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Chebyshev cardinal functions for solving volterra-fredholm integro-differential equations using operational matrices

M. Heydari, Z. Avazzadeh and G. B. Loghmani*

Department of Mathematics, Yazd University, Yazd, Iran
E-mails: m.heydari85@gmail.com, z.avazzadeh@yahoo.com & loghmani@yazduni.ac.ir

Abstract

In this paper, an effective direct method to determine the numerical solution of linear and nonlinear Fredholm and Volterra integral and integro-differential equations is proposed. The method is based on expanding the required approximate solution as the elements of Chebyshev cardinal functions. The operational matrices for the integration and product of the Chebyshev cardinal functions are described in detail. These matrices play the important role of reducing an integral equation to a system of algebraic equations. Illustrative examples are shown, which confirms the validity and applicability of the presented technique.

***Keywords**:* Integro-differential equation; mixed Volterra-Fredholm integral equations; chebyshev cardinal functions; operational matrix of integration

1. Introduction

Integral equations provide an important tool for modeling of the numerous problems in engineering and science [1, 2]. These equations appear in the modeling of electromagnetic and electrodynamic, elasticity and dynamic contact, heat and mass transfer, fluid mechanic, acoustic, chemical and electrochemical process, molecular physics, population, medicine and in many other phenomena [3-9]. So, it is clear that solving integral equations can be used to describe many events in real world.

In [10], some traditional methods for solving integral equations are classified and described. Recently, many researchers have focused on finding efficient numerical or analytical methods to estimate the solution of integral equations such as collocation method [1, 11], Adomian decomposition method (ADM) [12], homotopy perturbation method(HPM) [13], He's variational iteration method [14], optimal control [15], wavelets [16-19], neural networks [20], simulation methods [21], block-pulse method [22], and some other new methods [23-27].

In this study, Chebyshev cardinal functions are introduced as the efficient basis to approximate the solution of integral equations [28]. Also, the technique of solving is involved in operational matrices as a powerful tool to reduce an integral equation to a system of algebraic equations. However, operational matrices are applied with other basis [29, 30]. The coupling Chebyshev cardinal functions with operational matrices provide high accurate solutions using simple computations [31, 32].

Moreover, the new defined operational matrix utilizes the computations such that the proposed method reduces the integral equation to an algebraic system of equations without using the collocation scheme.

A nonlinear Fredholm-Volterra integro-differential equation can be considered in the following general form

$$\sum_{j=0}^{m}\alpha_j(t)u^{(j)}(t) = f(t)+\lambda_1\int_a^t k_1(t,s)\left[u(s)\right]^p ds + \lambda_2\int_a^b k_2(t,s)\left[u(s)\right]^q ds, \qquad (1)$$

under the initial conditions

$$u^{(j)}(a)=\mu_j, \qquad j=0,1,\dots,m-1, \qquad (2)$$

where $u(t)$ is an unknown function, the functions $f(t)$, $k_1(t,s)$ and $k_2(t,s)$ are defined on an interval $a \le t,s \le b$ and also λ_1,λ_2 and μ_j, $j=0,1,\dots,m-1$, are constants. Though different choices of the parameters lead to various problems, the method can afford to approximate the solution.

The presented paper is organized in 6 sections. In Section 2, we introduce the Chebyshev cardinal functions and the matrix form of an approximated function. Section 3 includes some useful property of the Chebyshev cardinal functions. In Section 4,

*Corresponding author
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we show how to approximate the solutions of the integral equation by the mentioned basis through the operational matrices. Numerical results are shown in Section 5. Finally, Section 6 concludes this paper with a brief summary and more discussion of the numerical results.

2. Chebyshev cardinal functions

Chebyshev cardinal functions of order N in $[-1,1]$ are defined as [28]

$$C_j(x) = \frac{T_{N+1}(x)}{T_{N+1,x}(x_j)(x-x_j)}, \quad j = 1,2,\ldots,N+1, \tag{3}$$

where $T_{N+1}(x)$ is the first kind Chebyshev function of order $N+1$ in $[-1,1]$ defined by

$$T_{N+1}(x) = \cos((N+1)\arccos(x)), \tag{4}$$

subscript x denotes x-differentiation and x_j, $j = 1,2,\ldots,N+1$, are the zeros of $T_{N+1}(x)$ defined by $\cos(\frac{(2j-1)\pi}{2N+2})$, $j = 1,2,\ldots,N+1$, with the Kronecker property

$$C_j(x_i) = \delta_{ji} = \begin{cases} 1, & if \ j = i, \\ 0, & if \ j \neq i, \end{cases} \tag{5}$$

where δ_{ji} is the Kronecker delta function. We change the variable $t = \frac{b-a}{2}x + \frac{b+a}{2}$ to use these functions on $[a,b]$. Now any function $g(t)$ on $[a,b]$ can be approximated as

$$g(t) \approx \sum_{j=1}^{N+1} g(t_j)C_j(t) = G^T\Theta_N(t), \tag{6}$$

where t_j, $j = 1,2,\ldots,N+1$, are the shifted points of x_j, $j = 1,2,\ldots,N+1$, by transforming $t = \frac{b-a}{2}x + \frac{b+a}{2}$,

$$G = [g(t_1), g(t_2), \ldots, g(t_{N+1})]^T, \tag{7}$$

and

$$\Theta_N(t) = [C_1(t), C_2(t), \ldots, C_{N+1}(t)]^T. \tag{8}$$

Also, we choose t_j so that, $t_1 < t_2 < \cdots < t_{N+1}$.

3. Some new properties of chebyshev cardinal functions

In this section, some operational matrices of integration and product will be derived.

Lemma 1. The integration of the vector $\Theta_N(t)$ defined in (8) can be approximated as

$$\int_a^t \Theta_N(s)ds \approx P\Theta_N(t), \tag{9}$$

where P is the $(N+1)\times(N+1)$ operational matrix of integration for Chebyshev cardinal functions.

