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Separation of the two dimensional Laplace operator by the disconjugacy property

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Abstract

In this paper we have studied the separation for the Laplace differential operator of the form

$$P[u]=-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+q(x,y)u(x,y)$$

in the Hilbert space $H=L^2(\Omega)$, with potential $q(x,y)\in C^1(\Omega)$. We show that certain properties of positive solutions of the disconjugate second order differential expression *P[u]* imply the separation of minimal and maximal operators determined by P i.e, the property that $P(u)\in L^2(\Omega)\Rightarrow qu\in L^2(\Omega),\Omega\in R^2$. A property leading to a new proof and generalization of a 1971 separation criterion due to Everitt and Giertz. This result will allow the development of several new sufficient conditions for separation and various inequalities associated with separation. A final result of this paper shows that the disconjugacy of $P-\lambda q^2$ for some $\lambda>0$ implies the separation of P.

Keywords: Separation; Laplace differential operator; Disconjugacy; Hilbert space

1. Introduction

The concept of separation of differential operators was first introduced by Everitt and Giertz in [1]. Mohamed and Atia [2] have studied the separation property of the Sturm-Liouville differential operator of the form

$$Ly(x)=-\frac{d}{dx}[\mu(x)\frac{dy}{dx}]+Q(x)y(x)$$

in the space $H_p(R)$, for $p=1,2,$ where $Q(x)\in L(\ell_p)$ is an operator potential which is a bounded linear operator on ℓ_p, and $\mu(x)\in C^1(R)$ is a positive continuous function on R.
Mohamed and Atia[3] have studied the separation of the Schrodinger operator of the form

$$Su(x)=-\vartriangle u(x)+V(x)u(x),$$

with the operator potential $V(x)\in C^1(R^n,L(H_1))$, in the Hilbert space $L_2(R^n,H_1)$, where $\vartriangle=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in R^n.

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Mohamed and Atia[4] have studied the separation of the Laplace-Beltrami differential operator of the form

$$Au=-\frac{1}{\sqrt{\det g(x)}}\frac{\partial}{\partial x_i}\left[\sqrt{\det g(x)}\,g^{-1}(x)\frac{\partial u}{\partial x_i}\right]+V(x)u(x),$$

for every $x\in\Omega\subset R^n$, in the Hilbert space $H=L_2(\Omega,H_1)$ with the operator potential $V(x)\in C^1(\Omega,L(H_1))$, where $L(H_1)$ is the space of all bounded linear operators on the Hilbert space H_1, $g(x)=g_{ij}(x)$ is the Riemannian matrix and $g^{-1}(x)$ is the inverse of the matrix $g(x)$.

In [5] Brown has shown that certain properties of positive solutions of discongugate second order differential expressions

$$M[y]=-(py')'+qy$$

imply the separation of the minimal and maximal operators determined by M in $L(I_a)$, where $I_a=[a,\infty)$ and $a\succ-\infty$. More fundamental results of separation have been obtained by Brown [6] and [7].

In this paper we have generalized this work to prove the separation of the two dimensional Laplace operator.

Consider the two dimensional Laplace differential operator of the form

$$P[u] = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + q(x,y)u(x,y) \qquad (1)$$

P is said disconjugate on Ω if and only if there exists a positive solution $u(x,y)$ on the interior of Ω. For additional discussions see [8]. We show that properties of positive solutions of disconjugate second order differential operator (1) [9], imply the separation of minimal and maximal operators determined by P in $L^2(\Omega)$ i.e, the property that $P[u] \in L^2(\Omega) \Rightarrow qu \in L^2(\Omega)$. In particular, the preminimal and maximal operators L_o' and L are given by $P[u]$ for u in domain $D_0' = C_0^\infty(\Omega)$, the space of infinitely differential functions with compact support in the interior of Ω and

$$D = \{u \in L^2(\Omega) \cap C_{loc}(\Omega) \mid u_{xx} + u_{yy} \in C_{loc}(\Omega), P[u] \in L^2(\Omega)\}$$

where $C_{loc}(\Omega)$ stands for the real locally absolutely continuous functions on Ω, and $L^2(\Omega)$ denotes the usual Hilbert space associated with equivalence classes of Lebesgue square integrable functions f and g having norm

$$\|f\| = \left(\iint_\Omega |f(x,y)|^2 \; dxdy\right)^{\frac{1}{2}},$$

and inner product

$$[f,g] = \left(\iint_\Omega f(x,y)\,\overline{g(x,y)} \; dxdy\right)^{\frac{1}{2}}.$$

The minimal operator L_o with domain D_o is defined as the closure of L_o'.

