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Multiplier in BL-algebras

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Abstract

In this paper, we introduce the notion of multiplier in BL-algebra and study relationships between multipliers and some special mappings, likeness closure operators, homomorphisms and $(\odot, \vee)$-derivations in BL-algebras. We introduce the concept of idempotent multipliers in BL-algebra and weak congruence and obtain an interconnection between idempotent multipliers and weak congruences. Also, we introduce the special multiplier α_p and study some properties. Finally, we show that if A is a boolean algebra, then the set of all multipliers of A is a BL-algebra under some conditions.

Keywords: BL−algebra; MV−algebra; MV-center; multiplier; closure operator; Godel algebra

1. Introduction

BL-algebras were invented by (H'ajek, 1998) in order to prove the completeness theorem of "Basic Logic" (BL, for short) arising from the continuous triangular norms, familiar in the fuzzy logic framework. The language of propositional H'ajek basic logic contains the binary connectives $\odot$and $\Rightarrow$and the constant 0.

Axioms of BL are:

$(A1)(\varphi \Rightarrow \psi) \Longrightarrow ((\psi \Rightarrow w) \Rightarrow (\varphi \Rightarrow w))$,
$(A2)(\varphi \odot \psi) \Longrightarrow \varphi$,
$(A3)(\varphi \odot \psi) \Longrightarrow (\psi \odot \varphi)$,
$(A4)(\varphi \odot (\varphi \Rightarrow \psi)) \Longrightarrow (\psi \Rightarrow (\psi \Rightarrow \varphi))$,
$(A5a)(\varphi \Rightarrow (\psi \Rightarrow w)) \Longrightarrow ((\varphi \odot \psi) \Rightarrow w))$,
$(A5b)((\varphi \odot \psi) \Longrightarrow w) \Longrightarrow (\psi \Rightarrow (\psi \Rightarrow w))$,
$(A6)((\varphi \Rightarrow \psi) \Longrightarrow w) \Longrightarrow (((\psi \Longrightarrow \varphi) \Longrightarrow w) \Longrightarrow w)$,
$(A7)\bar{0} \Longrightarrow w$.

The main example of a BL-algebra is the interval $[0,1]$ endowed with the structure induced by a continuous t-norm. MV-algebras, Godel algebras and product algebras are the most known classes of BL-algebras.

The concept of multiplier for distributive lattices was defined by (Cornish, 1974).Multipliers are used in order to give a non standard construction of the maximal lattice of quotients for a distributive lattice, (Schmid, 1980). A partial multiplier in a commutative semigroup $(A,\cdot)$ has been introduced as a function f from a nonempty subset $D_f$of A into$A$ such that

$f(x) \cdot y = x \cdot f(y)$ for all $x, y \in D_f$, (Larsen, 1971).

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In this paper, we introduce the concept of multiplier for a BL-algebra and study some properties. Then we study the relationships between multipliers and some special mappings, likeness closure operators, homomorphisms and $(\odot, \vee)$-derivations in BL-algebras. Next we study the relationships between cardinal of a BL-algebra and number of multipliers in BL-algebras. Effect of a multiplier on some special filter, likeness Boolean filter, prime filter, ... in BL-algebras is also studied. We introduce the concept of idempotent multipliers in BL-algebra and weak congruence and obtain an interconnection between idempotent multipliers and weak congruences. The special multiplier α_p and some properties are studied. Finally, we show that if A be a boolean algebra, then $M(A)$ will be a BL-algebra under the conditions.

2. Preliminaries

In this section, we present some definitions and results about BL-algebra and MV–algebra and closure operator.

Definition 2.1. (H'ajek, 1998). A BL-algebra is an algebra$(A, \wedge, \vee, \odot, \rightarrow, 0,1)$of type $(2,2,2,2,0,0)$ such that:

(BL1) $(A, \vee, \wedge, 0,1)$ is a bounded lattice,
(BL2) $(A, \odot, 1)$ is an abelian monoid,
(BL3) $x \odot z \leq y$ if and only if $z \leq x \rightarrow y$,
(BL4) $x \odot (x \rightarrow y) = x \wedge y$,
(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,
for all $x, y, z \in A$.

A BL-algebra is called an MV-algebra if $x^{**} = x$, for all x ∈ A, where $x^* = x \to 0$.

Theorem 2.2. (H'ajek, 1998). In any BL-algebra (A,∧,∨,⊙,→,0,1) the following properties are valid:
(1) $x \leq y$ if and only if $x \to y = 1$,
(2) $1 \to x = x$,
(3) $x \wedge y \leq x$,
(4) $x \to (y \to z) = y \to (x \to z)$,
(5) $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$,
(6) $(x \to z) \wedge (y \to z) = (x \vee y) \to z$,
(7) $(x \to z) \vee (y \to z) = (x \wedge y) \to z$,
(8) $x \wedge (\vee_{i\in I}\, y_i) = \vee_{i\in I}\, (x \wedge y_i)$,
(9) if A is BL-chain, then we have: $x \vee (\wedge_{i\in I}\, y_i) = \wedge_{i\in I}\, (x \vee y_i)$.

Definition 2.3. (H'ajek, 1998). Let A be a BL-algebra. A nonempty subset F of A is called a filter of A if F satisfies the following conditions:
(F1) if $x \in F$ and $x \leq y,\ y \in A$, then $y \in F$,
(F2) $x \odot y \in F$ for every $x, y \in F$, that is, F is a subsemigroup of A.