Proof: Let

$$\int_a^t \Theta_N(s)ds = \left[\int_a^t C_1(s)ds, \int_a^t C_2(s)ds, \ldots, \int_a^t C_{N+1}(s)ds\right]^T. \tag{10}$$

Using (6), any function $\int_a^t C_j(s)ds$ can be approximated as

$$\int_a^t C_j(s)ds \approx \sum_{k=1}^{N+1} \alpha_{jk}C_k(t), \tag{11}$$

where

$$\alpha_{jk} = \int_a^{t_k} C_j(s)ds = \frac{\beta}{T_{N+1,s}(t_j)} \int_a^{t_k} \prod_{i=1, i\neq j}^{N+1} (s-t_i)ds, \quad j,k = 1,2,\ldots,N+1. \tag{12}$$

and $\beta = \frac{2^{2N+1}}{(b-a)^{N+1}}$. Comparing (9) and (11), we obtain

$$P = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,N+1} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N+1,1} & \alpha_{N+1,2} & \cdots & \alpha_{N+1,N+1} \end{bmatrix} \tag{13}$$

Remark: The elements of the matrix P can be also found without integration equivalent to (12). Let $L_{M+1}(x)$ be the Legendre polynomial of order $M+1$ on $[-1,1]$. Then the Legendre-Gauss nodes are

$$-1 < \tau_1 < \tau_2 < \ldots < \tau_{M+1} < 1, \tag{14}$$

where $\{\tau_i\}_{i=1}^{M+1}$ are the zeros of $L_{M+1}(x)$. No explicit formulas are known for the points τ_i, and so they are computed numerically using subroutines [33]. Also, we approximate the integral of f on $[-1,1]$ as

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{M+1} w_i f(\tau_i), \tag{15}$$

where τ_i are Legendre-Gauss nodes in (14) and the weights w_i given in [33]

$$w_i = \frac{2}{(1-\tau_i^2)[L'_{M+1}(\tau_i)]^2}, \quad i = 1,2,\dots,M+1. \tag{16}$$

It is well known that the integration in (15) is exact whenever $f(x)$ is a polynomial of degree smaller than $2M+1$.

By change of variable $s = \frac{t_k - a}{2}\tau + \frac{t_k + a}{2}$, (12) can be written as

$$\alpha_{jk} = \frac{\beta}{T_{N+1,s}(t_j)} \frac{t_k - a}{2} \int_{-1}^{1} \prod_{i=1, i\neq j}^{N+1} \left(\frac{t_k - a}{2}\tau + \frac{t_k + a}{2} - t_i\right) d\tau. \tag{17}$$

Now, using the Gaussian integration formula in (15) with $M = \left[\frac{N}{2}\right]$ leads to

$$\alpha_{jk} = \frac{\beta}{T_{N+1,s}(t_j)} \frac{t_k - a}{2} \sum_{l=1}^{M+1} w_l \prod_{i=1, i\neq j}^{N+1} \left(\frac{t_k - a}{2}\tau_l + \frac{t_k + a}{2} - t_i\right), \tag{18}$$

for $j,k = 1,2,\dots N+1$.

Lemma 2. Assume $\Theta_N(t)$ in (8) and $F = [f_1, f_2, \dots, f_{N+1}]^T$ as the column vectors, then

$$\Theta_N(t)\Theta_N^T(t)F \approx \tilde{F}\Theta_N(t), \tag{19}$$

where $\tilde{F}$ is a $(N+1)\times(N+1)$ product operational matrix as follows

$$\tilde{F} = diag[f_1, f_2, \dots, f_{N+1}]. \tag{20}$$

Proof: First, by using the definition of $\Theta_N(t)$ in (8) we obtain

$$\Theta_N(t)\Theta_N^T(t) = \begin{bmatrix} C_1(t)C_1(t) & C_1(t)C_2(t) & \cdots & C_1(t)C_{N+1}(t) \\ C_2(t)C_1(t) & C_2(t)C_2(t) & \cdots & C_2(t)C_{N+1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ C_{N+1}(t)C_1(t) & C_{N+1}(t)C_2(t) & \cdots & C_{N+1}(t)C_{N+1}(t) \end{bmatrix}. \tag{21}$$

Using (6), any function $C_j(t)C_k(t)$, $j,k = 1,2,\dots,N+1$ can be approximated as

$$C_j(t)C_k(t) \approx \sum_{i=1}^{N+1} \theta_{jk}^{(i)} C_i(t), \tag{22}$$

where

$$\theta_{jk}^{(i)} = C_j(t_i)C_k(t_i) = \delta_{ji}\delta_{ki}. \tag{23}$$

So, from (21) and (22), we have

$$\Theta_N(t)\Theta_N^T(t) \approx \begin{bmatrix} C_1(t) & 0 & \cdots & 0 \\ 0 & C_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{N+1}(t) \end{bmatrix}. \tag{24}$$

Clearly, by using the vector F, we find the $(N+1)\times(N+1)$ matrix $\tilde{F}$ as follows

$$\tilde{F} = diag[f_1, f_2, \dots, f_{N+1}]. \tag{25}$$

Lemma 3. Assume $\Lambda_{(N+1)\times(N+1)}$ is an arbitrary matrix, then

$$\Theta_N^T(t)\Lambda\Theta_N(t) \approx \hat{\Lambda}\Theta_N(t), \tag{26}$$

where $\Theta_N(t)$ is defined in (8) and $\hat{\Lambda}$ is a ($N+1$)-row vector including elements equal to the diagonal entries of Λ matrix.

Proof: To prove the identity, we expand the formula as follows

$$\Theta_N^T(t)\Lambda\Theta_N(t) = \Theta_N^T(t)\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1,N+1} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2,N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{N+1,1} & \Lambda_{N+1,2} & \cdots & \Lambda_{N+1,N+1} \end{bmatrix}\begin{bmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_{N+1}(t) \end{bmatrix}$$

$$= \Theta_N^T(t)\begin{bmatrix} \sum_{j=1}^{N+1}\Lambda_{1j}C_j(t) \\ \sum_{j=1}^{N+1}\Lambda_{2j}C_j(t) \\ \vdots \\ \sum_{j=1}^{N+1}\Lambda_{N+1,j}C_j(t) \end{bmatrix} = \left[\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\Lambda_{ij}C_i(t)C_j(t)\right] \overset{(22)}{\approx} \left[\sum_{i=1}^{N+1}\Lambda_{ii}C_i(t)\right] = \hat{\Lambda}\Theta_N(t).$$

Lemma 4. Suppose $U^T = [u_1, u_2, ..., u_{N+1}]$ and $e_j^T = [\mu_j, \mu_j, ..., \mu_j]$. If we consider

$$u^{(m)}(t) \approx U^T \Theta_N(t), \qquad j = 0,1,...,m-1, \quad (27)$$

then

$$\begin{aligned} u^{(m-l)}(t) &\approx \left[U^T P^l + \sum_{k=1}^{l} e_{m-k}^T P^{l-k} \right] \Theta_N(t) \\ &= L_l \Theta_N, \quad l = 0,1,...,m. \end{aligned} \quad (28)$$

In particular, for $l = m$, we have

$$u(t) \approx \left[U^T P^m + e_{m-1}^T P^{m-1} + ... + e_1^T P + e_0^T \right] \Theta_N^T(t) = L_m \Theta_N, \quad (29)$$

where L_l, $l = 0,1,...,m$, are the $N+1$-row vectors.