With the above definitions one can show that:

(i) $C_0^\infty(\Omega) \subset D_0' \subset D_0 \subset D$. (ii) D_o', D_o and D are dense in $L^2(\Omega)$.

P is a limit point of L_P at ∞ if there is at most one solution of $P[u]=0$ which is in $L^2(\Omega)$.

Proposition 1. If P is separated on D_0 then it is separated on D if P is L_P at ∞.

We now turn to the central concern of this paper.

Theorem 2. Let $q(x,y)$ be C^1 functions. Suppose the laplace differential operator of the form (1) has a positive solution on the interior of Ω such that:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)u + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)u \equiv qu^2 \leq 2\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2, \qquad (2)$$

$$(1-\delta)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \leq \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)u + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)u, \delta \in \left[0, \frac{1}{3}\right). \qquad (3)$$

Then $q \geq 0$ and P is separated on $L_2(\Omega)$.

Proof: For the separation proof we need only show that u satisfy an inequality of the form $\|qu\|^2 \leq c\|u\|^2 + d\|P[u]\|^2$, where c, d are positive constants.

First, we prove that

$$z = \frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}}{u},$$

satisfies the P.D.E. of the form

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z^2 - q. \qquad (4)$$

We have,

$$\frac{\partial z}{\partial x} = \frac{-u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}}{u^2} = \frac{\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)}{u^2},$$

and

$$\frac{\partial z}{\partial y} = \frac{-u\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y}}{u^2} = \frac{\frac{\partial^2 u}{\partial y^2} - u\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)}{u^2}.$$

By substituting in (4), we get

$$\frac{-u\frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial^2 u}{\partial x \partial y} - u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}{u^2} = \frac{\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\right)^2}{u^2} - q$$

Hence

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = qu, \qquad (5)$$

since

$$b^2 - 4ac = 0,$$

so it is a parabolic equation.

The solution of the equation (5) is as follows:

$$\gamma^2+2\gamma+1=0 \Rightarrow \gamma_{1,2}=-1,$$
$$\frac{dy}{dx}-1=0 \quad \Rightarrow z=y-x.$$

Suppose that $w=y$, so

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=u_z+u_w,$$
$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial z}(u_z+u_w)\frac{\partial z}{\partial y}+\frac{\partial}{\partial w}(u_z+u_w)\frac{\partial w}{\partial y}$$
$$=\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial w\partial z}+\frac{\partial^2 u}{\partial w^2}, \tag{6}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=-u_z,$$
$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial z}(-u_z)\frac{\partial z}{\partial y}+\frac{\partial}{\partial w}(-u_z)\frac{\partial w}{\partial y}$$
$$=-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial z\partial w}, \tag{7}$$

And

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial z}(-u_z)\frac{\partial z}{\partial x}+\frac{\partial}{\partial w}(-u_z)\frac{\partial w}{\partial x}$$
$$=\frac{\partial^2 u}{\partial z^2} \tag{8}$$

By substituting from (6), (7) and (8) into (5), we get

$$\frac{\partial^2 u}{\partial w^2}=qu.$$

Hence

$$u=\varphi_1(y)exp(\sqrt{q}x)+\varphi_2(y)exp(-\sqrt{q}x).$$

The conditions (2) and (3) are equivalent to the conditions

$$-z^2\le \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y} \tag{9}$$

and

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\le \delta z^2. \tag{10}$$

To see this, note that from the definition of z and (6), (7), we get

(2) $\Leftrightarrow$

$$-2\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)^2}{u^2} \le \frac{-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u}{u^2}$$

$\Leftrightarrow$

$$-\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)^2}{u^2} \le \frac{-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u+\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)^2}{u^2}$$