Denote by $\mathcal{F}(A)$ the set of all filters of a BL-algebra A. Clearly, $\{1\}$ and A are respectively, the smallest and the largest elements of $\mathcal{F}(A)$. Moreover, the following result gives an equivalentversion of the concept of filters.

Theorem 2.4. (H'ajek, 1998). Let A be a BL-algebra. Then a nonempty subset F of A is a filter of A if and only if it satisfies the following conditions:
(F3) $1 \in F$,
(F4) $x, x \to y \in F$ imply $y \in F$.

If F is satisfied in $F3$, $F4$, then F is called a deductive system or Ds for short.

The MV-center of A, denoted by $MV(A)$ is defined as

$$MV(A) = \{x \in A : x^{**} = x\}.$$

Hence, a BL-algebra A is an MV-algebra iff $A = MV(A)$.

In the rest of this paper by $B(A)$ we denote the Boolean algebra of all complemented elements in $L(A)$ (hence $B(A) = B(L(A))$).

Remark 2.5. (H'ajek, 1998). If $x \in A$ and $e \in B$, then $e \odot x = e \wedge x,\ x \to e = (x \odot e^*)^* = x^* \vee e$.

Definition 2.6. (Burris, 1981). If we are given a set A, a mapping $f: Su(A) \to Su(A)$ is called a closure operator if for all $X, Y \subseteq A$, it satisfies the following conditions:
(c1) $X \subseteq f(X)$,
(c2) if $X \subseteq Y$, then $f(X) \subseteq f(Y)$,
(c3) $f^2(X) = f(X)$.

Definition 2.7. (Torkzadeh, 2012). Let A be a BL-algebra and $d: A \to A$ be a function. We call d a (⊙,∨)-derivation on A, if d satisfies the following condition:

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)),$$

for all $x, y \in A$.

3. Multipliers in BL-algebras

In this paper, we denote BL-algebra $(A,\wedge,\vee,\odot,\to,0,1)$ with A.

Definition 3.1. $f: A \to A$ is called a multipliers in A, if

$$f(x \to y) = x \to f(y)$$

for all $x, y \in A$.

We denote the set of all multiplier in A with $M(A)$.

Example 3.2. (a) $f(x) = 1,\ g(x) = x$ are multipliers in any BL-algebra.
(b) $\alpha_p(x) = p \to x$ is multiplier in every BL-algebra. α_p is called the simple multiplier.
(c) Let $I = [0;1]$ be the unit interval. We define $\odot, \to$ on $[0;1]$ as follows:
$x \odot y = x \wedge y,\ x \to y = 1$ if $x \leq y$, otherwise $x \to y = y$.

Then $(I,\wedge,\vee,\odot,\to,0,1)$ is a BL-algebra. Now, we define $f: I \to I$ as follows:

$$f(x) = \begin{cases} x & \text{if } x < 0.5 \\ 1 & \text{if } x \geq 0.5 \end{cases} \qquad (3.1)$$

then f is a multiplier.

(d) Suppose $0 < a < b < 1$ and let $A = \{0, a, b, 1\}$. For all $x, y \in A$, we define, ⊙ and → as follows:

⊙	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

→	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

then $(A,\vee,\wedge,\odot,\to,0,1)$ is a BL-algebra. Define a self map f as follows $f(0) = 0,\ f(a) = a, f(b) = f(1) = 1$, then f is a multiplier.

Proposition 3.3. If A has n elements, then it has at least n multipliers.

Proof: Since for every $p \in A, \alpha_p$ is a multiplier, so A has at least n multipliers.

Theorem 3.4. If f is a multiplier in A, then
(i) $f(1) = 1$,
(ii) $x \le f(x)$, for all $x \in A$,
(iii) if $f_1,\ f_2$ are two multipliers in A, then $f_1 \circ f_2$ is a multiplier in A.

Proof: (i) For all $x \in A$ we have $0 \to x = 1$, so $f(1) = f(0 \to x) = 0 \to f(x) = 1$. Thus $f(1) = 1$.
(ii) Let $x \le y$. So $x \to y = 1$, thus $1 = f(1) = f(x \to y) = x \to f(y)$.
So $x \le f(y)$, i.e. if $x \le y$, then $x \le f(y)$. Now since $x \le x$, so $x \le f(x)$.
(iii)
$$\begin{aligned}(f_1 \circ f_2)(x \to y) &= f_1(f_2(x \to y))\\ &= f_1(x \to f_2(y))\\ &= x \to f_1(f_2(y))\\ &= x \to (f_1 \circ f_2)(y).\end{aligned}$$

Proposition 3.5. $(M(A), \circ, I)$ is a monoid, where I is an identity function.
Let f be a self map on $A$and $x, y \in A$. We define

$$x \sqcup y = (y \to x) \to x,$$
$$F_f = \{x \in A : f(x)\},$$
$$F_1 = \{x \in A : f(x) = 1\}.$$

Example 3.6. In Example 3.2(d), consider $F_f = \{0, a, 1\}$. We have $a \in F_f$ but $b \notin F_f$.Then in general F_f is not a filter of A.

Theorem 3.7. Let f be a multiplier in $A$and $x \in F_f$. Then for all $y \in A$, $x \sqcup y \in F_f$.

Proof:

$$\begin{aligned}f(x \sqcup y) &= f((y \to x) \to x)\\ &= (y \to x) \to f(x)\\ &= (y \to x) \to x\\ &= x \sqcup y,\end{aligned}$$

so $x \sqcup y \in F_f$.