Proof: Clearly, by integration of (27) we will have

$$\int_a^t u^{(m)}(s) ds \approx u^{(m-1)}(t) - \mu_{m-1}. \quad (30)$$

Also, from (9), we get

$$\int_a^t U^T \Theta_N(s) ds \approx U^T P \Theta_N(t), \quad (31)$$

which concludes

$$u^{(m-1)}(t) \approx U^T P \Theta_N(t) + \mu_{m-1}. \quad (32)$$

From (6), μ_{m-1} as the constant function has the vector form $e_{m-1}^T \Theta_N(t)$ which gives

$$u^{(m-1)}(t) \approx U^T P \Theta_N(t) + e_{m-1}^T \Theta_N(t) = \left[U^T P + e_{m-1}^T \right] \Theta_N(t). \quad (33)$$

Similarly, we can obtain vector forms of other differentiations of $u(t)$.

Lemma 5. If we consider

$$u(t) \approx U^T \Theta_N(t), \quad (34)$$

then for every $n \in N$ we have

$$[u(t)]^n \approx U^T (\tilde{U})^{n-1} \Theta_N(t), \quad (35)$$

or

$$[u(t)]^n \approx [u_1^n, u_2^n, \cdots, u_{N+1}^n] \Theta_N(t), \quad (36)$$

where $\tilde{U} = diag(u_1, u_2, ..., u_{N+1})$.

Proof: By induction, for $k = n+1$, we will have

$$\begin{aligned} [u(t)]^{n+1} &= [u(t)]^n u(t) \approx U^T (\tilde{U})^{n-1} \overbrace{\Theta_N(t) \Theta_N^T(t) U}^{\tilde{U}\Theta_N(t)} \\ &= U^T (\tilde{U})^n \Theta_N(t) \\ &= [u_1^{n+1}, u_2^{n+1}, \cdots, u_{N+1}^{n+1}] \Theta_N(t). \end{aligned}$$

Lemma 6. If we consider

$$u^{(m)}(t) \approx U^T \Theta_N(t), \quad (37)$$

then for every $q \in N$ we have

$$[u(t)]^q \approx U_q^T \Theta_N(t), \quad (38)$$

where $(U_q)_i = [(L_m)_i]^q$ and L_m is defined in Lemma 3.

Proof: According to Lemma 4, hypostasis lead to $u(t) \approx L_m \Theta_N(t)$.

So, considering the previous Lemma 5 gives

$$[u(t)]^q \approx [(L_m)_1^q, (L_m)_2^q, \cdots, (L_m)_{N+1}^q] \Theta_N(t),$$

and the proof is completed.

4. Direct method to solve integral equations

In this section, by using results obtained in the previous section concerning Chebyshev cardinal functions, an effective and accurate direct method for solving nonlinear Fredholm, Volterra and Fredholm-Volterra integral and integro-differential equations is presented.

Consider the following integral equation

$$\sum_{j=0}^{m} \alpha_j(t) u^{(j)}(t) = f(t) + \lambda_1 \int_a^t k_1(t,s) [u(s)]^p ds + \lambda_2 \int_a^b k_2(t,s) [u(s)]^q ds, \quad (39)$$

under the initial conditions

$$u^{(j)}(a) = \mu_j, \qquad j = 0,1,...,m-1, \quad (40)$$

as before.

We first reform the proposed integral equation to utilize for solving. This process is described in the following steps.

Step 1.

Since $f(t)$ is a given function, according to (6) we can approximate it as follows

$$f(t) \approx \sum_{j=1}^{N+1} f(t_j) C_j(t) = F^T \Theta_N(t), \quad (41)$$

where $F^T = [f(t_1), f(t_2), ..., f(t_{n+1})]$ and $t_j, j = 1,..., N+1,$ are shifted points of $x_j, j = 1,..., N+1,$ by $t = \frac{b-a}{2}x + \frac{b+a}{2}$.

Step 2.
From (6), $\alpha_j(t)$ as the given functions can be rewritten in the following form

$$\alpha_j(t) \approx \hat{A}_j^T \Theta_N(t) = \Theta_N^T(t) \hat{A}_j, \quad j = 0, 2, ..., m, \quad (42)$$

where $\hat{A}_j^T = [\alpha_j(t_1), \alpha_j(t_2), ..., \alpha_j(t_{N+1})]$. Also, by Lemma 4 we have

$$u^{(j)}(t) = L_{m-j} \Theta_N(t), \qquad j = 0,1,..., m.$$

Therefore, the right side of the equation can be approximated as

$$\sum_{j=0}^{m} \alpha_j(x) u^{(j)}(t) \approx \sum_{j=0}^{m} L_j \overbrace{\Theta_N(t) \Theta_N^T(x) \hat{A}_j}^{\tilde{A}_j \Theta_N(t)} \approx \left[\sum_{j=0}^{m} L_j \tilde{A}_j \right] \Theta_N(t), \quad (43)$$

where $\tilde{A}_j$ is the $(N+1) \times (N+1)$ diagonal matrix whose entries correspond to the elements of $\hat{A}_j$. Note that the points t_j related to x_j, $j = 1,2,..., N+1,$ and the vectors L_j, $j = 0,2,..., m$, are defined in (28).

Step 3.
Now we focus on the Integral part with the constant limits of integration of the proposed integral equation. It is clear that $k_2(x,t)$, as a kernel of the Fredholm part, can be written as follows

$$k_2(t,s) \approx \sum_{j=1}^{N+1} \sum_{i=1}^{N+1} k_{ij}^f C_i(t) C_j(s)$$
$$= [C_1(t), C_2(t), ..., C_{N+1}(t)] K^f [C_1(s), C_2(s), ..., C_{N+1}(s)]^T$$
$$= \Theta_N^T(t) K^f \Theta_N(s), \quad (44)$$

where $k_2(t_i, s_j) = k_{ij}^f = (K^f)_{ij}$.

