IJST (2012) A1: 47-50
Iranian Journal of Science & Technology

http://www.shirazu.ac.ir/en

Pantic B-spline wavelets and their application for solving linear integral equations

A. Askari-Hemmat[1]* and Z. Rahbani[2]

[1] *Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran*
[1] *Department of Mathematics, Kerman Graduate University of Technology, Kerman, Iran*
[1] *International Center for Science, High Technology and Environmental Sciences, Kerman, Iran*
[2] *Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran*
E-mails: askari@mail.uk.ac.ir, zrahbani@mail.vru.ac.ir

Abstract

In this work we deal with the question: how can one improve the approximation level for some nonlinear integral equations? Good candidates for this aim are semi orthogonal B-spline scaling functions and their duals. Although there are different works in this area, only B-spline of degree at most 2 are used for this approximation. Here we compute B-spline scaling functions of degree 4 and their duals, then we will show that, by using them, one can have better approximation results for the solution of integral equations in comparison with less degrees or other kinds of scaling functions. Some numerical examples show their attractiveness and usefulness.

***Keywords**:* Integral equations; B-spline scaling function; dual functions; wavelets

1. Introduction

The theory and application of integral equation is an important subject in many areas of engineering, signal processing, physics and applied mathematics. One can also consider them as a reformulation of other mathematical problems such as partial and ordinary differential equations, which are of interest to solve. In this paper we try to solve a large class of these equations called Fredholm integral equations.

Although some these equations are solved analytically, the rest are solved numerically. Several numerical methods for approximating the solution of integral equations are known. Some of them for Fredholm-Hammerstein integral equations are: variation method, collocation-type method and iterated collocation method [1]. Some of these methods transform a given integral equation into a system of nonlinear equations, while others like applying orthonormal bases, reduce them to a linear system of algebraic equations [2, 3, 4].

On the other hand, wavelets theory is a relatively new and emerging area in mathematical research which is used in a width range of applications such as signal processing, time-frequency analysis and segmentation. Also, wavelets as orthonormal bases are good candidates for providing fast algorithms in numerical aspects in approximating, because of their vanishing moment and small supports which lead to a sparse matrix.

In this work we apply compactly supported semi orthogonal B-spline father wavelets, specially constructed for the bounded interval [0, 1] to solve the second kind of linear

Fredholm integral equations of the form

$$y(x) = f(x) + \int_0^1 k(x,t)y(t)dt, \qquad 0 \le x \le 1, \tag{1}$$

where f and k are given continuous functions and y is an unknown function to be determined.

It is known that semi orthogonal wavelets are best suited for integral equation applications than orthogonal ones [5]. Semi orthogonal compactly supported B-spline wavelets have interesting properties such as, in a bounded interval, they behave better and easier than other wavelets in boundary conditions, and they have close-form expressions. Based on our method, (1) will be reduced to a set of algebraic equations by expanding y in terms of pantic B-spline scaling functions, with unknown coefficients. These coefficients will then be determined from the properties of the desired B-splines. The method is computationally attractive and applications are described through illustrative examples.

This paper is organized as follows: in section 2 we present pantic B-spline functions and some of

*Corresponding author
Received: 11 January 2011 / Accepted: 23 July 2011

their properties which will guide us to find their dual. In section 3, the method which helps finding algebraic equations corresponding to the integral equation is described. In section 4 we give our numerical computations and demonstrate the accuracy of this method and functions by considering numerical examples.

2. Pantic B-spline scaling functions supported in [0, 1]

When we work with semi orthogonal B-spline scaling functions on the entire real line, we should notice that they may be outside the domain of problem. In this article, to avoid this matter, construction of compactly supported B-spline functions will be on the interval [0, 1]. Also, to ensure that there exist at least one complete inner scaling function, the condition $2^J \geq n$ must full fill for B-spline scaling functions of order n. Here we use B-spline order 5 (degree 4), so the pantic B-spline of the lowest level, which has to be integer, is $J = 3$. This contains all octave levels to $J = 3$. B-spline functions of order n can be determined by formula $B_n = B_{n-1} * B_1$ where $B_1(x) = \chi_{[0,1]}$ and * denotes the convolution of functions. The two scale relation $\varphi_{jk}(x) = \varphi(2^j x - k)$ describes all scaling B-spline functions, and characterization of a function with these scaling functions is well known, [6]. Now we introduce the pantic B-spline functions which are computed from characteristic function and the convolution mentioned above, and also applying boundary conditions:

$$\varphi(x) = B_5(x) = \begin{cases} \frac{1}{24}x^4 & 0 \le x < 1 \\ \frac{-1}{6}x^4 + \frac{5}{6}x^3 - \frac{4}{5}x^2 + \frac{5}{6}x - \frac{5}{24} & 1 \le x < 2 \\ \frac{1}{4}x^4 - \frac{15}{6}x^3 + \frac{35}{4}x^2 - \frac{75}{6}x + \frac{155}{22} & 2 \le x < 3 \\ \frac{-1}{6}x^4 + \frac{15}{6}x^3 - \frac{55}{4}x^2 + \frac{195}{6}x - \frac{655}{24} & 3 \le x < 4 \\ \frac{1}{24}x^4 - \frac{5}{6}x^3 + \frac{25}{4}x^2 - \frac{125}{6}x + \frac{625}{24} & 4 \le x < 5 \\ 0 & \text{otherwise..} \end{cases}$$

So the corresponding scaling function is:

$$\begin{aligned} \varphi_{jk}(x) = {} & \underbrace{\left\{1/24(2^j x - k)^4\right\} \chi_{[k,k+1]}}_{\mathrm{I}} \\ & + \underbrace{\left\{-1/6(2^j x - k)^4 + 5/6(2^j x - k)^3 - 4/5(2^j x - k)^2 + 5/6(2^j x - k) - 5/24\right\} \chi_{[k+1,k+2]}}_{\mathrm{II}} \\ & + \underbrace{\left\{1/4(2^j x - k)^4 - 15/6(2^j x - k)^3 + 35/4(2^j x - k)^2 - 75/6(2^j x - k) + 155/22\right\} \chi_{[k+2,k+3]}}_{\mathrm{III}} \\ & + \underbrace{\left\{-1/6(2^j x - k)^4 + 15/6(2^j x - k)^3 - 55/4(2^j x - k)^2 + 195/6(2^j x - k) - 655/24\right\} \chi_{[k+3,k+4]}}_{\mathrm{IV}} \\ & + \underbrace{\left\{1/24(2^j x - k)^4 - 5/6(2^j x - k)^3 + 25/4(2^j x - k)^2 - 125/6(2^j x - k) + 625/24\right\} \chi_{[k+4,k+5]}}_{\mathrm{V}} \\ & , k = 0, 1, 2, \ldots, 2^j - 4, \end{aligned}$$

otherwise they are zero. Their left and right-hand side boundary conditions are given. In the following formulas, for convenience, we have used the Greek numbers I, II, III, IV, V that were introduced above. Note that x^j denotes $2^j x$.

$$k = -4: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{V} & 0 \le x_j < 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$k = -3: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{IV} & k+3 \le x_j < k+4 \\ \mathrm{V} & k+4 \le x_j < k+5 \\ 0 & \text{otherwise,} \end{cases}$$

$$k = -2: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{III} & k+2 \le x_j < k+3 \\ \mathrm{IV} & k+3 \le x_j < k+4 \\ \mathrm{V} & k+4 \le x_j < k+5 \\ 0 & \text{otherwise,} \end{cases}$$

$$k = -1: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{II} & k+1 \le x_j < k+2 \\ \mathrm{III} & k+2 \le x_j < k+3 \\ \mathrm{IV} & k+3 \le x_j < k+4 \\ \mathrm{V} & k+4 \le x_j < k+5 \\ 0 & \text{otherwise,} \end{cases}$$

$$k = 2^j - 4: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{IV} & k+3 \le x_j < k+4 \\ \mathrm{III} & k+2 \le x_j < k+3 \\ \mathrm{II} & k+1 \le x_j < k+2 \\ \mathrm{I} & k \le x_j < k+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$k = 2^j - 3: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{III} & k+2 \le x_j < k+3 \\ \mathrm{II} & k+1 \le x_j < k+2 \\ \mathrm{I} & k \le x_j < k+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$k = 2^j - 2: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{II} & k+1 \le x_j < k+2 \\ \mathrm{I} & k \le x_j < k+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$k = 2^j - 1: \quad \varphi_{jk}(x) = \begin{cases} \mathrm{I} & k \le x_j < k+1 \\ 0 & \text{otherwise,} \end{cases}$$