Theorem 3.8. Let f be a multiplier in A.
(i) If F_f be a filter of A, then $f(F_f)$ is a filter of A,
(ii) $f(F_1)$ is the trivial filter of A,
(iii) if f is a homomorphism of A, then F_f is a filter of A,
(iv) if $x \in D(A)$, ($D(A)$ is all dense elements of A), then $f(x) \in D(A)$.

Proof: (iv) Let $x \in D(A)$. Then $x^* = 0$. Since $x \le f(x)$, then $f(x) \to 0 \le x \to 0 = x^* = 0$, so $(f(x))^* = 0$, thus $f(x) \in D(A)$.

Example 3.9. Let $A = \{0, a, b, c, d, 1\}$, with $0 < a < b < 1$, $0 < c < d < 1$, but a, c and respectively b, d are incomparable. For all $x, y \in A$, we define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(A, \vee, \wedge, \odot, \to, 0, 1)$ is a non-linearly ordered BL-algebra. We define $f: A \to A$ as follows:

$$f(x) = \begin{cases} c, & if\ \mathrm{x = 0, c} \\ d, & if\ \mathrm{x = a, d} \\ 1, & \mathrm{if\ \ x = b, 1} \end{cases} \qquad (3.2)$$

Then f is a multiplier in A. All nilpotent elements of A, are $Nil(A) = \{0, a\}$.
We have $0 \in Nil(A)$, but $f(0) = c \notin Nil(A)$. So if x is a nilpotent element of A and f is amultiplier, then $f(x)$ is not necessarily nilpotent element of A.

Example 3.10. Let $A = \{0, a, b, c, d, e, f, g, 1\}$, with $0 < a < b < e < 1$, $0 < c < f < g < 1$, $a < d < g$, $c < d < e$but $\{a, c\}$, $\{b, d\}$, $\{d, f\}$, $\{b, f\}$, $\{e, g\}$ are incomparable. For all $x, y \in A$, we define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
B	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

$\rightarrow$	0	a	B	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

then$(A,\vee,\wedge,\odot,\rightarrow,0,1)$ is a non-linearly ordered BL-algebra. We define $f: A \rightarrow A$ as follows:

$$f(x)=\begin{cases} c, & \text{if } x=0 \\ d, & \text{if } x=a \\ e, & \text{if } x=b \\ f, & \text{if } x=c,f \\ g, & \text{if } x=d,g \\ 1, & \text{if } x=e,1 \end{cases} \tag{3.3}$$

Then f is a multiplier in A. All of idempotent and Boolean elements of A, are respectively, $Idem(A)=\{0,b,f,1\}$, $Bool(A)=\{0,b,f,1\}$. We have $b \in Idem(A)$, but $f(b)=e \notin Idem(A)$. So if x is an idempotent element of A and f is a multiplier, then $f(x)$ is not necessarily an idempotent element of A.

Also we have $b \in Bool(A)$, but $f(b)=e \notin B(A)$. So if x is a Boolean element of A and f is a multiplier, then $f(x)$ is not necessarily Boolean element of A.

Example 3.11. (a) In Example 3.10, $F=\{c,d,e,f,g,1\}$ is prime filter, but $f(F)=\{f,g,1\}$ is not, because $1=a \vee c \in f(F)$, but $a,c \notin f(F)$.
(b) In Example 3.10, $F=\{d,e,g,1\}$ is Boolean filter, but $f(F)=\{g,1\}$ is not.
(c) In Example 3.10, $F=\{b,e,1\}$ is maximal filter, but $f(F)=\{g,1\}$ is not maximal filter.

Theorem 3.12. Let f be a multiplier in A. Then:
(i) for all $x,y \in A$, $f(x) \rightarrow y \leq x \rightarrow f(y)$,
(ii) $(f(x))^* \leq f(x^*)$,
(iii) $f(x \rightarrow y) \geq f(x) \rightarrow f(y)$.

Proof: (i) We have
$(f(x) \rightarrow y) \odot x \leq (x \rightarrow y) \odot x = x \wedge y \leq y \leq f(y)$, so

$$(f(x) \rightarrow y) \odot x \leq f(y),$$

thus $f(x) \rightarrow y \leq x \rightarrow f(y)$.
(ii) We have $(f(x))^*=f(x) \rightarrow 0$ and $f(x^*)=f(x \rightarrow 0)=x \rightarrow f(0)$, so by (i) we get, $f(x) \rightarrow 0 \leq x \rightarrow f(0)=f(x^*)$, so $(f(x))^* \leq f(x^*)$.
(iii) $f(x \rightarrow y)=x \rightarrow f(y) \geq f(x) \rightarrow f(y)$.

If $f(x)=1$ and $y \neq 1$, then $f(x) \rightarrow y < x \rightarrow f(y)$. So equality in part (i) in the above theorem is not always valid.

In general every multiplier in BL-algebra is not homomorphism and conversely.

Example 3.13. (a) In Example 3.2(b), f is a multiplier but is not homomorphism.
(b) Let X be a nonempty set and $P(X)$ a family of all subset of X. For each $A,B \in P(X)$,
we define the operations $\odot$ and $\rightarrow$ by
$A \rightarrow B = A^C \cup B$, $A \odot B = A \cap B$.
Then $(P(X),\subseteq,\cup,\cap,\odot,\rightarrow,\emptyset,A)$ is a BL-algebra.

Now, let $X=\{x_1,x_2,x_3\}$ and $Y=\{y_1,y_2\}$ be two sets. Define $f: P(X) \rightarrow P(Y)$ as follows:

$$f(\emptyset)=\emptyset, \quad f(X)=X,$$
$$f(\{x_1\})=\{y_1\}, \quad f(\{x_2\})=\{y_2\},$$
$$f(\{x_3\})=\emptyset,$$
$$f(\{x_1,x_2\})=Y, \quad f(\{x_1,x_3\})=\{y_1\},$$
$$f(\{x_2,x_3\})=\{y_2\}.$$

Then f is BL-homomorphism, but is not multiplier.
By the following example, we show that every multiplier is not isotone.

