,\frac{1-\gamma\rho+\gamma\rho e^{i\Delta x\xi}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}}\,|\le 1$$

$$(\frac{\frac{\lambda}{\Delta x}}{1+\gamma\rho-\gamma\rho e^{-i\Delta x\xi}})^2\le\frac{\lambda^2}{\Delta x^2}.$$

If we assume

$$\Delta x\ge\frac{\lambda^2}{e^{1+e^{\lambda^2}}},$$

then we get

$$E\,|\,g(\Delta x\xi,\Delta t,\Delta x)\,|^2\le 1+\frac{\lambda^2}{\Delta x}\Delta t\le 1+e^{1+e\lambda^2}\Delta t$$

so we have

$$\mathrm{E}\,|\,\mathrm{g}(\Delta x\xi,\Delta t,\Delta x)\,|^2\le 1+\mathrm{K}\Delta t.$$

For $K=e^{1+e^{\lambda 2}}$ Therefore, $\Delta x\ge\frac{\lambda^2}{e^{1+e^{\lambda 2}}}$ is a sufficient condition for stability of the stochastic Saul'yev scheme applying to stochastic diffusion equation with multiplicative noise.

2.2. Consistency condition

In general, consistency implies that the solution of stochastic partial differential equations is an approximation of the considered stochastic finite difference. Consider a stochastic partial differential equation:

$$Lv = G$$

where L denotes the differential operator and $G \in L^2(R)$ is an inhomogeneity. Assuming u_k^n is the solution that is approximated by a stochastic finite difference scheme denoted by L_k^n, and applying the stochastic scheme to the SPDE, we have $L_k^n u_k^n = G_k^n$, whereby G_k^n is the approximation of the inhomogeneity.

Definition 1. (Consistency of an SDS) The finite stochastic difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the stochastic partial differential equation $Lv = G$ at point (x, t), if for any continuously differentiable function Φ=Φ(x, t), in mean square

$$E \,\|\, (L\Phi - G)_k^n - [L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n] \,\|^2 \to 0 \qquad (10)$$

as $\Delta x \to 0, \Delta t = t$, and $(k\Delta x, (n+1)\Delta t)$ converges to (x, t).

Theorem 2. The stochastic Saul'yev scheme is consistent in mean square for the stochastic diffusion equation (1) with multiplicative noise.

Proof: Let Φ(x, t) be a smooth function (at least continuously differentiable in x and continuous in t), then we have

$$L(\Phi)_k^n = \int_{n\Delta t}^{(n+1)\Delta t}\int_{k\Delta x}^{(k+1)\Delta x} \Phi_t(r,s)drds \quad -\gamma\int_{n\Delta t}^{(n+1)\Delta t}\int_{k\Delta x}^{(k+1)\Delta x} \Phi_{xx}(r,s)drds$$
$$-\lambda\int_{n\Delta t}^{(n+1)\Delta t}\int_{k\Delta x}^{(k+1)\Delta x} \Phi(r,s)dW(r,s)$$

and

$$-\gamma \frac{\Delta t}{\Delta x^2}[\Phi((k+1)\Delta x, n\Delta t) - |\,\Phi(k\Delta x, n\Delta t)$$
$$L_k^n(\Phi) = (\Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t))\Delta x$$
$$-\Phi(k\Delta x, (n+1)\Delta t) + \Phi((k-1)\Delta x, (n+1)\Delta t)]\Delta x$$
$$-\lambda\Phi(k\Delta x, n\Delta t)[W((k+1)\Delta x, (n+1)\Delta t) - W(k\Delta x, (n+1)\Delta t)$$
$$-W[(k+1)\Delta x, n\Delta t) - W(k\Delta x, n\Delta t)].$$

