IJST (2014) 38A2: 117-121
Iranian Journal of Science & Technology
http://ijsts.shirazu.ac.ir

On Covers of Acts over Semigroups

M. A. Naghipoor* and M. Ershad

Department of Mathematics, College of Sciences, Shiraz University, P. O. Box 71454, Shiraz, Iran
E-mail: ma_naghipoor@yahoo.com

Abstract

Let S be a monoid and X a class of S-acts which is closed under coproducts. The object of this article is to find conditions under which all S-acts have X-precovering. We have shown that the existence of torsion-free precovering implies the existence of torsion-free covering. This work is an attempt to further facilitate the study of the conjecture that all S-acts have flat cover.

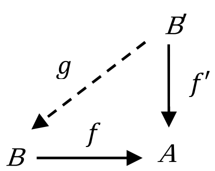
***Keywords**:* Precover of act; cover of act; torsion free act

1. Introduction

Throughout this note, unless otherwise stated, S is a monoid. A right S-act A_S is a non-empty set on which S acts unitarily on the right, that is, there is a function,
$A \times S \to A$,
$(a, s) \to as$,
for each $s \in S$ and $a \in A$, with the following properties:
(i) $(at)s = a(ts)$, for all $t, s \in S$ and $a \in A$;
(ii) $a1_S = a$, for all $a \in A$.

The notion of a left S-act is defined dually. To simplify, by an S-act we mean a right S-act. Also, we shall write all maps on the left and hence gf means that f followed by g. The reader is referred to (Clifford and Preston, 1961) and (Kilp et al. 2000) for basic definitions and results related to act and semigroup theory.

Let S be a monoid and X be a class of S-acts that is closed under coproducts. An S-act $B \in X$ is called an *X-precover* for an S-act A if there is a homomorphism $f: B \longrightarrow A$ such that for any $B' \in X$ and any homomorphism $f': B' \longrightarrow A$ there exists a homomorphism $g: B' \longrightarrow B$ with $f' = fg$, that is, the following diagram of right S-homomorphisms is commutative,



*Corresponding author
Received: 6 March 2013 / Accepted: 5 November 2013

In addition, in the above diagram, B is called an *X-cover* of A if g is an isomorphism for any endomorphism $g: B \longrightarrow B$ with $f = fg$.
We say that A has a torsion free (resp. flat, strongly flat, torsion free injective) cover if X is the class of torsion free (resp. flat, strongly flat, torsion free injective) acts.

It is easily seen that all X-covers of an S-act are isomorphic, so they are unique up to isomorphisms.
Another definition for covering is used in the literature. An S-act $B \in X$ is called an X-cover of A if there exists an epimorphism $f: B \longrightarrow A$ such that $f|_C$ is not an epimorphism for every proper subact C of B. For projective covering these two definitions are the same as it is shown in (Mahmoudi and Renshaw, 2008). But for strongly flat covering we do not know whether they coincide. There has been some progress but this is still an open question (see (Ershad and Khosravi, 2011)).

A great deal of work has been done on the module version of covering. As a pioneering work, Bass (1960) used the later definition of covering and showed that projective covers exist for all modules over a ring R if and only if R is a perfect ring. Also, in (Bican et al. 2001), the authors, proved the conjecture of Enochs in module theory, that all unitary modules over a ring with a unit have a flat cover (in the first sense). This work is an attempt to further facilitate the study of this conjecture in act theory.

The definition of cover used in almost all other works in act theory is the same as in (Bass, 1960) (see for example, (Khosravi et al. 2010) and (Mahmoudi and Renshaw, 2008)). In this article we use the first definition (the Enochs's sense) to study the existence problem of *X*-cover, especially when *X* is the class of torsion free or flat acts.

After some preliminaries and definitions on acts and monoids, in Section 2, we will find conditions (on *S*) under which all acts have torsion free cover. Also, we have shown that if *S* contains a zero, all *S*-acts have *X*-precover provided that all injective *S*-acts have *X*-precover.