3. Algebraic equations corresponding with an integral equation

Considering pantic B-spline scaling functions with compact support [0, 1], as an

Semi orthonormal set, the function f(x) defined over [0, 1] can be presented as

$$f(x) = \sum_{k=-4}^{2^L - 1} a_k \phi_{Lk} = A\Phi_L, \tag{2}$$

where

$$A = [a_{-4}, a_{-3}, \ldots, a_{2^L - 1}] \tag{3}$$

and

$$\Phi_L = [\phi_{L,-4}, \phi_{L,-3}, \ldots, \phi_{L,2^L-1}]^T, \tag{4}$$

where L is an arbitrary integer such that $L \geq J$. We show the dual of Φ_L with the matrix $\tilde{\Phi}_L$, so

$$\tilde{\Phi}_L = [\tilde{\phi}_{L,-4}, \tilde{\phi}_{L,-3}, \ldots, \tilde{\phi}_{L,2^L-1}]^T .$$

If $\tilde{\Phi}_{L,k}$ are the dual functions, then from semi orthogonality condition

$$\int_0^1 \tilde{\Phi}_L \Phi_L{}^T dx = I_{2^L+3} \tag{5}$$

we have

$$a_k = \int_0^1 f(x)\tilde{\Phi}_{L,k}(x)dx, \qquad k = -3, \ldots, 2^L - 1 \tag{6}$$

Let

$$P_L = \int_0^1 \Phi_L \Phi_L{}^T dx, \tag{7}$$

where $(P_L)_{ij} = \int_0^1 \phi_{Li}\phi_{Lj}^T dx$ is the ij-th entry of matrix P_L. From (5) and (7) we have

$$\tilde{\Phi}_L = (P_L)^{-1}\Phi_L ,$$

which is used for finding dual functions. Now we are ready to find the desired algebraic equations. Considering Fredholm integral equation (1), first we expand y with scaling functions, i.e.

$$y(x) = A_y \Phi_L(x), \tag{8}$$

where Φ_L is given in (4) and A_y is a $1 \times 2^{L+1}$ matrix of unknown coefficients, similar to A. Also, we describe *f(x)* and *k(x, t)* by B-spline dual functions:

$$f(x) = A_f \tilde{\Phi}_L, \qquad k(x,t) = \tilde{\Phi}_L(t)\Gamma\tilde{\Phi}_L(x) , \tag{9}$$

where

$$\Gamma_{ij} = \int_0^1 \left(\int_0^1 k(x,t)\Phi_i(t)dt\right)\Phi_j(x)dx . \tag{10}$$

By substituting (8) and (9) we get

$$\int_0^1 k(x,t)y(t)dt = \int_0^1 \tilde{\Phi}_L^T(t)\Gamma\tilde{\Phi}_L(x)A_y\Phi_L(t)dt = A_y\Gamma\tilde{\Phi}_L(x).$$

So by applying all these equations in (1), the corresponding integral equation will transform to

$$A_y\Phi_L(x) = A_y\Gamma\tilde{\Phi}_L(x) + A_f\tilde{\Phi}_L(x)$$

Now, we multiply $\tilde{\Phi}_L(x)$ by this equation and integrate with respect to x, so

$$A_y P_L - A_y\Gamma = A_f ,$$

or

$$A_y = A_f(P_L - \Gamma)^{-1} .$$

The last equation provides desired algebraic equations for solving $y(x) = A_y\Phi_L(x)$.