Example 3.14. Let A be the BL-algebra in Example 3.2(d) and

$$f(x)=\begin{cases} a, & \text{if } x=0 \\ b, & \text{if } x=b \\ 1, & \text{if } x=a,1 \end{cases} \tag{3.4}$$

Then f is multiplier, but is not isotone.

Theorem 3.15. If $f: A \rightarrow A$ is an isotone multiplier and $ff(x) \leq f(x)$ for all $x \in A$, then f is a closure operator on A.

Proof: 1) Since f is multiplier, then $x \leq f(x)$,
2) since f is isotone, if $x \leq y$, then $f(x) \leq f(y)$,
3) by(1), $f(x) \leq f(f(x))$, also by hypothesis $f^2(x) \leq f(x)$, so $f^2(x)=f(x)$.

Example 3.16. α_p is multiplier and isotone but $\alpha_p^2 \geq \alpha_p$. So α_p is not a closure operator.

Theorem 3.17. If $f: A \rightarrow A$ is a closure operator and homomorphism, then f is a multiplier.

Proof:

$$\begin{aligned} f(x \rightarrow y) &= f(x) \rightarrow f(y) \\ &\leq x \rightarrow f(y), \\ x \rightarrow f(y) &\leq f(x \rightarrow f(y)) \\ &= f(x) \rightarrow f^2(y) \\ &= f(x) \rightarrow f(y) \\ &= f(x \rightarrow y). \end{aligned}$$

Example 3.18. Let $I = [0;1]$ be the unit interval. Define $\odot, \rightarrow$ on $[0;1]$ as follows:
$x \odot y = x \wedge y$, $x \rightarrow y = 1$ if $x \leq y$, otherwise $x \rightarrow y = y$.

Then $(I, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Now define $f: I \rightarrow I$ as follows:

$$f(x) = \begin{cases} a, & \text{if } x = 0 \\ b, & \text{if } x = b \\ 1, & \text{if } x = a, 1 \end{cases} \quad (3.5)$$

We can see that f is a closure operator and f is not homomorphism, so f is not a multiplier.

Lemma 3.19. Let $f: A \rightarrow A$ be a multiplier and $(\odot, \vee)$-derivation in A. If $x \in A$ and $x^2 = x$, then $f(x) = x$.

Proof: Since f is multiplier, then $x \leq f(x)$. On the other hand,

$$f(x) = f(x \odot x) = (f(x) \odot x) \vee (x \odot f(x)) \\ = f(x) \odot x \leq x,$$

so $f(x) \leq x$, thus $f(x) = x$.

Theorem 3.20. Let $f: A \rightarrow A$ be a multiplier and $(\odot, \vee)$-derivation on G-algebra A. Then f is identity.

Theorem 3.21. A multiplier $f: A \rightarrow A$ is an identity map if it satisfies $x \rightarrow f(y) = f(x) \rightarrow y$, for all $x, y \in A$.

Proof: Let $x, y \in A$ be such that $x \rightarrow f(y) = f(x) \rightarrow y$. Now

$$f(x) = f(1 \rightarrow x) = 1 \rightarrow f(x) = f(1) \rightarrow x = 1 \rightarrow x = x.$$

Therefore f is identity.

In general, every multiplier in BL-algebra need not be identity. However, in the following, we derive a set of conditions which are all together equivalent to that f being an identity multiplier.

Theorem 3.22. A multiplier f is an identity map if and only if the following conditions are satisfied for all $y \in A$:
(i) $f^2(x) = f(x)$,
(ii) $f(x \rightarrow y) = f(x) \rightarrow f(y)$,
(iii) $f^2(x) \rightarrow y = f(x) \rightarrow f(y)$.

Proof: The conditions for necessary are trivial. For sufficiency, assume the conditions (i), (ii) and (iii). Then for any $x, y \in A$, we can obtain

$$f(x) \rightarrow y = f^2(x) \rightarrow y = f(x) \rightarrow f(y) = f(x \rightarrow y).$$

Also by the definition of multiplier, we have $f(x \rightarrow y) = x \rightarrow f(y)$. Hence $f(x \rightarrow y) = x \rightarrow f(y) = f(x) \rightarrow y$. Therefore by the previous theorem, f is identity multiplier in A.

Theorem 3.23. If multiplier f is a monomorphism and closure operator, then f is identity map.

Proof: We prove that $x \rightarrow f(y) = f(x) \rightarrow y$.

$$\begin{aligned} f(x \rightarrow f(y)) &= x \rightarrow f^2(y) \\ &= x \rightarrow f(y) \\ &= f(x \rightarrow y) \\ &= f(x) \rightarrow f(y) \\ &= f^2(x) \rightarrow f(y) \\ &= f(f(x) \rightarrow y). \end{aligned}$$

So $x \rightarrow f(y) = f(x) \rightarrow y$. Therefore f is an identity map.

Let A_1 and A_2 be two BL-algebras. Then $A_1 \times A_2$ is also a BL-algebra with respect to the point-wise operations given by

$$(a, b) \odot (c, d) = (a \odot c, b \odot d),$$
$$(a, b) \rightarrow (c, d) = (a \rightarrow c, b \rightarrow d).$$

Theorem 3.24. Let A_1 and A_2 be two BL-algebras. Define a map $f: A_1 \times A_2 \rightarrow A_1 \times A_2$ by $f(x, y) = (x, 1)$ for all $(x, y) \in A_1 \times A_2$. Then f is a multiplier in $A_1 \times A_2$ with respect to point-wise operations.