Let *S* be a semigroup and *T* be a submonoid of *S*. The right semigroup of fractions of *S* relative to *T*, when it exists, is denoted by ST^{-1}, and is the quotient $S \times T/\sim$, where $\sim$ is the equivalence relation defined by $(s,t)\sim(s',t')$ if and only if there exist $u \in S$ and $v \in T$, such that $su = s'v$ and $tu = t'v$. When T is the set of all cancellable elements of *S,* then ST^{-1} is called *the right quotient semigroup*. Indeed, if the equivalence class determined by (s,t) is denoted by s/t, then ST^{-1} is a semigroup under the operation $(s/t)(s'/t') = su/t'v$, where $tu = s'v$ for $u \in S$ and $v \in T$. Moreover, a semigroup Q containing S is a right quotient semigroup of S if every cancellable element of S has a two sided inverse in *Q,* and every element of Q is of the form st^{-1} for some $s,t \in S$, where *t* is cancellable.

A semigroup *S* is said to have *right Ore condition* (or *common right multiple property*, CRM for short) if for every $s,t \in S$ with *t* cancellable, there exist $s',t' \in S$ where t' is cancellable and $st' = ts'$. It is easy to see that a monoid *S* has a right quotient semigroup if *S* has Ore condition. In fact, if *S* has Ore condition and *T* is the set of all cancellable elements of S, then it suffices to check whether ST^{-1} is a semigroup. It is a routine matter to verify that the stated relation $\sim$ is an equivalence relation. Since *S* has Ore condition, for each $s',t \in S$ with t cancellable, there exist $u,v \in S$ with v cancellable such that, $s'v = tu$. So for each $s/t, s'/t' \in ST^{-1}$, the operation $(s/t)(s'/t') = su/t'v$ is well-defined. Thus ST^{-1} is the right quotient semigroup of *S*.

As in (Feller and Gantos, 1969), an *S*-act *A* is called torsion free if *as=bs* with *a, b*∈A and *s*∈*S* are cancellable, implying *a=b*. This definition is slightly different to the one used in (Kilp et al. 2000) which considers the right cancellable element *s*∈*S* instead of cancelable. Also the relation τ, defined by $a\, \tau\, b$, for each $a,b \in A$, if and only if $as = bs$, for some cancellable element *s*∈*S*, is called the *torsion relation* on *A*. Clearly an *S*-act *A* is torsion free if and only if $\tau = \Delta_A$. Moreover, we have the following two lemmas that are used frequently in this paper:

Lemma 1.1. ((Feller and Gantos, 1969), Theorem 6.7) Let S be a semigroup with a zero and a cancellable element. Then S satisfies CRM if and only if for every *S*-act *A,* the torsion relation τ is a congruence.

Lemma 1.2. ((Feller and Gantos, 1969), Theorem 6.8) Let S be a semigroup which has a right quotient semigroup. Then for an *S*-act A, A/τ is torsion free, where τ is the torsion relation.

2. The Existence of Torsion Free Covering of Acts

In (Enochs, 1981), it is shown that in the category of modules over a ring, flat covers exist whenever flat precovers exist. Since the proof is categorical, by a similar proof we may have the same result for the category **Act-S**, that is, for every semigroup *S* and every *S*-act, *A,* if *A* has a flat (resp. strongly flat) precover then it has a flat (resp. strongly flat) cover. But in general the existence of precover does not imply the existence of cover. For example, clearly every *S*-act has a projective precover, but every *S*-act has a projective cover only if *S* is a perfect semigroup (see (Isbel, 1981)). In the first part of this section we will show that the existence of torsion free precovers implies the existence of torsion free covers. Then we will find conditions on a monoid *S* by which every *S*-act has a torsion free precover.

The module version of the following proposition has appeared in (Xu, 1981).

Proposition 2.1. For any torsion free precovering for an *S*-act *B* there exists a torsion free precovering $f: A \longrightarrow B$ such that for no nontrivial congruence $\rho \subseteq ker(f)$, A/ρ is torsion free.