Now we describe the most important part of the process

$$\int_a^b k_2(t,s)[u(s)]^q ds \approx \int_a^b \Theta_N^T(t) K^f \Theta_N(s) \Theta_N^T(s) U_q ds$$
$$= \Theta_N^T(t) K^f \left(\int_a^b \Theta_N(s) \Theta_N^T(s) ds \right) U_q$$
$$= \Theta_N^T(t) K^f S U_q, \quad (45)$$

where S is defined as follows

$$S = \int_a^b \Theta_N(s) \Theta_N^T(s) ds \approx \text{diagonal}[\int_a^b C_1(s) ds, \int_a^b C_2(s) ds, \cdots, \int_a^b C_{N+1}(s) ds],$$

which is obtained from (24).

Note that the integral has the scalar value. So it is equal to its transpose. Finally, we have

$$\int_a^b k_2(t,s)[u(s)]^q ds \approx \left[K^f S U_q\right]^T \Theta_N(t). \quad (46)$$

Step 4.
This step explains how to reduce the Volterra Integral part of the proposed mixed Volterra-Fredholm integral equation. Similarly, $k_1(t,s)$, as a kernel of the Volterra part, can be approximated as

$$k_1(t,s) \approx \sum_{j=1}^{N+1} \sum_{i=1}^{N+1} k_{ij}^v C_i(t) C_j(s) = \Theta_N^T(t) K^v \Theta_N(s), \quad (47)$$

where $k_1(x_i, t_j) = k_{ij}^v = (K^v)_{ij}$. Therefore, we have

$$\int_a^t k_2(t,s)[u(s)]^p ds \approx \int_a^t \Theta_N^T(t) K^v \overbrace{\Theta_N(s) \Theta_N^T(s) U_p}^{\tilde{U}_p \Theta_N^T(s)} ds \approx \Theta_N^T(t) K^v \tilde{U}_p \int_a^t \Theta_N^T(s) ds$$
$$\approx \Theta_N^T(t) K^v \tilde{U}_p P \Theta_N(t) \approx \hat{H} \Theta_N(t), \quad (48)$$

Where, according to Lemma 2 $\tilde{U}_p$ is the diagonal matrix constructed by entries of U_p. Also, $\hat{H}$ is a column vector including the entries of the main diagonal of $K^v \tilde{U}_p P$. It is a direct result of Lemma 3.

Step 5.
Now we substitute the obtained reformed parts into (39). So, we have

$$\left[\left(\sum_{j=0}^{m} L_j \tilde{A}_j \right) - F^T - \lambda_1 \hat{H} - \lambda_2 \left(K^f S U_q \right)^T \right] \Theta_N(t) = 0. \quad (49)$$

Since the above equation is satisfied for every $t \in [a,b]$, we have

$$\left(\sum_{j=0}^{m} L_j \tilde{A}_j \right) - F^T - \lambda_1 \hat{H} - \lambda_2 \left(K^f S U_q \right)^T = 0. \quad (50)$$

Solving the obtained system included $N+1$ unknowns and $N+1$ equations, leading to the solution of the integral equation. Newton's method can fulfill the accurate solutions of nonlinear systems.

5. Numerical results

In this section, some different examples which have been solved with other usual methods are considered. Thus, the obtained numerical results can be compared with other methods. The associated computations with the examples were performed using MAPLE 13 with 64 digit precision on a Personal Computer. Although 64 digits were used in the computations, only 2 digits are legit in the illustrative examples.

Example 1. As the first example, consider the following integral equation of the first kind [34]

$$\frac{e^{t^2+1}-1}{t^2+1}-\int_0^1 e^{t^2 s}u(s)ds=0,\quad t\in[0,1],$$

where the exact solution is $u(t)=e^t$. We proceed with the process of solving corresponding to the described steps for $N=4$. According to (39)

$$k_2(t,s)=e^{t^2 s},\qquad q=1,\qquad \lambda_1=0,\qquad \lambda_2=-1,$$

$$f(t)=\frac{e^{t^2+1}-1}{t^2+1},$$

and we approximate the solution as follows

$$u(t)\approx U^T\Theta_N(t).$$

Now, we obtain the needed vectors and matrices as follows

$$K^f=\begin{bmatrix}2.53&2.13&1.61&1.22&1.02\\1.85&1.65&1.37&1.14&1.02\\1.28&1.22&1.13&1.05&1.01\\1.04&1.03&1.02&1.01&1.0\\1.0&1.0&1.0&1.0&1.0\end{bmatrix},\qquad F=\begin{bmatrix}3.09\\2.52\\1.99\\1.76\\1.72\end{bmatrix},$$

$$S=\begin{bmatrix}0.08&0&0&0&0\\0&0.26&0&0&0\\0&0&0.31&0&0\\0&0&0&0.26&0\\0&0&0&0&0.08\end{bmatrix},\qquad U_q=\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{bmatrix}.$$

Substituting these values in (53) gives the following system of linear equations

$$F^T-(K^f SU_q)^T=0,$$

which is equivalent to

$$\begin{cases}0.21u_1+0.56u_2+0.49u_3+0.32u_4+0.09u_5-3.09=0,\\0.16u_1+0.43u_2+0.42u_3+0.30u_4+0.09u_5-2.52=0,\\0.11u_1+0.32u_2+0.35u_3+0.28u_4+0.08u_5-1.99=0,\\0.09u_1+0.27u_2+0.31u_3+0.27u_4+0.08u_5-1.76=0,\\0.08u_1+0.26u_2+0.31u_3+0.26u_4+0.08u_5-1.72=0.\end{cases}$$

From this system, the coefficients u_i, $i=1,2,\dots,5$ are computed as

$$U=[2.64\quad 2.22\quad 1.64\quad 1.24\quad 1.01],$$

and the approximate solution of the integral equation is obtained by

$$u(t)\approx\sum_{i=1}^{5}u_i C_i(t),$$

which is fitted on the exact solution demonstrated in Fig. 1. In addition, error function is shown for $N=10$. Further investigations will be described in the conclusion.