Proof: Let $(a, b), (c, d) \in A_1 \times A_2$. Then we get

$$\begin{aligned} f((a, b) \rightarrow (c, d)) &= f(a \rightarrow c, b \rightarrow d) \\ &= (a \rightarrow c, 1) \\ &= (a \rightarrow c, b \rightarrow 1) \\ &= (a, b) \rightarrow (c, 1) \\ &= (a, b) \rightarrow f(c, d). \end{aligned}$$

Therefore f is multiplier in the direct product $A_1 \times A_2$.

Theorem 3.25. If BL-algebra $A \neq \{0\}$, then there is no nilpotent multiplier in A.

Proof: For every multiplier f, we have $f^n(x) \geq f^{n-1}(x) \geq \ldots \geq f(x) \geq x$, for all $x \in A$. Now if there is a natural number n such that $f^n = 0$, so $f^n(x) = 0$, for all $x \in A$. Thus $x = 0$, for all $x \in A$, which is a contradiction. Then there is no nilpotent multiplier in A.

Definition 3.26. A multiplier f in A is called idempotent, if $f^2(x) = f(x)$, for all $x \in A$.

Example 3.27. (a) Let A be a BL-algebra in Example 3.2(d) and

$$f(x) = \begin{cases} a, & if\ x = 0 \\ 1, & if\ x = a, b, 1 \end{cases} \quad (3.6)$$

Then f is a multiplier but is not idempotent. Because $f^2(0) \neq f(0)$.
(b) In Example 3.2(c), f is idempotent multiplier.
If f is an idempotent multiplier, then it can be easily observed that $f(x) \in F_f$, for all $x \in A$.

Theorem 3.28. Let f and g be two idempotent multipliers in A such that $f \odot g = g \odot f$. Then the following conditions are equivalent:
(i) $f = g$,
(ii) $f(A) = g(A)$,
(iii) $F_f(A) = F_g(A)$.

Proof: (1) ⇒ (2): It is obvious.
(2) ⇒(3): Assume that $f(A) = g(A)$. Let $x \in F_f(A)$. Then we get $x = f(x) \in f(A) = g(A)$. Hence $x = g(y)$, for some $y \in A$. Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Thus $x \in F_g(A)$. Therefore $F_f(A) \subseteq F_g(A)$. Similarly, we can obtain $F_g(A) \subseteq F_f(A)$. Therefore $F_f(A) = F_g(A)$.
(3) ⇒ (1): Assume that $F_f(A) = F_g(A)$. Let $x \in A$. Since $f(x) \in F_f(A) = F_g(A)$, we can obtain $g(f(x)) = f(x)$. Also we have $g(x) \in F_f(A) = F_g(A)$. Hence we get $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x).$$

Therefore f and g are equal in the sense of mappings.

Definition 3.29. An equivalence relation θ on A is called a weak congruence, if $(x, y) \in \theta$ implies that $(a \to x, a \to y) \in \theta$, for any $a \in A$.
Clearly every congruence on A is a weak congruence on A. In the following, we have an example of a weak congruence in terms of multipliers.

Theorem 3.30. Let f be a multiplier in A. Define a binary relation θ_f on A as follows:
$(x, y) \in \theta_f$ if and only if $f(x) = f(y)$ for all $x, y \in A$.
Then θ_f is a weak congruence on A.

Proof: Clearly θ_f is an equivalence relation on the BL-algebra A. Let $(x, y) \in \theta_f$. Then we get $f(x) = g(x)$. Now, for any $a \in A$, we have
$f(a \to x) = a \to f(x) = a \to f(y) = f(a \to y)$.
Hence $(a \to x, a \to y) \in \theta_f$. Therefore θ_f is a weak congruence on A.

Example 3.31. Let A be a BL-algebra and f be a multiplier in Example 3.2(d). We have $\theta_f = \{(0,0), (a, a), (b, b), (1,1), (b, 1), (1, b)\}$. Then θ_f is weak congruence and is not congruence.

Because θ_f is not a congruence relation, then the quotient cannot form a BL-algebra.

Theorem 3.32. Let f be an idempotent multiplier in A. Then we have the following:
(i) $f(x) = x$, for all $x \in f(A)$,
(ii) if$(x, y) \in \theta_f$ and $x, y \in f(A)$, then $x = y$.

Proof: (i) Let $x \in f(A)$. Then $x = f(a)$ for some $a \in A$. Now
$x = f(a) = f^2(a) = f(f(a)) = f(x)$.
(ii) Let $(x, y) \in \theta_f$ and $x, y \in f(A)$. Then by (i), $x = f(x) = f(y) = y$.

4. Simple Multipliers

Theorem 4.1. (i) The simple multiplier α_1 is an identity function on A,
(ii) if $p \le q$, then $\alpha_q \le \alpha_p$,
(iii) if $p \neq q$, then $\alpha_q \neq \alpha_p$,
(iv) $\alpha_p(x \sqcup p) = 1$, for all $p \in A$.