Proof: Let $f': A' \longrightarrow B$ be a torsion free precovering and Σ be the set of all congruences σ on A' contained in $ker(f')$ with A'/σ torsion free. By Zorn's Lemma, there is a maximal element $\rho \in \Sigma$. Put $A = A'/\rho$, and consider $f: A \longrightarrow B$ as the induced map. Note that since $\rho \subseteq ker(f')$, f is well-defined. Then one can easily see that *f* is a torsion free precovering with the needed property, since *f'* is a torsion free precovering and $\rho \in \Sigma$ is the maximal element.

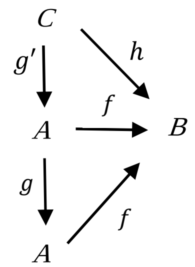
Remark 2.2. Note that the above proposition is also true for strongly flat (flat) precovers instead of torsion free precovers.

The following theorem shows the torsion free covers exist for acts over monoids which have torsion free precovers.

Theorem 2.3. The torsion free precovering $f: A \longrightarrow B$ in the Proposition 2.1 is a torsion free covering of *B*. Conversely, if $f: A \longrightarrow B$ is a torsion free cover, then it satisfies the condition in Proposition 2.1.

Proof: It suffices to show that the endomorphism $g: A \longrightarrow A$ with f=fg is an isomorphism. First note that since A is torsion free, $A/ker(g)$ is torsion free. Also f=fg implies $ker(g) \subseteq ker(f)$. So by Proposition 2.1, $ker(g) = \Delta_A$, that is, g is a monomorphism.

Now we show that g is onto. Let $X \supset A$ be a set with a larger cardinality and consider $\Sigma = \{(A_i, g_i) \mid i \in I\}$, to be the set of all proper subsets of X with the same property as A. Define the relation $\leq$ on Σ by $(A_i, g_i) \leq (A_j, g_j)$ iff $A_i \subseteq A_j$ and $g_j|_{A_i} = g_i$. Note that the union$(\bigcup A_i, \bigcup g_i)$, of any chain in Σ is also a torsion free precover and if ρ is a non-identity congruence on $\bigcup A_i$ such that $\rho \subset ker(\bigcup g_i)$, then $\bigcup A_i / \rho$ is not torsion free, since each (A_i/ρ_i) is not torsion free, where $\rho_i = \rho \bigcap (A_i \times A_i) \neq \Delta_{A_i}$. Thus Σ has a maximal element, (C, h). Since h is a torsion free precover, there exists $g': C \longrightarrow A$ with $fg' = h$, which is a monomorphism by the same proof as for g. We have the following commutative diagram,

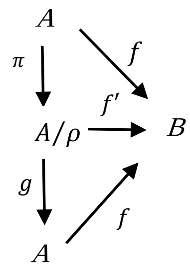


We will show that gg' is an isomorphism, which implies g is onto. Clearly gg' is a monomorphism. If gg' is not onto, then put $D = A \setminus gg'(C)$, $A' = C \bigcup D$ and $k: A' \longrightarrow A$, with $k = gg'$ on C and the identity map on D. Then k is a bijection map and gives A' the same S-act structure as A. Now consider (A', fk). Then it is easy to see that (A', fk) is a torsion free precover of B, and $(C, h) \leq (A', fk)$. Moreover, for each congruence $\rho \subset ker(fk)$ on A',

$$k(\rho) = \{(k(x), k(y)) \mid (x, y) \in \rho\}$$

is a congruence on A contained in $ker(f)$, and if A'/ρ is torsion free then $A/k(\rho)$ is torsion free. So $k(\rho)$ is trivial, which implies $\rho = \Delta_{A'}$. Thus $(A', fk) \in \Sigma$, which contradicts the maximality of (C, h). So $D = \emptyset$, that is, gg' is onto.