Therefore in mean square, we obtain:

$$E\,|\,L(\Phi)\,|_k^n - L_k^n(\Phi)\,|^2 \le E\,|\int_{n\Delta t}^{(n+1)\Delta t}\int_{k\Delta x}^{(k+1)\Delta x} \Phi_t(r,s)$$
$$-\frac{\Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)}{\Delta t} dtds\,|^2$$
$$+2\gamma^2 E\,|\int_{n\Delta t}^{(n+1)\Delta t}\int_{k\Delta x}^{(k+1)\Delta x} \Phi_{xx}(r,s)$$
$$-\frac{1}{\Delta x^2}((\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)$$
$$-\Phi(k\Delta x, (n+1)\Delta t) + (\Phi((k-1)\Delta x, (n+1)\Delta t))drds\,|^2$$
$$+2\sigma^2 E\,|\int_{k\Delta x}^{(k+1)\Delta x}\int_{n\Delta t}^{(n+1)\Delta t} (\Phi(r,s) - \Phi(k\Delta x, n\Delta t))dW(r,s)\,|^2,$$

$$E\,|\,L(\Phi)\,|_k^n - L_k^n(\Phi)\,|^2 \le \int_{n\Delta t}^{(n+1)\Delta t}\int_{k\Delta x}^{(k+1)\Delta x} \Phi_t(r,s)$$
$$-\frac{\Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)}{\Delta t} drds\,|^2$$
$$+2\gamma^2\,|\int_{n\Delta t}^{(n+1)\Delta t}\int_{k\Delta x}^{(k+1)\Delta x} \Phi_{xx}(r,s)$$
$$-\frac{1}{\Delta x^2}(\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)$$
$$-\Phi(k\Delta x, (n+1)\Delta t) + (\Phi((k-1)\Delta x, (n+1)\Delta t))drds\,|^2$$
$$+2\lambda^2\,|\int_{k\Delta x}^{(k+1)\Delta x}\int_{n\Delta t}^{(n+1)\Delta t} (\Phi(r,s) - \Phi(k\Delta x, n\Delta t))\,|^2\, drds.$$

Since $\Phi(x, t)$ is only a deterministic function as, we have

$$E\,|\,L(\Phi)\,|_k^n\,(\Phi)\,|^2 \to 0,$$

when $n, k \to \infty$. This proves the consistency.

Essentially, it is extremely important for the solution of stochastic difference schemes (SDS) to converge to the solution of the stochastic partial differential equations or SPDEs.

Definition 2. (Convergence of an SDS) A stochastic difference scheme $L_k^n u_k^n = G_k^n$ approximating the stochastic partial differential equation $Lv = G$ is convergent in mean square at time t if, as $(n+1)\Delta t$ converges to t,

$$E\,\|\,u_{n+1} - v_{n+1}\,\|^2 \to 0 \qquad (11)$$

$$\text{for } (n+1)\Delta t = t, \quad \text{and} \quad \Delta x \to 0$$

where u^{n+1} and v^{n+1} are infinite dimensional vectors

$$u^{n+1} = \left(\ldots, u_{k-2}^{n+1}, u_{k-1}^{n+1}, u_k^{n+1}, u_{k+1}^{n+1}, u_{k+2}^{n+1}, \ldots\right)^T,$$
$$v^{n+1} = \left(\ldots, v_{k-2}^{n+1}, v_{k-1}^{n+1}, v_k^{n+1}, v_{k+1}^{n+1}, v_{k+2}^{n+1}, \ldots\right)^T.$$

According to the theorems proved about the stability and consistency of the stochastic Saul'yev scheme and the stochastic version of the Lax-Richtmyer theorem [20], stochastic Saul'yev method is convergent for solving stochastic diffusion equation (1) with multiplicative noise.

3. Additive noise

Applying the stochastic implicit Crank-Nicolson to the stochastic diffusion equation (1) with Additive noise $(\sigma(u)=1)$ we have