Proof: (i) For all $x \in A$, we have $\alpha_1(x) = 1 \to x = x$.
(ii) Let $p \le q$. So for all $x \in A$, we have $q \to x \le p \to x$, thus $\alpha_q(x) \le \alpha_p(x)$, for all $x \in A$, therefore $\alpha_q \le \alpha_p$.
(iii) Let $\alpha_q = \alpha_p$. So $\alpha_p(x) = \alpha_q(x)$, for all $x \in A$. Thus $p \to x = q \to x$, for all $x \in A$. Now, if $x := p$, then $p \to p = q \to p$, so $q \to p = 1$, hence $q \le p$. If $x := q$, then $p \to q = q \to q$, so $p \to q = 1$, thus $p \le q$. We get $p = q$, which is a contradiction. Therefore if $p \neq q$, then $\alpha_q \neq \alpha_p$.
(iv) For all $p \in A$, we have

$$\begin{aligned}\alpha_p(x \sqcup p) &= \alpha_p((p \to x) \to x)\\ &= p \to ((p \to x) \to x)\\ &= (p \to x) \to (p \to x) = 1.\end{aligned}$$

Put $S = \{\alpha_p : p \in A\}$, now we define:

$$(\alpha_p \wedge \alpha_q)(x) = \alpha_p(x) \wedge \alpha_q(x),$$
$$(\alpha_p \vee \alpha_q)(x) = \alpha_p(x) \vee \alpha_q(x),$$
$$(\alpha_p \odot \alpha_q)(x) = \alpha_p(x) \odot \alpha_q(x),$$
$$(\alpha_p \to \alpha_q)(x) = \alpha_p(x) \to \alpha_q(x).$$

Lemma 4.2. Let α_p and $\alpha_q \in S$.
(i) $\alpha_p \wedge \alpha_q \in S$,
(ii) if $p, q \in B(A)$, then $\alpha_p \vee \alpha_q \in S$.

Proof: (i)

$$\begin{aligned}(\alpha_p \wedge \alpha_q)(x) &= \alpha_p(x) \wedge \alpha_q(x)\\ &= (p \to x) \wedge (q \to x)\\ &= (p \vee q) \to x\\ &= \alpha_{(p \vee q)}(x),\end{aligned}$$

since $p \vee q \in A$, $\alpha_{(p \vee q)} \in S$, therefore $\alpha_p \wedge \alpha_q \in S$.
(ii)

$$(\alpha_p \vee \alpha_q)(x) = \alpha_p(x) \vee \alpha_q(x)$$

$$
\begin{aligned}
&= (p \to x) \vee (q \to x)\\
&= (p^* \vee x) \vee (q^* \vee x)\\
&= (p^* \vee q^*) \vee x\\
&= (p \wedge q)^* \vee x\\
&= (p \wedge q) \to x\\
&= \alpha_{(p\wedge q)}(x),
\end{aligned}
$$

since $p \wedge q \in A$, then $\alpha_{(p\wedge q)} \in S$, therefore $\alpha_p \vee \alpha_q \in S$.

Lemma 4.3. Let A be a Boolean algebra and α_p, $\alpha_q \in S$. Then:
(i) $\alpha_p \odot \alpha_q \in S$,
(ii) $\alpha_p \to \alpha_q \in S$.

Proof: (i)

$$
\begin{aligned}
(\alpha_p \odot \alpha_q)(x) &= \alpha_p(x) \odot \alpha_q(x)\\
&= (p \to x) \odot (q \to x)\\
&= (p \to x) \wedge (q \to x)\\
&= (p \vee q) \to x\\
&= \alpha_{(p\vee q)}(x),
\end{aligned}
$$

since $p \vee q \in A$, $\alpha_{(p\vee q)} \in S$, therefore $\alpha_p \odot \alpha_q \in S$.
(ii)

$$
\begin{aligned}
(\alpha_p \to \alpha_q)(x) &= \alpha_p(x) \to \alpha_q(x)\\
&= (p \to x) \to (q \to x)\\
&= (p^* \vee x) \to (q^* \vee x)\\
&= (p^* \to q^* \vee x) \wedge (x \to q^* \vee x)\\
&= (p \vee q^* \vee x) \wedge (x^* \vee q^* \vee x)\\
&= (p^* \wedge q)^* \vee x\\
&= (p^* \wedge q) \to x\\
&= \alpha_{(p^*\wedge q)}(x),
\end{aligned}
$$

since $p^* \wedge q \in A$, $\alpha_{(p^*\wedge q)} \in S$, therefore $\alpha_p \to \alpha_q \in S$.

Theorem 4.4. S is bounded $\wedge$-semi lattice with top element α_0 and bottom element α_1.

Proposition 4.5. If A is a BL-chain, then $S^* = \{\alpha_p : p \in B(A)\}$ is a distributive bounded lattice.

Proof: α_0, $\alpha_1 \in S^*$. By Theorem 2.2(8), we have: $x \wedge (\vee_{i\in I} y_i) = \vee_{i\in I} (x \wedge y_i)$.
Also, by Theorem 2.2(9), if A is BL-chain, then we have: $x \vee (\wedge_{i\in I} y_i) = \wedge_{i\in I} (x \vee y_i)$.

Theorem 4.6. If A is a G-algebra, then:
(i) $\alpha_p(x \to y) = \alpha_p(x) \to \alpha_p(y)$,
(ii) $\alpha_p(x \odot y) = \alpha_p(x) \odot \alpha_p(y)$,
(iii) $\alpha_p(x \wedge y) = \alpha_p(x) \wedge \alpha_p(y)$.