Conversely, suppose that $f: A \longrightarrow B$ is a torsion free cover of B and $\rho \subset ker(f)$, is a nontrivial congruence such that A/ρ is torsion free. Since f is a torsion free precover, there exists a homomorphism $g: A/\rho \longrightarrow A$, with $fg = f'$, where $f': A/\rho \longrightarrow B$, defined by $f'(a\rho) = f(a)$. So f' is a torsion free precover for B. Now consider the following commutative diagram of S-homomorphisms,



where π is the canonical projection. Since f is a torsion free cover, $g\pi$ is an isomorphism, that is, π is a monomorphism. So $\rho = ker\,\pi = \Delta_A$.

Theorem 2.4. An S-act B has a torsion free cover if and only if there is a torsion free precovering $f: A \longrightarrow B$ of B, such that $ker(f)$ contains a maximal congruence ω, such that A/ω is torsion free.

Proof: First suppose that $f: A \longrightarrow B$ is a torsion free cover of B. Then by Theorem 2.3, $ker(f)$ contains no nontrivial congruence $\rho \subseteq ker(f)$, with A/ρ torsion free. So $\omega = \Delta_A$ is a maximal congruence in $ker(f)$ such that A/ω is torsion free. Conversely, suppose that $f: A \longrightarrow B$ is a torsion free precovering of B, and $ker(f)$ contains a maximal congruence ω, such that, A/ω is torsion free. Thus there exists a homomorphism $g: A/\omega \longrightarrow A$, with $fg = f'$, where $f': A/\omega \longrightarrow B$, defined by $f'(a\omega) = f(a)$. Then f' is a torsion free precover of B. So by Theorem 2.3, f' is a torsion free cover of B, by maximality of ω.

In the following, we will find conditions on S on which every S-act has a torsion free precover.

Lemma 2.5. Let X be a class of S-acts which is closed under subacts. If $A \in X$ and $f: A \longrightarrow B$ is an X-precover and C is a proper subact of B, then $g: f^{-1}(C) \longrightarrow C$ is an X-precover for C. In particular, the result is true if X is the class of torsion free S-acts.

Proof: First note that $f^{-1}(C) \in X$, by assumption. For each homomorphism $h: D \longrightarrow C$, with $D \in X$, there exists a homomorphism $\bar{h}: D \longrightarrow A$, such that $h = f\bar{h}$, since f is an X-precover. Now clearly $\bar{h}(D) \subseteq f^{-1}(C)$. So $\bar{h}$ may be considered from D into $f^{-1}(C)$. Thus $h = g\bar{h}$, that is, g is an X-precover.

The proof of the second part is clear, since any subact of any torsion free right act is torsion free.

Lemma 2.6. Let S be a semigroup with a right quotient semigroup and B be an injective S-act. Then $f: A \longrightarrow B$ is a torsion free precover if and only if for each homomorphism $f': A' \longrightarrow B$ with A' torsion free and injective, there exists $g: A' \longrightarrow A$ such that $f' = fg$.

Proof: The necessity part is clear.

Sufficiency. Let C be a torsion free right act and $h: C \longrightarrow B$ be a homomorphism. First note that $E(C)$, the injective envelope of C, is torsion-free. To prove this we show that $E(C)$ is isomorphic to $E(C)/\tau$, where τ is the torsion congruence on $E(C)$. Consider the natural epimorphism $\pi: E(C) \longrightarrow E(C)/\tau$. Since C is torsion free, the restriction $\pi|_C$ is monomorphism. Now since C is essential in $E(C), \pi$ must be a monomorphism.

Since B is injective, there exists $\bar{h}: E(C) \longrightarrow B$ such that $\bar{h}|_C = h$. By assumption there exists a homomorphism $g: E(C) \longrightarrow A$ such that $\bar{h} = fg$. So we have,

$h = \bar{h}|_C = fg|_C = f(gi)$,

where i is the inclusion map from C into $E(C)$. Thus f is a precover.

Remark 2.7. The condition in the previous lemma is also sufficient for $f: A \longrightarrow B$ to be an strongly flat (a flat) precover. This condition is also necessary if the injective envelope of any strongly flat (resp. flat) S-act is strongly flat (resp. flat).