4. Examples and numerical results

In this section, using the above method we approximate the solution of Fredholm integral equation, numerically. First, set L = 3. By (7), we have the following matrix:

$$P_3 = \begin{bmatrix}
0.000 & 0.0004 & 0.0005 & 0.0001 & 0.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0004 & 0.0079 & 0.0152 & 0.0045 & 0.0002 & 0.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0005 & 0.0152 & 0.0459 & 0.0300 & 0.0050 & 0.0002 & 0.0000 & 0 & 0 & 0 & 0 & 0 \\
0.0001 & 0.0045 & 0.0300 & 0.0538 & 0.0304 & 0.0050 & 0.0002 & 0.0000 & 0 & 0 & 0 & 0 \\
0.0000 & 0.0002 & 0.0050 & 0.0304 & 0.0538 & 0.0304 & 0.0050 & 0.0002 & 0.0000 & 0 & 0 & 0 \\
0 & 0.0000 & 0.0002 & 0.0050 & 0.0304 & 0.0538 & 0.0304 & 0.0050 & 0.0002 & 0.0000 & 0 & 0 \\
0 & 0 & 0.0000 & 0.0002 & 0.0050 & 0.0304 & 0.0538 & 0.0304 & 0.0050 & 0.0002 & 0.0000 & 0 \\
0 & 0 & 0 & 0.0000 & 0.0002 & 0.0050 & 0.0304 & 0.0538 & 0.0304 & 0.0050 & 0.0002 & 0.0000 \\
0 & 0 & 0 & 0 & 0.0000 & 0.0002 & 0.0050 & 0.0304 & 0.0538 & 0.0300 & 0.0045 & 0.0001 \\
0 & 0 & 0 & 0 & 0 & 0.0000 & 0.0002 & 0.0050 & 0.0300 & 0.0459 & 0.0152 & 0.0005 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0000 & 0.0002 & 0.0045 & 0.0152 & 0.0079 & 0.0004 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0000 & 0.0001 & 0.0005 & 0.0004 & 0.0000
\end{bmatrix}$$

From (3) and (10), we compute A_f and Γ, so we have Ay . Finally, $y(x) = A_y\Phi_L(x)$.

Numerical results are computed for some integral equations via the following examples and absolute errors at different points with L=3, 4 given in Table1.

One can see better results in comparison with what is presented in [2].

Table1. Numerical results

x	Ex.1, L=3	Ex.1, L=4	Ex.2, L=3	Ex.2, L=4
0.0	0.113×10−15	0.1475×10−17	0.0022×10−8	000.0073×10−9
0.1	0.113×10−15	0.1585×10−17	0.0022×10−8	00.00870×10−9
0.2	0.091×10−15	0.3162×10−17	0.0015×10−8	000.0131×10−9
0.3	0.101×10−15	0.2131×10−17	0.0020×10−8	0.003615×10−9
0.4	0.149×10−15	0.2346×10−17	0.0025×10−8	0.001770×10−9
0.5	0.231×10−15	0.0072×10−17	0.0044×10−8	00.01821×10−9
0.6	0.233×10−15	0.0736×10−17	0.0045×10−8	00.02212×10−9
0.7	0.400×10−15	0.3166×10−17	0.0070×10−8	00.05097×10−9
0.8	0.732×10−15	0.2108×10−17	0.0137×10−8	00.01287×10−9
0.9	1.732×10−15	0.1354×10−17	0.0312×10−8	0000.188×10−9
1.0	6.617×10−15	0.0289×10−17	0.1213×10−8	0000.245×10−9

Example 4.1. In this example let the integral equation be

$$y(x)=\int_0^1 \frac{-1}{3} e^{2x-\frac{5}{3}t} y(t)dt + e^{2x+1/3}$$

with exact solution $y(x)=e^{2x}$. The corresponding errors are given in Table1.

Example 4.2. Consider the equation

$$\mathrm{y(x)} = \mathrm{e^x} - \frac{\mathrm{e^{x+1}}-1}{\mathrm{x}+1} + \int_0^1 \mathrm{e^{2t}y(t)dt}, \qquad 0 \le \mathrm{t} \le 1,$$

with exact solution $y(x)=e^{x}$. Errors found with our method can be seen in the last two columns of the Table 1.

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