Proof:
(i)
$$
\begin{aligned}
\alpha_p(x \to y) &= p \to (x \to y)\\
&= (p \odot x) \to y\\
&= (p \wedge x) \to y\\
&= (p \odot (p \to x)) \to y\\
&= (p \to x) \to (p \to y)\\
&= \alpha_p(x) \to \alpha_p(y).
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\alpha_p(x \odot y) &= p \to (x \odot y)\\
&= p \to (x \wedge y)\\
&= (p \to x) \wedge (p \to y)\\
&= (p \to x) \odot (p \to y)\\
&= \alpha_p(x) \odot \alpha_p(y).
\end{aligned}
$$

(iii)
$$
\begin{aligned}
\alpha_p(x \wedge y) &= \alpha_p(x \odot y)\\
&= \alpha_p(x) \odot \alpha_p(y)\\
&= \alpha_p(x) \wedge \alpha_p(y).
\end{aligned}
$$

Theorem 4.7. For all $p \in B(A)$, we have the following:
(i) $\alpha_p(x \wedge y) = \alpha_p(x) \wedge \alpha_p(y)$,
(ii) $\alpha_p(x \vee y) = \alpha_p(x) \vee \alpha_p(y)$,
(iii) $\alpha_p(x \to y) = \alpha_p(x) \to \alpha_p(y)$,
(iv) $\alpha_p(x \odot y) = \alpha_p(x) \odot \alpha_p(y)$.

Proof:
(i)
$$
\begin{aligned}
\alpha_p(x \wedge y) &= p \to (x \wedge y)\\
&= (p \to x) \wedge (p \to y)\\
&= \alpha_p(x) \wedge \alpha_p(y).
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\alpha_p(x \vee y) &= p \to (x \vee y)\\
&= p^* \vee (x \vee y)\\
&= (p^* \vee p^*) \vee (x \vee y)\\
&= (p^* \vee x) \vee (p^* \vee y)\\
&= (p \to x) \vee (p \to y)\\
&= \alpha_p(x) \vee \alpha_p(y).
\end{aligned}
$$

(iii)
$$
\begin{aligned}
\alpha_p(x \to y) &= p \to (x \to y)\\
&= (p \to x) \to (p \to y)\\
&= \alpha_p(x) \to \alpha_p(y).
\end{aligned}
$$

(iv)
$$
\begin{aligned}
\alpha_p(x \odot y) &= p \to (x \odot y)\\
&= (p \to x) \odot (p \to y)\\
&= \alpha_p(x) \odot \alpha_p(y).
\end{aligned}
$$

Theorem 4.8. For all $p \in B(A)$, we have the following:
(i) $\alpha_p \vee \alpha_{p^*} = \alpha_0$,
(ii) $\alpha_p \wedge \alpha_{p^*} = \alpha_1$.

Proof:
(i)
$$
\begin{aligned}
(\alpha_p \vee \alpha_{p^*})(x) &= (p \to x) \vee (p^* \to x)\\
&= (p \wedge p^*) \to x\\
&= 0 \to x = \alpha_0(x).
\end{aligned}
$$

(ii)
$$
\begin{aligned}
(\alpha_p \wedge \alpha_{p^*})(x) &= (p \to x) \wedge (p^* \to x)\\
&= (p \vee p^*) \to x\\
&= 1 \to x = \alpha_1(x).
\end{aligned}
$$

Proposition 4.9. S^* is complement lattice.

Proof: For every $\alpha_p \in S^*$, α_{p^*} is complemented from α_p.

5. Multipliers in MV-center of BL-algebras

In this section we restrict multiplier f to $MV(A)$.

Theorem 5.1. Let f be a multiplier in $MV(A)$. Then:
(i) $f(x \oplus y) = x \oplus f(y)$,
(ii) if $x \leq y$, then $f(x) \leq f(y)$,
(iii) $f(x \odot y) \geq x \odot f(y)$,
for all $x, y \in MV(A)$.

Proof: (i) $f(x \oplus y) = f(x^* \to y) = x^* \to f(y) = x \oplus f(y)$.
(ii) If $x, y \in MV(A)$and $x \leq y$, then there exists $z \in MV(A)$ such that $z \oplus x = y$, so $f(y) = f(z \oplus x) = z \oplus f(x)$, thus $f(x) \leq f(y)$.
(iii) We have $x \odot y \leq z$ if and only if $y \leq x \to z$, so by (ii), $f(y) \leq f(x \to z) = x \to f(z)$, so $f(y) \leq x \to f(z)$, thus $x \odot f(y) \leq f(z)$. Now, put $z = x \odot y$, so $x \odot f(y) \leq f(x \odot y)$.

Proposition 5.2. Let f be a multiplier in MV(A). If $x \in F_1$ and x ≤ y, then $y \in F_1$.

Proof: Since $1 \in A$ and $f(1) = 1$, so $1 \in F_1$, then $F_1 \neq \emptyset$. Let $x \in F_1$. So $f(x) = 1$. Since $x \leq y$ so $f(x) \leq f(y)$, thus $f(y) = 1$, then $y \in F_1$.

Theorem 5.3. Let f be a multiplier in $MV(A)$ and f be a homomorphism of A. Then:
(i) F_1 is a filter of $MV(A)$,
(ii) $F_f \cap MV(A)$ is a filter of $MV(A)$.

Proof: (i)
(1) $f(1) = 1$, so $1 \in F_1$,
(2) by previous theorem, if $x \leq y$ and $x \in F_1$, then $y \in F_1$,
(3) let $x, y \in F_1$. So $f(x) = f(y) = 1$, thus $f(x \odot y) = f(x) \odot f(y) = 1 \odot 1 = 1$, therefore $x \odot y \in F_1$. Thus F_1 is a filter of $MV(A)$.
(ii) (1) $f(1) = 1$, so $1 \in F_f \cap MV(A)$,
(2) let $x \in F_f \cap MV(A)$ and $x \leq y$. Then there exists $z \in MV(A)$ such that $z \oplus x = y$, so $f(y) = f(z \oplus x) = z \oplus f(x) = z \oplus x = y$, then $f(y) = y$, therefore $y \in F_f \cap MV(A)$,
(3) let $x, y \in F_f \cap MV(A)$. Then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, thus $x \odot y \in F_f \cap MV$(A).