$\Leftrightarrow$

$$-z^2\le \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}.$$

(3)$\Leftrightarrow$

$$-(1-\delta)\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)^2}{u^2} \ge \frac{-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u}{u^2}$$

$\Leftrightarrow$

$$\delta\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)^2}{u^2} \ge \frac{-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)u+\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)^2}{u^2}$$

$\Leftrightarrow$

$$\delta z^2\ge \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}.$$

Next we define the operators

$$L(v)=\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+zv,$$

and

$$L^*(v)=-\frac{\partial v}{\partial x}-\frac{\partial v}{\partial y}+zv,$$

where $v\in C_0^\infty(\Omega)$ and $\Omega\in R^2$.

Now we derive sufficient conditions for the separation of L^* as follows:
We have

$$\|L^*(v)\|^2=[L^*(v),L^*(v)]=[LL^*(v),v]$$

and

$$LL^*(v)=L\left(-\frac{\partial v}{\partial x}-\frac{\partial v}{\partial y}+zv\right)$$
$$=\frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}-\frac{\partial v}{\partial y}+zv\right)$$
$$+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}-\frac{\partial v}{\partial y}+zv\right)$$
$$+z\left(-\frac{\partial v}{\partial x}-\frac{\partial v}{\partial y}+zv\right).$$

So

$$\|L^*(v)\|^2 = \left[-\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z^2\right) v, v\right].$$

Using (9), we obtain

$$\|L^*(v)\|^2 = \left[\frac{\partial v}{\partial x}, \frac{\partial v}{\partial x}\right] + \left[\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right] + \left[\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}\right] + \left[\frac{\partial v}{\partial y}, \frac{\partial v}{\partial y}\right]$$
$$\geq \left\|\frac{\partial v}{\partial x}\right\|^2 + 2\left\|\frac{\partial v}{\partial x}\right\| \left\|\frac{\partial v}{\partial y}\right\| + \left\|\frac{\partial v}{\partial y}\right\|^2$$
$$= \left(\left\|\frac{\partial v}{\partial x}\right\| + \left\|\frac{\partial v}{\partial y}\right\|\right)^2.$$

By the triangle inequality it also follows that

$$\|zv\|^2 \leq 4\|L^*(v)\|^2.$$

The remaining step is to use the separation of L^* to show that M, which is restricted to $C_0^\infty(\Omega)$ is also separated.

We first observe that

$$L^*L(v) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + zv\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + zv\right) + z\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + zv\right)$$
$$= -\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial y^2} - \frac{\partial z}{\partial x} v - \frac{\partial z}{\partial y} v + z^2 v.$$

Since

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z^2 - q.$$

So

$$L^*L(v) == -\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial y^2} + qv.$$

Suppose that

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y \partial x},$$

then

$$L^*L(v) = -\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + qv = M[v].$$

A consequence of (9) and (10) is that

$$-\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z^2 \geq -\delta z^2 + z^2 = z^2(1-\delta).$$

Then

$$q \geq 0.$$

Now, also

$$\|P[u]\|^2 = [L^*L(u), L^*L(u)] = \|L^*L(u)\|^2$$

Since

$$\|zL(u)\| = 2\|L^*L(u)\|.$$

So

$$\|P[u]\|^2 \geq \frac{1}{4}\|zL(u)\|^2 = \frac{1}{4}[zL(u), zL(u)] = \frac{1}{4}\left[L^*\left(z^2 L(u)\right), u\right] \qquad (11)$$

and

$$\left[L^*\left(z^2 L(u)\right), u\right] = \left[-\frac{\partial}{\partial x}\left(z^2\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + zu\right)\right), u\right] + \left[-\frac{\partial}{\partial y}\left(z^2\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + zu\right)\right), u\right] + \left[z\left(z^2\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + zu\right)\right), u\right]$$
$$= \left[z^2 \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right] + \left[-\frac{\partial}{\partial x}\left(z^2\left(\frac{\partial u}{\partial y} + zu\right)\right), u\right] + \left[z^2 \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right] + \left[-\frac{\partial}{\partial y}\left(z^2\left(\frac{\partial u}{\partial x} + zu\right)\right), u\right] + \left[z^3\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + zu\right), u\right]$$
$$= z^2\left(\left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right] + \left[\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right]\right) + \left[-\frac{\partial}{\partial x}\left(z^2 \frac{\partial u}{\partial y}\right), u\right] + \left[-\frac{\partial}{\partial x}(z^3 u), u\right] + \left[-\frac{\partial}{\partial y}\left(z^2 \frac{\partial u}{\partial x}\right), u\right] + \left[-\frac{\partial}{\partial y}(z^3 u), u\right] + \left[z^3 \frac{\partial u}{\partial x} + z^3 \frac{\partial u}{\partial y}, u\right] + [z^4 u, u] \qquad (12)$$