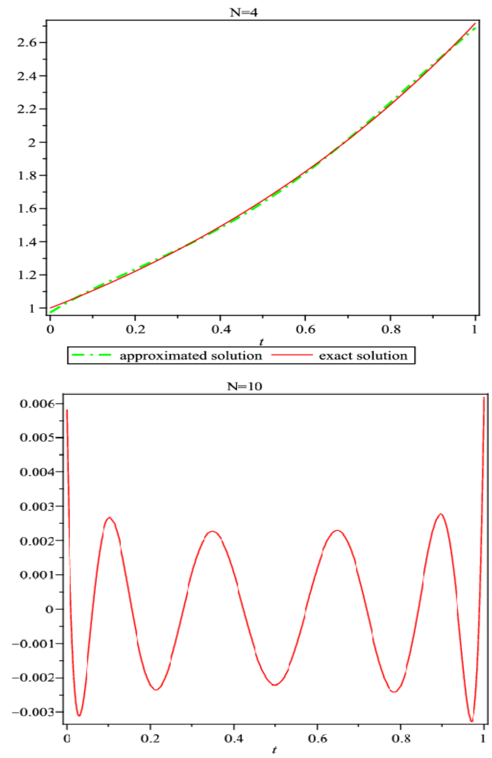

Fig. 1. The exact and approximated solution cover each other for $N=4$. The error function for $N=10$ is also shown. See the Example 1

Example 2. Consider the following integral equation of the first kind [17]

$$\frac{-1}{15}\left(-8e^{2t}+6\sin t+3\cos t+5e^{-t}\right)-\int_0^t (e^{s-t}+\sin(t-s))u(s)ds=0,$$

where the exact solution is $u(t)=e^{2t}$. We describe the steps of solving for $N=4$. According to (39)

$$k_1(t,s)=e^{s-t}+\sin(t-s),\qquad p=1,\qquad \lambda_1=-1,\qquad \lambda_2=0,$$

$$f(t)=\frac{-1}{15}\left(-8e^{2t}+6\sin t+3\cos t+5e^{-t}\right),$$

and we approximate the solution as follows

$$u(t)\approx U^T\Theta_N(t).$$

Now we obtain the needed vectors and matrices as follows

$$K^v=\begin{bmatrix}1.0 & 1.01 & 1.08 & 1.16 & 1.20\\ 1.02 & 1.0 & 1.04 & 1.11 & 1.16\\ 1.15 & 1.05 & 1.0 & 1.04 & 1.08\\ 1.46 & 1.25 & 1.05 & 1.0 & 1.01\\ 1.77 & 1.46 & 1.15 & 1.02 & 1.0\end{bmatrix},\qquad F=\begin{bmatrix}3.18\\ 2.03\\ 0.88\\ 0.26\\ 0.03\end{bmatrix},$$

$$P=\begin{bmatrix}0.06 & -0.01 & 0.01 & -0.0 & 0.0\\ 0.27 & 0.13 & -0.02 & 0.01 & -0.0\\ 0.30 & 0.33 & 0.15 & -0.03 & 0.0\\ 0.26 & 0.25 & 0.29 & 0.13 & -0.0\\ 0.08 & 0.09 & 0.08 & 0.10 & 0.03\end{bmatrix},\qquad \tilde{U}_p=\begin{bmatrix}u_1 & 0 & 0 & 0 & 0\\ 0 & u_2 & 0 & 0 & 0\\ 0 & 0 & u_3 & 0 & 0\\ 0 & 0 & 0 & u_4 & 0\\ 0 & 0 & 0 & 0 & u_5\end{bmatrix},$$

and $\hat{H}$ constructed by $K^v\tilde{U}_pP$

$$\hat{H}^T=\begin{bmatrix}0.06u_1+0.27u_2+0.33u_3+0.31u_4+0.10u_5\\ -0.01u_1+0.13u_2+0.35u_3+0.28u_4+0.10u_5\\ 0.01u_1-0.02u_2+0.15u_3+0.30u_4+0.08u_5\\ -0.01u_1+0.02u_2-0.03u_3+0.13u_4+0.10u_5\\ 0.03u_5\end{bmatrix}.$$

Substituting these values in (53) gives the following system of linear equations

$$F^T-\hat{H}^T=0,$$

which is equivalent to

$$\begin{cases}0.06u_1+0.27u_2+0.33u_3+0.31u_4+0.10u_5-3.18=0,\\ -0.01u_1+0.13u_2+0.35u_3+0.28u_4+0.10u_5-2.03=0,\\ 0.01u_1-0.02u_2+0.15u_3+0.30u_4+0.08u_5-0.88=0,\\ -0.01u_1+0.02u_2-0.03u_3+0.13u_4+0.10u_5-0.26=0,\\ 0.03u_5-0.03=0.\end{cases}$$

From this system, the coefficients u_i, $i=1,2,\ldots,5$ are computed as

$$U=[6.99\quad 4.91\quad 2.71\quad 1.52\quad 1.05],$$

and the approximate solution of the integral equation is obtained by

$$u(t)\approx\sum_{i=1}^{5}u_iC_i(t),$$

which is fitted on the exact solution demonstrated in Fig. 2. In addition, error function is shown for $N=15$.

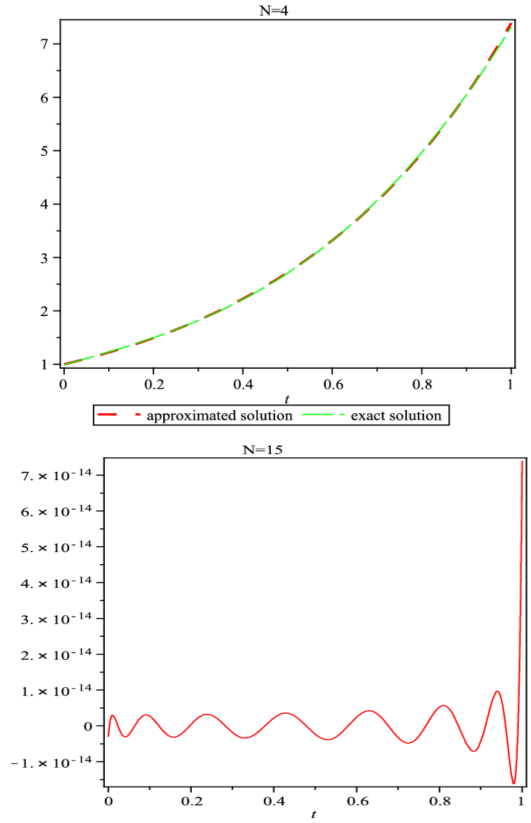

Fig. 2. The exact and approximated solution coincided with each other for $N=4$. The error function for $N=15$ is also shown. See the Example 2.