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{\gamma}{2}\left(D^2u_j^n+D^2u_j^{n+1}\right)+\lambda\frac{\partial^2\hat{W}}{\partial t\partial x}|_{jn}$$
$$=\frac{\gamma}{2\Delta x^2}\left(u_{j-1}^n-2u_j^n+u_{j+1}^n+u_{j-1}^{n+1}-2u_j^{n+1}+2u_{j+1}^{n+1}\right)+\lambda\frac{\partial^2\hat{W}}{\partial t\partial x}|_{jn},$$

which can be written as

$$-\gamma ru_{j-1}^{n+1}+(1+2\gamma u)u_j^{n+1}-\gamma ru_{j+1}^{n+1}=$$
$$\gamma ru_{j-1}^n+(1-2\gamma r)u_j^n+\gamma ru_{j+1}^n+\frac{\lambda}{\Delta x}W_j^n, \qquad (12)$$

where $r=\frac{\Delta t}{\Delta x^2}$. Assuming u_j^n is the approximation of SPDE (1) at $(j\Delta x,n\Delta t)$ and $W_j^n=W(R_{j,n})$ where $R_{j,n}$ is the rectangle $[j\Delta x,(j+1)\Delta x]\times[n\Delta t,(n+1)\Delta t]$. We want to investigate the qualification of this implicit stochastic difference scheme in the viewpoint of stability, consistency and convergence.

3.1. Stability analysis

Definition 3. (Stability of an SDS) A stochastic difference scheme is said to be stable with respect to a norm in mean square if there exist some positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and non negative constants K and β such that

$$\mathrm{E}\left\|u^{n+1}\right\|^2\le k e^{\beta t}\mathrm{E}\left\|u^0\right\|^2. \qquad (13)$$

For all $0\le t=(n+1)\Delta t, 0\le\Delta x\le\overline{\Delta x_0}, 0\le\Delta t\le\overline{\Delta t_0}$

Theorem 3. The stochastic Crank-Nicolson scheme is stable in mean square with respect to $|.|\infty=\sqrt{\sup_k|.|^2}$ -norm for the stochastic diffusion equation (1) with additive noise.

Proof: Applying $\mathrm{E}|.|^2$ to (12), in mean square we get:

$$E\left|-\gamma ru_{j-1}^{n+1}+(1+2\gamma r)u_j^{n+1}-\gamma ru_{j+1}^{n+1}\right|^2$$
$$=E\left|\gamma ru_{j-1}^n+(1-2\gamma r)u_j^n+\gamma ru_{j+1}^n+\frac{\lambda}{\Delta x}W_j^n\right|^2$$
$$=E\left|\gamma ru_{j\_1}^n+(1-2\gamma r)u_j^n+\gamma ru_{j+1}^n\right|^2+\left|\frac{\lambda}{\Delta x}W_j^n\right|^2$$
$$\le(\gamma r)^2E|u_{j-1}^n+(1-2\gamma r)^2E|u_j^n|^2+(\gamma r)^2E|u_{j+1}^n|^2$$
$$+2|\gamma r||1-2\gamma r|E\left|u_{j-1}^nu_j^n\right|+2|\gamma r|^2E\left|u_{j-1}^n+u_{j+1}^n\right|$$
$$+2|\gamma r||1-2\gamma r|E\left|u_j^nu_{j+1}^n\right|+\frac{\lambda^2}{\Delta x^2}E\left|W_j^n\right|^2$$

so we have

$$\mathrm{E}\left|-\gamma ru_{j-1}^{n+1}+(1+2\gamma r)u_j^{n+1}-\gamma ru_{j+1}^{n+1}\right|^2$$
$$\le\{(\gamma r)^2+(1+2\gamma r)^2+(\gamma r)^2+2|\gamma r||1+2\gamma r|+2(\gamma r)^2$$
$$+2|\gamma r||1+2\gamma r|)\}\left(\sup_k\mathrm{E}\left|u_k^n\right|^2+\frac{\lambda^2}{\Delta x}\Delta t\right).$$

This holds for every j on the $(n+1)-$th time step, so we have

$$\{(\gamma r)^2+(1+2\gamma r)^2+(\gamma r)^2+2|\gamma r|\times|1+2\gamma r|$$
$$+2(\gamma r)^2+2|\gamma r|\times|1+2\gamma r|\}\sup_k E|u_k^{n+1}|^2$$
$$\le\{(\gamma r)^2+(1+2\gamma r)^2+(\gamma r)^2+2|\gamma r|\times|1+2\gamma r|$$
$$+2(\gamma r)^2+2|\gamma r|\times|1+2\gamma r|\}(\sup_k E|u_k^n|^2+\frac{\lambda^2}{\Delta x}\Delta t).$$