Corollary 2.8. Let S be a semigroup with a right quotient semigroup and B be an injective S-act. Then $f: A \longrightarrow B$ is a torsion free precover (cover) of B if it is a torsion free injective precover (cover).

As in (Kilp et al. 2000), by **Act$_0$-S** we mean the category of right acts containing the unique one element act, θ, over a semigroup S with a zero. Also $\overline{\textbf{Act}}$**-S** is the category of right S-acts containing $\emptyset_S$. Note that the pullbacks exist in these categories. So we have the following theorem and corollary in **Act$_0$-S** and $\overline{\textbf{Act}}$**-S**.

Theorem 2.9. Let X be a class of acts in **Act$_0$-S** or $\overline{\textbf{Act}}$**-S** which is closed under subacts. Then every S-act has an X-precover if and only if every injective S-act has an X-precover.

Proof: The necessity part is clear.

Let B be an S-act and $i: B \longrightarrow E(B)$, be the inclusion. By assumption $E(B)$ has an X-precover, say, $g: C \longrightarrow E(B)$. Put

$A = \{(b, c) \in B \times C \mid g(c) = b\}$.

Then $(A, (p_1, p_2))$ is the pullback of (i, g), where p_i are the restrictions of the canonical projections. We claim that $p_1: A \longrightarrow B$ is an X-precover for B. First note that $p_2: A \longrightarrow C$, is a monomorphism, for if $p_2(b, c) = p_2(b', c')$, then $c = c'$ which implies $b = b'$, by definition of A. Thus A is isomorphic with a subact of C. So $A \in X$, by assumption. Moreover, if $A' \in X$ and $p': A' \longrightarrow B$ is a homomorphism, then there exists a homomorphism $h: A' \longrightarrow C$, such that $gh = ip'$. Now there exists a homomorphism $\overline{p'}: A' \longrightarrow A$ such that, $p_1\overline{p'} = p'$. Indeed $\overline{p'}$ is defined by $\overline{p'}(a') = (p'(a'), c')$, where $h(a') = c'$.

Corollary 2.10. Let S be a semigroup with a zero. Then every S-act has a torsion free precover if and only if every injective S-act has a torsion free precover.

Note that if S is a group, then every S-act A is torsion free. If $A = \bigcup_{i \in I} A_i$ is a decomposition of A into indecomposable subacts, A_i, then each A_i is cyclic as an indecomposable act over a group.

Proposition 2.11. Let S be a semigroup with a zero and a right quotient semigroup Q. If Q is a group, then every right S-act B has a torsion free precover.

Proof: Since $B \subseteq E(B)$, by Lemma 2.5, it suffices to find a torsion free precover for injective acts. Moreover, by Lemma 2.6, it suffices to find a torsion free right act A and $f: A \longrightarrow B$ such that for each homomorphism $f': A' \longrightarrow B$ with A' torsion free and injective, there exists $g: A' \longrightarrow A$ such that $f' = fg$.

First note that the class of $\mathcal{F}$ of cyclic torsion free right S-acts forms a set. Suppose that T is the set of all homomorphisms from S-acts in $\mathcal{F}$ into B and put $A = \bigcup_T F_i$, where F_i's are the domains of the homomorphisms in T. Define $f: A \longrightarrow B$ by $f(a_t) = ta_t$, for each $t \in T$. It is easy to see that every torsion free injective S-act A' is a unitary Q-act (Lemma 7.4 of (Feller and Gantos, 1969)). Since Q is a group, $A' = \bigcup_{T' \subseteq T} F_i'$ for some cyclic torsion free Q-acts, F_i', which are also cyclic torsion free S-acts. Then any homomorphism $f': A' = \bigcup_{T' \subseteq T} F_i' \longrightarrow B$, is defined by $f'(a') = \bigcup_{t \in T' \subseteq T} t'(a')$, for each $a' \in A'$. Hence there exists $g: A' \longrightarrow A$ such that $f' = fg$.

One can easily see that by using Remark 2.7, the above proposition is also true for strongly flat (resp. flat) precover whenever any inverse image of any strongly flat (resp. a flat) act is strongly flat (resp. flat).