Theorem 5.4. If $f: A \to A$ is a multiplier in $MV(A)$, then f is a closure operator on $MV(A)$ if $f(f(x)) \leq f(x)$, for all $x \in MV(A)$.

Remark: In BL-algebra, we have
$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
and in MV-algebra, we have
$x \vee y = ((x \to y) \to y) = ((y \to x) \to x)$.

Theorem 5.5. Let f be a multiplier in $MV(A)$. For all $x, y \in MV(A)$ such that $x \in F_f$, then $x \vee y \in F_f$.

Proof:
$$\begin{aligned} f(x \vee y) &= f((y \to x) \to x) \\ &= (y \to x) \to f(x) \\ &= (y \to x) \to x \\ &= x \vee y, \end{aligned}$$
so $x \vee y \in F_f$.

Lemma 5.6. Let f be a multiplier in A. If $x, y \in B(A)$, then:
(i) $f(x \vee y) = x \vee f(y)$,
(ii) $f(x \wedge y) \geq x \wedge f(y)$.

Proof: (i) In $B(A)$, we have $x \oplus y = x \vee y$, so
$f(x \vee y) = f(x \oplus y) = x \oplus f(y) = x \vee f(y)$.
(ii) In $B(A)$, we have $x \odot y = x \wedge y$, so $f(x \wedge y) = f(x \odot y) \geq x \odot f(y) = x \wedge f(y)$.
If A is a BL-algebra and f_1, f_2 are two multipliers in A, we define:
$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$
$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$
$$(f_1 \to f_2)(x) = f_1(x) \to f_2(x),$$
$$(f_1 \odot f_2)(x) = f_1(x) \odot f_2(x).$$

Theorem 5.7. If f_1, $f_2 \in M(A)$, then:
(i) $f_1 \wedge f_2 \in M(A)$,
(ii) $f_1 \to f_2 \in M(A)$.

Proof: (i)
$$\begin{aligned} (f_1 \wedge f_2)(x \to y) &= f_1(x \to y) \wedge f_2(x \to y) \\ &= (x \to f_1(y)) \wedge (x \to f_2(y)) \\ &= x \to (f_1(y) \wedge f_2(y)) \\ &= x \to (f_1 \wedge f_2)(y). \end{aligned}$$
(ii)
$$\begin{aligned} (f_1 \to f_2)(x \to y) &= (f_1(x \to y)) \to (f_2(x \to y)) \\ &= (x \to f_1(y)) \to (x \to f_2(y)) \\ &= x \to (f_1(y) \to f_2(y)) \\ &= x \to (f_1 \to f_2)(y). \end{aligned}$$

Theorem 5.8. If A is a MV-algebra and for all $x \in A$, $x \odot x = x$ and $f_1, f_2 \in M(A)$, then $f_1 \oplus f_2 \in M(B(A))$.

Theorem 5.9. If A is a G-algebra and $f_1, f_2 \in M(A)$, then $f_1 \odot f_2 \in M(A)$.

Proof:
$$\begin{aligned} (f_1 \odot f_2)(x \to y) &= (f_1(x \to y)) \odot (f_2(x \to y)) \\ &= (x \to f_1(y)) \odot (x \to f_2(y)) \end{aligned}$$

$$= (x \to f_1(y)) \wedge (x \to f_2(y))$$
$$= x \to (f_1(y) \wedge f_2(y))$$
$$= x \to (f_1 \odot f_2)(y).$$

Theorem 5.10. Let $f_1, f_2 \in M(B(A))$. Then $f_1 \odot f_2$, $f_1 \to f_2$, $f_1 \vee f_2$, $f_1 \wedge f_2 \in M(B(A))$.

Theorem 5.11. Let A be a BL-algebra. Then $M(A)$ is a meet lattice with top element $f(x) = 1$.

Theorem 5.12. If A is a Boolean algebra and $M(A)$ has a bottom element, then $M(A)$ is a BL-algebra.

Proof: We prove the adjointness property:
let $f \odot g \leq h$. We have for all $x \in A$, $(f \odot g)(x) \leq h(x)$, so $f(x) \odot g(x) \leq h(x)$, then $f(x) \leq g(x) \to h(x)$ for all $x \in A$, therefore $f \leq g \to h$. The converse is similarly.

6. Conclusion and future research

BL-algebras are the algebraic structures for H'ajek basic logic (BL, for short), arising from the continuous triangular norms (t-norm), familiar in the frameworks of fuzzy set theory. The concept of multiplier for a commutative semigroup, Implicative algebra, distributive lattice and BE-algebra are introduced.

In this paper, we introduced the concept of the multiplier in BL-algebra, MV-center of BL-algebra and studied some properties. Then we studied relationships between multipliers and some special mappings, likeness closure operators, homomorphisms and $\odot$-derivations in BL-algebras. One of the interesting results is "If A be a Boolean algebra and $M(A)$ has a bottom element, then $M(A)$ is a BL-algebra."

Some important issues for future work include:

(i) Developing the properties of the multiplier in BL-algebra,

(ii) finding useful results on other algebraic structures,

(iii) constructing the related logical properties of such structures.

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