Therefore we have

$$\sup_k E|u_k^{n+1}|^2\le\sup_k E|u_k^n|^2+\frac{\lambda^2}{\Delta x}\Delta t$$
$$\le\sup_k E|u_k^0|^2+(n+1)\frac{\lambda^2}{\Delta x}\Delta t,$$

and

$$\|u^{n+1}\|_\infty^2\le\|u^0\|_\infty^2+\frac{\lambda^2t}{\Delta x}$$
$$\le\|u^0\|_\infty^2\{1+\frac{\lambda^2t}{\|u^0\|_\infty^2\Delta x}\}$$
$$\le\|u^0\|_\infty^2\{1+\frac{\lambda^2t}{\|u^0\|_\infty^2\Delta x}+\frac{1}{2!}(\frac{\lambda^2t}{\|u^0\|_\infty^2\Delta x})^2+\ldots\}.$$

Assuming $\Delta x\ge\lambda^2\big/(e^{1+e^{\lambda^2}}\|u^0\|_\infty^2)$, we have

$$\| u^{n+1} \|_\infty^2 \le \| u^0 \|_\infty^2 \{1 + e^{1+e^{\lambda 2}} + \frac{e^{2(1+e^{\lambda 2})}}{2!} + \frac{e^{3(1+e^{\lambda 2})}}{3!} + \dots\},$$

$$\| u^{n+1} \|_\infty^2 \le | e^{e^{1+\lambda^2}} \| u^0 \|_\infty^2,$$

$$\| u^{n+1} \|_\infty \le | e^{\frac{e^{1+e^{\lambda 2}}}{2}} \| u^0 \|_\infty .$$

Therefore, the stochastic Crank-Nicolson scheme is stable for $| \Delta x \ge \dfrac{\lambda^2}{e^{1+e^{\lambda 2}} \| u^0 \|_\infty^2}$ applying to stochastic diffusion equation with additive noise.

3.2. Consistency condition

Theorem 4. The stochastic Crank-Nicolson scheme is consistent in mean square for the stochastic diffusion equation (1) with additive noise.

Proof: If $\Phi(x,t)$ be a smooth function, then we have:

$$L(\Phi)_k^n = \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta x}^{(k+1)\Delta x} \Phi_t(r,s) dr\, ds$$
$$-\gamma \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta x}^{(k+1)\Delta x} \Phi_{xx}(r,s) dr\, ds$$
$$-\lambda \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta x}^{(k+1)\Delta x} dW(r,s)$$

Therefore, in mean square we get

$$L_k^n(\Phi) = (\Phi(k\Delta x,(n+1)\Delta t)) - \Phi(k\Delta x, n\Delta t))\Delta x$$
$$-\gamma \frac{\Delta t}{2\Delta x^2}[\Phi((k-1)\Delta x, n\Delta t) - 2\Phi(k\Delta x, n\Delta t) + \Phi((k+1)\Delta x, n\Delta t)$$
$$+\Phi((k-1)\Delta x,(n+1)\Delta t) - 2\Phi(k\Delta x,(n+1)\Delta t) + \Phi((k+1)\Delta x,(n+1)\Delta t)]\Delta x$$
$$-\lambda[W((k+1)\Delta x,(n+1)\Delta t) - W(k\Delta x,(n+1)\Delta t)$$
$$-W((k+1)\Delta x, n\Delta t) - W(k\Delta x, n\Delta t)].$$