Corollary 2.12. Let S be a monoid with a zero and a right quotient semigroup Q. If Q is a group, then every right S-acts has a torsion free cover.

In the proof of Lemma 2.6, it is shown that for a semigroup S with a right quotient semigroup, the injective envelope of a torsion free S-act is torsion free. In the following theorem, we will show that in this case, torsion free cover of an injective S-act is injective.

Theorem 2.13. If S is a semigroup with a right quotient semigroup, then every torsion free cover of every injective right S-act (if exists) is injective.

Proof: Let B be an injective S-act and $f: A \longrightarrow B$ be a torsion free cover of B. Since B is injective, there exists $g: E(A) \longrightarrow B$ such that $gi = f$, where $i: A \longrightarrow E(A)$ is the inclusion map. We shall show that $E(A)$ is also a torsion free cover of B and hence it is isomorphic to A, that is, A is injective. It is easily seen that $g: E(A) \longrightarrow B$ is a torsion free precovering. Now let $\rho \subseteq ker\, g$ be a right congruence such that $E(A)/\rho$ is torsion free. By Theorem 2.3, it is enough to prove that ρ is trivial.

Put $f' = \pi|_A$, where $\pi: E(A) \longrightarrow E(A)/\rho$ is the canonical projection with $ker\, \pi = \rho$. So,

$$\begin{aligned} ker\, f' &= \{(a, a') \in A \times A | f'(a) = f'(a')\} \\ &= \{(a, a') \in A \times A | (a, a') \in \rho \subseteq ker\, g\} \\ &\subseteq \{(a, a') \in A \times A | g(a) = g(a')\} \\ &= \{(a, a') \in A \times A | g(i(a)) = g(i(a'))\} \\ &= \{(a, a') \in A \times A | f(a) = f(a')\} \\ &\subseteq ker\, f. \end{aligned}$$

Now $A/ker\, f'$ is torsion free as it is isomorphic to a subact of $E(A)/\rho$. Since f is a torsion free covering, by Theorem 2.3, $ker\, f' = \Delta_A$ as a subcongruence of $ker\, f$. Therefore, $f' = \pi|_A$ is a monomorphism. So π must be a monomorphism, that is, $\rho = ker\, \pi = \Delta_{E(A)}$.

The converse of the above theorem is true for weakly injective torsion free precovers by the following proposition.

Proposition 2.14. If an S-act B has a weakly injective torsion free precover A, then B is weakly injective.

Proof: Suppose $f: A \longrightarrow B$ is a torsion free precover. Let I be a right ideal of S and $g: I \longrightarrow B$ be a homomorphism. It suffices to find a homomorphism $\bar{g}: S \longrightarrow B$ such that $g|_I = g$.

Clearly I is torsion free. So there exists $f': I \longrightarrow A$ such that $ff' = g$. On the other hand since A is weakly injective, there exists $\overline{f'}: S \longrightarrow A$ with $\overline{f'}|_I = f'$. Now put $\bar{g} = f\overline{f'}$.

References

Bass, H. (1960). Finitistic Dimension and a Homological Generalization of Semi-primary Rings. *Trans. Amer. Math. Soc*. 95, 466–488.

Bican, L., El Bashir, R., & Enochs, E. (2001). All Modules Have Flat Covers. *Bull. Lond. Math. Soc*., 33, 385–390.

Clifford, A. H., & Preston, G. B. (1961). *The algebraic theory of semigroups*. Vols. 1, 2, Math. Surveys, No. 7, Amer. Math. Soc. Providence, R. I.

Enochs, E. (1981). Injective and Flat Covers, Envelopes and Resolvents. Is. J. Math. 39, 189–209.

Ershad, M., & Khosravi, R. (2011). On the Uniqueness of Strongly Flat Covers of Cyclic Acts. *Turk. J. Math*., 35, 437–442.

Feller, E. H., & Gantos, R. L. (1969). Indecomposable and Injective S-systems with Zero. *Mathematische Nachrichten*, 41, 37–48.