Example 3. Consider the Volterra integral equation of the second kind as follows [35, 36]

$$u(t)=\cos(t)-\int_0^t (t-s)\cos(t-s)u(s)ds,$$

where the exact solution is $u(t)=\frac{1}{3}\left(2\cos\sqrt{3}t+1\right)$.

Corresponding to (39)

$$k_1(t,s)=(t-s)\cos(t-s), \qquad p=1,$$
$$\lambda_1=-1, \qquad \lambda_2=0 \qquad m=0,$$

$$f(t)=\cos t, \qquad \alpha_0(t)=1,$$

and we approximate the solution as follows

$$u(t)\approx U^T\Theta_N(t).$$

Similarly, we obtain the needed matrices and vectors for $N=4$ as follows

$$\hat{A}_0=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\quad \tilde{A}_0=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\quad L_0=\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{bmatrix},\quad F=\begin{bmatrix}0.56\\0.70\\0.88\\0.98\\1.0\end{bmatrix},$$

$$K^v=\begin{bmatrix}0.0&0.18&0.42&0.55&0.55\\-0.18&0.0&0.28&0.49&0.55\\-0.42&-0.28&0.0&0.28&0.42\\-0.55&-0.49&-0.28&0.0&0.18\\-0.55&-0.55&-0.42&-0.18&0.0\end{bmatrix},\quad P=\begin{bmatrix}0.06&-0.01&0.01&-0.0&0.0\\0.27&0.13&-0.02&0.01&-0.0\\0.30&0.33&0.15&-0.03&0.0\\0.26&0.25&0.29&0.13&-0.0\\0.08&0.09&0.08&0.10&0.03\end{bmatrix},$$

$$\tilde{U}_p=\begin{bmatrix}u_1&0&0&0&0\\0&u_2&0&0&0\\0&0&u_3&0&0\\0&0&0&u_4&0\\0&0&0&0&u_5\end{bmatrix},\quad \hat{H}^T=\begin{bmatrix}0.05u_2+0.13u_3+0.15u_4+0.05u_5\\0.09u_3+0.12u_4+0.05u_5\\0.01u_2+0.08u_4+0.03u_5\\-0.01u_2+0.01u_3+0.02u_5\\0.0\end{bmatrix}.$$

Substituting these values in (53) gives the nonlinear system of equations

$$\begin{cases}u_1+0.05u_2+0.13u_3+0.15u_4+0.05u_5-0.56=0,\\u_2+0.09u_3+0.12u_4+0.05u_5-0.70=0,\\u_3+0.01u_2+0.08u_4+0.03u_5-0.88=0,\\u_4-0.01u_2+0.01u_3+0.02u_5-0.98,\\u_5-1.0=0.\end{cases}$$

From this system, the coefficients u_i, $i=1,2,...,5$ are computed as

$$U=[0.25\quad 0.46\quad 0.77\quad 0.96\quad 1.0]^T.$$

and the approximate solution of the mixed Volterra-Fredholm integral equation is obtained by

$$u(t)\approx\sum_{i=1}^{5}u_iC_i(t),$$

which coincides for 3-digit arithmetic with the exact solution. The method for $N=10$ gives 11 true digits. The results are reported in Table1.

Table 1. The infinity norm of error functions for different N.

Example	$N=5$	$N=15$	$N=30$
1	2.05×10^{-2}	7.82×10^{-3}	8.74×10^{-4}
2	1.17×10^{-3}	7.52×10^{-14}	5.05×10^{-36}
3	2.41×10^{-5}	1.04×10^{-17}	8.01×10^{-41}
4	2.01×10^{-52}	3.57×10^{-57}	4.51×10^{-59}
5	3.51×10^{-4}	1.37×10^{-13}	2.91×10^{-30}
6	3.10×10^{-5}	1.94×10^{-17}	2.73×10^{-40}
7	4.51×10^{-5}	1.48×10^{-18}	1.44×10^{-43}
8	4.06×10^{-6}	3.82×10^{-20}	1.12×10^{-48}
9	8.31×10^{-2}	5.59×10^{-5}	3.61×10^{-17}

Example 4. Consider the following mixed Volterra-Fredholm integral equation [37, 24]

$$u(t)=\frac{-1}{30}t^6+\frac{1}{3}t^4-t^2+\frac{5}{3}t-\frac{5}{4}$$
$$+\int_0^t(t-s)[u(s)]^2ds+\int_0^1(t+s)u(s)ds, \quad t,s\in[0,1],$$

where the exact solution is $u(t)=t^2-2$. According to (39)

$$k_1(t,s)=t-s, \quad k_2=t+s, \quad p=2, \quad q=1, \quad \lambda_1=\lambda_2=1 \quad m=0,$$

$$f(t)=\frac{-1}{30}t^6+\frac{1}{3}t^4-t^2+\frac{5}{3}t-\frac{5}{4}, \qquad \alpha_0(t)=1,$$

and we approximate the solution as follows

$$u(t)\approx U^T\Theta_N(t).$$

Then, for $N=4$, we obtain the vector and matrices as follows

$$\hat{A}_0=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\quad \tilde{A}_0=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\quad L_0=\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{bmatrix},\quad F=\begin{bmatrix}-0.30\\-0.43\\-0.65\\-0.95\\-1.21\end{bmatrix},$$

$$K^f=\begin{bmatrix}1.95&1.77&1.48&1.18&1.0\\1.77&1.59&1.29&1.0&0.82\\1.48&1.29&1.0&0.71&0.52\\1.18&1.0&0.71&0.41&0.23\\1.0&0.82&0.52&0.23&0.05\end{bmatrix},$$
$$K^v=\begin{bmatrix}0.0&0.18&0.48&0.77&0.95\\-0.18&0.0&0.29&0.59&0.77\\-0.48&-0.29&0.0&0.29&0.48\\-0.77&-0.59&-0.29&0.0&0.18\\-0.95&-0.77&-0.48&-0.18&0.0\end{bmatrix},$$