we find that

$$\left[-\frac{\partial}{\partial x}(z^3u),u\right]+\left[-\frac{\partial}{\partial y}(z^3u),u\right]+[z^4u,u]=$$
$$\left[-\frac{\partial z^3}{\partial x}u,u\right]+\left[-z^3\frac{\partial u}{\partial x},u\right]$$
$$+\left[-\frac{\partial z^3}{\partial y}u,u\right]+\left[-z^3\frac{\partial u}{\partial y},u\right]+z^4[u,u]$$

Since

$$-\frac{\partial z^3}{\partial x}-\frac{\partial z^3}{\partial y}+z^4=-3z^2\frac{\partial z}{\partial x}-3z^2\frac{\partial z}{\partial y}+z^4,$$

and

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\le \delta z^2.$$

Hence

$$-\frac{\partial z^3}{\partial x}-\frac{\partial z^3}{\partial y}+z^4\ge z^4(1-3\delta). \quad (13)$$

But

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\ge -z^2,$$

So

$$z^2=q+\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\ge q-z^2,$$

Hence

$$z^2\ge\frac{q}{2}.$$

Then (13) becomes

$$-\frac{\partial z^3}{\partial x}-\frac{\partial z^3}{\partial y}+z^4\ge\frac{q^2}{4}(1-3\delta). \quad (14)$$

From (11), (12) and (14), we obtain

$$\|P[u]\|^2\ge\frac{1}{8}\left(\left\|\sqrt{q}\frac{\partial u}{\partial x}\right\|+\left\|\sqrt{q}\frac{\partial u}{\partial y}\right\|\right)^2+\frac{1-3\delta}{16}\|qu\|^2.$$

This immediately yields the separation inequality

$$\frac{16}{1-3\delta}\|P[u]\|^2\ge\|qu\|^2.$$

The final result of this paper is quite different from Theorem 2, but it reinforces the connection between disconjugacy and separation. In addition, the proof is quite elementary.

Theorem 3. Suppose that $P^\lambda[u]=-\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)+(q-\lambda q^2)u$, is disconjugate on Ω for some $\lambda>0$. Then $P[u]=-\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)+qu$, is separated.

Proof: It is well known that the disconjugacy of P^λ is equivalent to the positive definiteness of the functional

$$Q^\lambda(u)=\iint_\Omega\left(\left|\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right|^2+(q-\lambda q^2)|u|^2\right)dxdy$$
for $u\in C_0^\infty(\Omega)$,

see for example [8, Theorem 6.2]. In other words, we must have the inequality

$$Q^0(u)=\iint_\Omega\left(\left|\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right|^2+qu^2\right)dxdy\ge\iint_\Omega q^2|u|^2\,dxdy, \quad (15)$$

with equality holding iff $u=0$.

Now consider the expression

$$P_{q^2}(u)=q^{-2}\left[-\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)+qu\right],$$

where u is an appropriate function in $L^2(q^2;\Omega)$. If $u\in C_0^\infty(\Omega)$, then the Cauchy-Schwrtz inequality and (15) yields that

$$\left\|P_{q^2}(u)\right\|_{q^2}\|u\|_{q^2}\ge Q^0(u)\ge\lambda\|u\|_{q^2}^2=\lambda\|qu\|^2.$$

It follows that the inequality

$$\|P(u)\|\ge\left\|P_{q^2}(u)\right\|_{q^2}\ge\lambda\|qu\|,$$

holds on the C_0^∞ functions, and therefore on D_0. Because P is L_P at ∞ we again conclude that it is separated on D. Hence the proof.

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