$$E | L(\Phi)_k^n - L_k^n(\Phi) |^2 \le E | \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta x}^{(k+1)\Delta x} \Phi_t(r,s)$$
$$- \frac{\Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)}{\Delta t} drds \Big|^2$$
$$+2\gamma^2 E | \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta x}^{(k+1)\Delta x} \Phi_{xx}(r,s)$$
$$-\frac{1}{2\Delta x^2}[\Phi((k-1)\Delta x, n\Delta t) - 2\Phi(k\Delta x, n\Delta t)$$
$$+\Phi((k+1)\Delta x, n\Delta t) + \Phi((k-1)\Delta x,(n+1)\Delta t) - 2\Phi(k\Delta x,(n+1)\Delta t)$$
$$+\Phi((k+1)\Delta x,(n+1)\gamma\Delta t)]drds \Big|^2$$
$$+2\lambda^2 E \Big| \int_{k\Delta x}^{(k+1)\Delta x} \int_{n\Delta t}^{(n+1)\Delta t} dW(r,s) \Big|^2 .$$

Therefore,

$$E | L(\Phi)_k^n - L_k^n(\Phi) |^2 \le | \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta x}^{(k+1)\Delta x} \Phi_t(r,s)$$
$$- \frac{\Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t)}{\Delta t} drds \Big|^2$$
$$+2\gamma^2 E \Big| \int_{n\Delta t}^{(n+1)\Delta t} \int_{k\Delta x}^{(k+1)\Delta x} \Phi_{xx}(r,s)$$
$$-\frac{1}{2\Delta x^2}[\Phi((k-1)\Delta x, n\Delta t) - 2\Phi(k\Delta x, n\Delta t)$$
$$+\Phi((k+1)\Delta x, n\Delta t) + \Phi((k-1)\Delta x,(n+1)\Delta t) - 2\Phi(k\Delta x,(n+1)\Delta t)$$
$$+\Phi((k+1)\Delta x,(n+1)\Delta t)]drds \Big|^2 + 2\lambda^2 \Delta t \Delta x$$

since $\Phi(x,t)$ is only a deterministic function as, we have

$$E \left| L(\Phi)_k^n - L_k^n(\Phi) \right|^2 \to 0,$$

when $\Delta x, \Delta t \to 0$. This proves the consistency.

As a result, the stochastic Crank-Nicolson method is convergent in its region of stability for approximating the solution of stochastic diffusion equation (1) with additive noise according to the stochastic Lax-Richtmyer theorem.

4. Numerical results

Computational efficiency is another important factor in evaluating the superiority of the numerical techniques. In this section, we perform some numerical tests for approximating the solutions of SPDE (1). We apply the two stochastic Saul'yev and Crank-Nicolson schemes to the stochastic diffusion equation driven by multiplicative and additive noise. In all our computations, the space domain is the interval Ω = [0, 1] and discretized into M uniform grid points. We carry out 10000 realizations for each test, and display the averaged solutions along with the considered simulations.

4.1. Example 1.

We examine the performance of the proposed Stochastic Saul'yev Scheme for stochastic diffusion equation with multiplicative noise of the form:

$$\frac{\partial u(x,t)}{\partial t} - \gamma \frac{\partial^2 u(x,t)}{\partial x^2} = \lambda u(x,t) \dot{W}(x,t) \qquad (14)$$

subject to the following initial condition:

$$u(0,x)=\exp(-\frac{(x-0.2)^2}{\gamma}),\ x\in[0,1],$$

$$u(0,x)=\exp(-\frac{(x-0.2)^2}{\gamma}),\ x\in[0,1],$$

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$$u(0,x)=\exp(-\frac{(x-0.2)^2}{\gamma}),\ x\in[0,1],$$

and the boundary conditions:

$$u(t,0)=\frac{1}{\sqrt{4t+1}}\exp(-\frac{0.04}{\gamma(4t+1)}),$$

$$u(t,1)=\frac{1}{\sqrt{4t+1}}\exp(-\frac{0.64}{\gamma(4t+1)}).$$

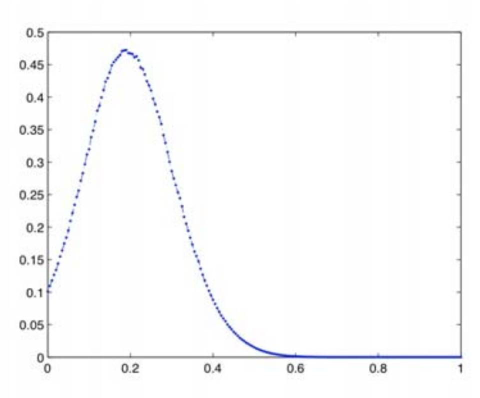

Fig. 1. Mean solution of stochastic diffusion equation driven by additive noise with γ =0.005 and λ =1.5 using Saul'yev method with 200 mesh points