$$P = \begin{bmatrix} 0.06 & -0.01 & 0.01 & -0.0 & 0.0 \\ 0.27 & 0.13 & -0.02 & 0.01 & -0.0 \\ 0.30 & 0.33 & 0.15 & -0.03 & 0.0 \\ 0.26 & 0.25 & 0.29 & 0.13 & -0.0 \\ 0.08 & 0.09 & 0.08 & 0.10 & 0.03 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.08 & 0 & 0 & 0 & 0 \\ 0 & 0.26 & 0 & 0 & 0 \\ 0 & 0 & 0.31 & 0 & 0 \\ 0 & 0 & 0 & 0.26 & 0 \\ 0 & 0 & 0 & 0 & 0.08 \end{bmatrix},$$

$$\tilde{U}_p = \begin{bmatrix} u_1^2 & 0 & 0 & 0 & 0 \\ 0 & u_2^2 & 0 & 0 & 0 \\ 0 & 0 & u_3^2 & 0 & 0 \\ 0 & 0 & 0 & u_4^2 & 0 \\ 0 & 0 & 0 & 0 & u_5^2 \end{bmatrix}, \quad U_q = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix},$$

and $\hat{H}$, which is constructed by entries of $K^v \tilde{U}_p P$

$$\hat{H}^T = \begin{bmatrix} 0.049u_2^2 + 0.145u_3^2 + 0.203u_4^2 + 0.079u_5^2 \\ 0.002u_1^2 + 0.098u_3^2 + 0.147u_4^2 + 0.067u_5^2 \\ -0.003u_1^2 + 0.007u_2^2 + 0.084u_4^2 + 0.037u_5^2 \\ 0.003u_1^2 - 0.007u_2^2 + 0.008u_3^2 + 0.018u_5^2 \\ 0.001u_2^2 - 0.001u_3^2 + 0.001u_4^2 \end{bmatrix}.$$

Substituting these values in (53) gives the nonlinear system of equations

$$\begin{cases} 0.84u_1 - 0.46u_2 - 0.45u_3 - 0.31u_4 - 0.08u_5 + 0.30 - 0.05u_2^2 - 0.14u_3^2 - 0.20u_4^2 - 0.08u_5^2 = 0, \\ -0.15u_1 + 0.58u_2 - 0.40u_3 - 0.26u_4 - 0.07u_5 + 0.43 - 0.10u_3^2 - 0.15u_4^2 - 0.07u_5^2 = 0, \\ -0.12u_1 - 0.34u_2 + 0.69u_3 - 0.19u_4 - 0.04u_5 + 0.65 - 0.01u_2^2 - 0.08u_4^2 - 0.04u_5^2 = 0, \\ -0.10u_1 - 0.26u_2 - 0.22u_3 + 0.89u_4 - 0.02u_5 + 0.95 + 0.01u_2^2 - 0.01u_3^2 - 0.02u_5^2 = 0, \\ -0.08u_1 - 0.22u_2 - 0.16u_3 - 0.06u_4 + 1.0u_5 + 1.21 = 0. \end{cases}$$

From this system, the coefficients u_i, $i = 1,2,...,5$ are computed as

$$U = [-1.05 \quad -1.37 \quad -1.75 \quad -1.96 \quad -2.0]^T,$$

and the approximate solution of the mixed Volterra-Fredholm integral equation is obtained by

$$u(t) \approx \sum_{i=1}^{5} u_i C_i(t).$$

The numerical results are reported in Table 1.

Example 5. Consider the following initial value problem [38, 39]

$$u'(t) + u(t) = 1 + 2t + \int_0^t t(1+2t)e^{s(t-s)}u(s)ds, \quad u(0) = 1, \quad t \in [0,1],$$

where $u(0) = 1$ and the exact solution is $u(t) = e^{t^2}$. With respect to (39),

$$k_1 = t(1+2t)e^{s(t-s)}, \quad p = 1, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad m = 1,$$

$$f(t) = 1 + 2t, \quad \alpha_0(t) = 1, \quad \alpha_1(t) = 1,$$

and we approximate the solution as follows:

$$u'(t) \approx U^T \Theta_N(t).$$

Then, for $N = 4$, we obtain the vector and matrices as follows:

$$\hat{A}_0 = \hat{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \tilde{A}_0 = \tilde{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2.95 \\ 2.59 \\ 2.0 \\ 1.41 \\ 1.05 \end{bmatrix}, \quad e_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$L_0^T = U^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}, \quad L_1^T = U^T P + e_0 = \begin{bmatrix} 0.06u_1 + 0.27u_2 + 0.30u_3 + 0.26u_4 + 0.08u_5 \\ -0.01u_1 + 0.13u_2 + 0.33u_3 + 0.25u_4 + 0.09u_5 \\ 0.01u_1 - 0.02u_2 + 0.15u_3 + 0.29u_4 + 0.08u_5 \\ 0.01u_2 - 0.03u_3 + 0.13u_4 + 0.10u_5 \\ 0.03u_5 \end{bmatrix},$$

$$K^v = \begin{bmatrix} 2.88 & 3.33 & 3.65 & 3.37 & 2.95 \\ 1.72 & 2.05 & 2.38 & 2.32 & 2.09 \\ 0.63 & 0.79 & 1.0 & 1.06 & 1.01 \\ 0.14 & 0.18 & 0.25 & 0.29 & 0.29 \\ 0.01 & 0.01 & 0.02 & 0.02 & 0.03 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.06 & -0.01 & 0.01 & -0.0 & 0.0 \\ 0.27 & 0.13 & -0.02 & 0.01 & -0.0 \\ 0.30 & 0.33 & 0.15 & -0.03 & 0.0 \\ 0.26 & 0.25 & 0.29 & 0.13 & -0.0 \\ 0.08 & 0.09 & 0.08 & 0.10 & 0.03 \end{bmatrix},$$

$$U_p = \begin{bmatrix} 0.06u_1 + 0.27u_2 + 0.30u_3 + 0.26u_4 + 0.08u_5 + 1 \\ -0.01u_1 + 0.13u_2 + 0.33u_3 + 0.25u_4 + 0.09u_5 + 1 \\ 0.01u_1 - 0.02u_2 + 0.15u_3 + 0.29u_4 + 0.08u_5 + 1 \\ 0.01u_2 - 0.03u_3 + 0.13u_4 + 0.10u_5 + 1 \\ 0.03u_5 + 1 \end{bmatrix},$$

$$\tilde{U}_p = diagonal((U_p)_{11}, (U_p)_{22}, (U_p)_{33}, (U_p)_{44}, (U_p)_{55}),$$

and $\hat{H}$, which is constructed by entries of $K^v \tilde{U}_p P$

$$\hat{H}^T = \begin{bmatrix} 3.30+0.15u_2+0.49u_3+0.70u_4+0.27u_5 \\ 0.02u_2+0.19u_3+0.36u_4+0.15u_5+1.81 \\ 0.52+0.01u_3+0.08u_4+0.04u_5 \\ 0.06 \\ 0.0 \end{bmatrix}.$$