In order to examine the behavior of the numerical solution with respect to the various values of the SPDE's coefficients, we used different values for diffusion constant γ and stochastic coefficient λ in our tests. Assuming Δt =0.01, λ =1.5, according to the stability conditions for approximating the solutions of stochastic diffusion equation (14) with multiplicative noise at time t = 1, we obtain:

$$\Delta x \geq \frac{\lambda^2}{e^{1+e^{\lambda 2}}} \rightarrow \Delta x \geq 6.27\times 10^{-5}$$

In order to qualify the numerical results of the considered stochastic diffusion equation, we plot, in Fig. 1, the stochastic solutions using stochastic Saul'yev Scheme (2) with γ = 0.005 on a mesh of 200 gridpoints. The computational results for approximating the solution of SPDE(14) is shown in Table 1 considering several values for time step and space size, γ and λ.

Table 1. Test of multiplicative SPDE by the stochastic Saul'yev method

γ	λ	Δt	Δx	$E(u(0.2,1))$	$E(u(0.2,1))^2$
0.005	2.5	0.005	0.005	0.480932	0.584977
0.05	1.5	0.01	0.01	0.499408	0.265886
0.1	2	0.0025	0.0025	0.472556	0.223612
0.2	1	0.01	0.025	0.506348	0.263021

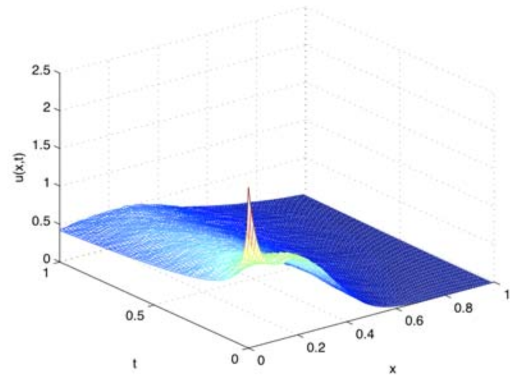

Fig. 2. Mean solutions of stochastic diffusion equation using stochastic Saul'yev method

The evolution in time of averaged solution for the multiplicative stochastic diffusion with γ = 0.04 and λ = 4 is shown in Fig. 2 during the time interval [0, 1]. In Fig. 3, we plot the results obtained by six different realizations (plotted by solid lines) at t=0.75, with the averaged solution plotted by dotted lines) for comparison reasons. As it can be seen, the computed stochastic solution preserves the symmetry in the computational domain and, at every realization, the simulated solution remains close to the averaged one.

4.2. Example 2.

We consider another test example for approximating the solution of stochastic diffusion equation driven by additive noise of the form

$$\frac{\partial u(x,t)}{\partial t}-\gamma\frac{\partial^2 u(x,t)}{\partial x^2}=\lambda\dot{W}(x,t) \qquad (15)$$

with initial condition

$$u(x,0)=1-4(x-\frac{1}{2})^2,$$

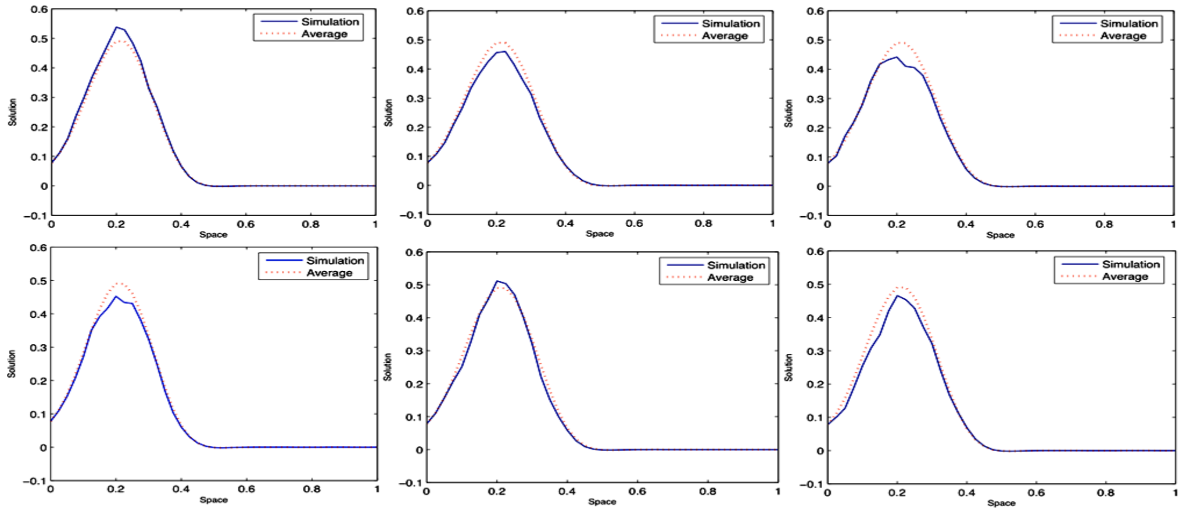

Fig. 3. Six different simulations with the stochastic Saul'yev scheme at t=0.75

and boundary condition $u(0,t) = u(1,t) = 0$, using stochastic Crank-Nicolson method. In order to examine the behavior of the numerical solutions, we provide, in Table 2, the averaged solution of (15) with some different values for diffusion and stochastic coefficients.

Table 2. Test of additive SPDE by the stochastic Crank-Nicolson method

γ	λ	Δt	Δx	E(u(0.5,0.8))	$E(u(0.5,0.8))^2$
0.5	1.5	0.01	0.01	0.011012	0.027708
0.05	2	0.005	0.005	0.638203	0.520909
0.1	2.5	0.00625	0.01250	0.394541	0.299308
0.005	3.5	0.003125	0.00625	0.988298	1.867629

In Fig. 4 we have represented the numerical solutions of SPDE (15) subject to the initial condition $u(x,0) = \sin(3\pi x) - 2\sin(5\pi x)$ and boundary conditions $u(x,0) = u(x,t) = 0$, with $\gamma = 0.005$ and $\lambda = 4$ during the time interval [0, 1].

5. Conclusion

This paper has provided two stochastic finite difference methods for the numerical solution of stochastic parabolic equations with space-time white noise process. The stable explicit Saul'yev and implicit Crank-Nicolson schemes are developed for the stochastic case for solving the parabolic SPDEs driven by multiplicative and additive noise. In this viewpoint, the most important properties of a stochastic finite difference scheme have been described and analyzed. Despite the fact that two explicit and implicit methods are unconditionally stable for solving deterministic diffusion equations, applying to the parabolic SPDEs with two-dimensional white noise process, the stochastic term limits the stability conditions.

The proposed methods have been illustrated by numerical examples and stochastic finite difference approximation for the stochastic diffusion equation has been demonstrated.

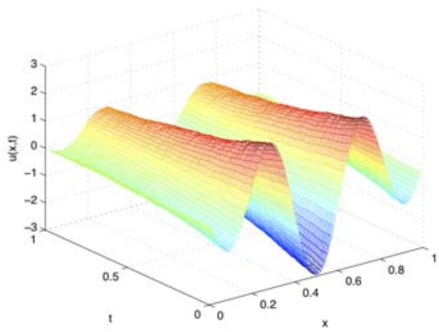

Fig. 4. Mean solution of stochastic diffusion with additive noise problem using stochastic Crank-Nicolson method

Another open question is how to extend such methods for non-uniform mesh, and how to define the mesh with regard to local truncation error at each grid point. This method appears to yeild a better approximation for computing the numerical solution of stochastic parabolic equations.

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