Substituting these values in (53) gives the linear system of equations

$$\begin{cases} 1.06u_1+0.12u_2-0.19u_3-0.43u_4-0.19u_5-5.25=0, \\ -0.01u_1+1.11u_2+0.14u_3-0.11u_4-0.06u_5-3.40=0, \\ 0.01u_1-0.02u_2+1.14u_3+0.21u_4+0.04u_5-1.52=0, \\ 0.01u_2-0.03u_3+1.12u_4+0.09u_5-0.47=0, \\ 1.03u_5-0.05=0. \end{cases}$$

From this system, the coefficients u_i, $i=1,2,...,5$, are computed as

$$U=[5.05 \quad 2.98 \quad 1.28 \quad 0.43 \quad 0.05]^T,$$

and the approximate solution of the mixed Volterra-Fredholm integral equation is obtained by

$$u(t) \approx \sum_{i=1}^{5} u_i C_i(t),$$

which coincides for 2-digit arithmetic with the exact solution. Also, the norm infinity of error function for $N=10$ shows 8 true digits with respect to the exact solution.

We are reminded that we used the 64 digits for solving the examples, however, we show the results with 2 digits. Compare the last equation in (54) with the obtained value for u_5. It confirms digits can have a serious effect on the results. The results are reported in Table 1.

Example 6. Consider the following nonlinear Volterra integral equation of the second kind [40, 24]

$$u(t) = 1+\sin^2 t - 3\int_0^t \sin(t-s)[u(s)]^2 ds, \quad t\in[0,1],$$

where the exact solution is $u(t)=\cos t.$ Corresponding to (39)

$$k_1(t,s)=\sin(t-s), \quad p=2, \quad \lambda_1=-3, \quad \lambda_2=0, \quad m=0,$$

$$f(t)=1+\sin^2 t, \qquad \alpha_0(t)=1,$$

and we approximate the solution as follows

$$u(t) \approx U^T\Theta_N(t).$$

The results are reported in Table 1.

Example 7. Consider the Fredholm integro-differential equation as follows [41]:

$$u''(t)+tu'(t)-tu(t)=e^t-2\sin t + \int_{-1}^{1}\sin(s)e^{-s}u(s)ds, \qquad t\in[-1,1],$$

$$u(0)=1, \qquad u'(0)=1,$$

where the exact solution is $u(t)=e^t$. According to (39)

$$k_1(t,s)=\sin(s)e^{-s}, \quad q=1, \quad \lambda_1=0, \quad \lambda_2=1, \quad m=2,$$

$$f(t)=e^t-2\sin t, \quad \alpha_0(t)=-t, \quad \alpha_1(t)=t, \quad \alpha_2(t)=1,$$

and we approximate the solution as follows

$$u''(t) \approx U^T\Theta_N(t).$$

The results are shown in Table 1.

Example 8. Consider the Volterra integro-differential equation as follows [42]

$$u''(t)=\sinh t+\frac{1}{2}\cosh t\sinh t-\frac{1}{2}t-\int_0^t u^2(s)ds, \qquad t\in[0,1],$$

$$u(0)=0, \quad u'(0)=1,$$

where the exact solution is $u(t)=\sinh t.$ According to (39)

$$k_1(t,s)=1, \quad q=2, \quad \lambda_1=-1, \quad \lambda_2=0, \quad m=2,$$

$$f(t)=\sinh t+\frac{1}{2}\cosh t\sinh t-\frac{1}{2}t,$$

$$\alpha_0(t)=0, \quad \alpha_1(t)=0, \quad \alpha_2(t)=1,$$

and we approximate the solution as follows

$$u''(t) \approx U^T\Theta_N(t).$$

The results are reported in Table 1.

Example 9. Consider the integral equation of the first kind as follows [17]

$$\frac{4\pi\cos(4\pi t)+\sin(4\pi t)-4\pi e^t}{(1+t^2)(1+16\pi^2)} - \int_0^t \frac{e^{t-s}}{1+s^2}u(s)ds = 0,$$

where the exact solution is $u(t)=\sin(4\pi t).$ According to (39)

$$k_1(t,s) = \frac{e^{t-s}}{1+s^2}, \qquad q = 1, \qquad \lambda_1 = -1, \qquad \lambda_2 = 0,$$

$$f(t) = \frac{4\pi\cos(4\pi t) + \sin(4\pi t) - 4\pi e^t}{(1+t^2)(1+16\pi^2)},$$

and we approximate the solution as follows:

$$u(t) \approx U^T \Theta_N(t).$$

We illustrate the numerical example in Table 1.

Conclusion

The Chebyshev cardinal functions and the associated operational matrices of integration P and product $\widetilde{F}$ are applied to solve the general type of linear and nonlinear integral equations. Moreover, the new defined matrix operations utilize the computations so that the proposed method can reduce the integral equation to an algebraic system without using the collocation scheme. The obtained results showed that this approach can solve the problem effectively with simple computations.

There are some notable points in the numerical results which we investigate in detail. Table 1 shows the maximum errors for $N = 5, 15$ and 30. Looking carefully at the results shows the different behavior of error functions. In particular, Examples 1 and 4 have an irregular rate of convergence. Experimental results show the Fredhlom integral equations of the first kind have the most ill-conditioned systems, which lead to weak accurate. Also, the best candidate integral equations for this method have the solution in the polynomial forms, even if the problem includes many terms or the higher order of differentiations.

The method of Chebyshev cardinal functions proposed in this paper can be extended to solve the more general type as follows

$$\sum_{j=0}^{m} \alpha_j(t) u^{(j)}(t) = f(t) + \lambda_1 \int_a^t k_1(t,s) F(t,s,u(s)) ds$$
$$+ \lambda_2 \int_a^b k_2(t,s) G(t,s,u(s)) ds,$$

under the initial conditions

$$u^{(j)}(a) = \mu_j, \qquad j = 0,1,\ldots,m-1.$$

Here we can use Taylor series of F and G.

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