
Fractional differential transform method for solving a class of weakly singular Volterra integral equations

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Abstract

A method for solving a class of weakly singular Volterra integral equations is given by using the fractional differential transform method. The approximate solution of these equations is calculated in the form of a finite series with easily computable terms. While in some examples this series solution increased up to the exact closed solution, in some other examples, we can see the accuracy and the reliability of the fractional differential transform method.

Keywords: Weakly singular Volterra integral equations; fractional differential transform

1. Introduction

Many physical and engineering problems lead to analysis of the nonlinear Weakly Singular Volterra Integral Equation (WSVIE) of the second kind. Such equations arise from many applications (Capobianco et al., 2009) such as reaction-diffusion problems in small cells (Dixon, 1987) or the semi-discretization in space of Volterra-Fredholm integral equations with weakly singular kernel and of partial Abel integral or integro-differential equations. The last kind of equations occurs as mathematical model in linear quasistatic viscoelasticity problems (Shaw et al., 1997; Shaw et al., 2001). In many of the cited examples, the spatial semi-discretization leads to Volterra Integral Equations (VIEs) with linear convolution kernel (Shaw et al., 1997; Shaw et al., 2001; Cuesta et al., 2006). Volterra-Fredholm integral equations with singular kernels occur, for example, in the modeling of the coding mechanism in the transmission of nervous signals among neurons (Giraudet al., 2002). Several numerical methods have been proposed for these equations (Brunner, 1983; Brunner, 1985; Brunner, 1986; Brunner, 2004; Capobianco et al., 2004; Capobianco et al., 2006; Capobianco et al., 2008; Diogo et al., 1994; Tang, 1992; Tang, 1993).

On the other hand, in recent years the Fractional Differential Transform Method (FDTM) has been developed for solving the differential and integral equations. For example in (Arikogl, 2007), FDTM

is used for fractional differential equations and in (Arikoglu, 2009) it is used for fractional integro-differential equations. In (Momani, et al., 2008) this method is applied to nonlinear fractional partial differential equations.

The main goal of the presented paper is applying FDTM for solving WSVIE of the form

$$y(x) = h(x) + \int_0^x (x-t)^{-q} K(x,t, y(t)) dt, x \in I \quad (1)$$

where $0 < q < 1$, $I = [0, T]$ and the given functions h and K are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $y \in C(I)$ (Atkinson, 1974; Cahlon, 1981)

The numerical treatment of (1) is not simple because, as it is well-known, the solutions of WSVIEs usually have a weak singularity at $x = 0$, even when the inhomogeneous term $h(x)$ is regular.

In this paper, we apply FDTM to solve the linear and nonlinear Volterra integral equations with separable kernels, i.e.

$$K(x,t, y(t)) = \sum_{j=0}^m f_j(x) g_j(t, y(t)).$$

In this case, the nonlinear WSVIE of the second kind can be written in the following general form:

$$y(x) = h(x) + \sum_{j=0}^m f_j(x) \int_0^x (x-t)^{-q} g_j(t, y(t)) dt. \quad (2)$$

Some necessary preliminaries and definitions are given in section 2, and is followed by a theorem which gives the main result of this paper. Then a description of the method and some numerical

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examples, which show the accuracy of our method, are given in section 3 and 4, respectively. We conclude our discussion in section 5.

2. Preliminaries

An analytical and continuous function can be presented in terms of an infinite power series expansion (Momani et al., 2006):

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^{k/\alpha}, \quad (3)$$

where α is the order of fraction and $F(k)$ is the fractional differential transform of $f(x)$. Since the initial conditions of fractional differential equations are implemented to the integer order derivatives, the transformation of the initial conditions can be written as follows.

$$F(k) = \begin{cases} \frac{1}{(\frac{k}{\alpha})!} \left[\frac{d^{\frac{k}{\alpha}} f(x)}{dx^{\frac{k}{\alpha}}} \right]_{x=x_0} & \text{if } \frac{k}{\alpha} \in \mathbb{Z}^+, \\ 0 & \text{if } \frac{k}{\alpha} \notin \mathbb{Z}^+, \end{cases} \quad (4)$$

for $k = 0, 1, 2, \dots, (n\alpha - 1)$ where n is the order of the considered fractional differential equation. In real applications, the function $f(x)$ is expressed by a finite series and equation (3) can be written as

$$f(x) = \sum_{k=0}^N F(k)(x - x_0)^{k/\alpha}, \quad (5)$$

where N is decided by the convergence of natural frequency.

In the following theorem, we summarize fundamental properties of the fractional differential transform (Arikoglu et al., 2007).

Theorem 1. If $F(k)$, $G(k)$ and $H(k)$ are the fractional differential transforms of the functions $f(x)$, $g(x)$ and $h(x)$, respectively, then:

- (a) If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$
- (b) If $f(x) = ag(x)$, then $F(k) = aG(k)$.
- (c) If $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^k G(l)H(k-l)$.
- (d) If $f(x) = (x - x_0)^r$, then $F(k) = \delta(k - r\alpha)$

where $\delta(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$ and $r\alpha \in \mathbb{Z}^+$.

(e) If $f(x) = g_1(x)g_2(x)\dots g_{m-1}(x)g_m(x)$, then

$$F(k) = \sum_{k_{m-1}=0}^k \sum_{k_{m-2}=0}^{k_{m-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2 - k_1)\dots G_{m-1}(k_{m-1} - k_{m-2})G_m(k - k_{m-1}).$$

A special function, which is connected to the Gamma function in a direct way, is given by the Beta function and defined as follows.

Definition 1. The Beta function $B(a, b)$ in two variables $a, b \in \mathbb{C}$ is defined by

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Now, we are ready to state the main theorem of this paper on which a weakly singular integral can be stated as a series of fractional differential transform. Later, we will use the result of this theorem to solve WSVIEs.

Theorem 2. Suppose that $G(k)$ and $W(k)$ are the fractional differential transforms of the functions $g(t, y(t))$ and $w(x)$ such that

$$w(x) = \int_0^x (x-t)^{p-1} g(t, y(t)) dt, \quad 0 < p < 1. \quad (6)$$

Then, by choosing a suitable $\alpha \in \mathbb{Z}^+$ such that $\alpha p \in \mathbb{Z}^+$, we have

$$W(k) = \sum_{l=0}^k B\left(\frac{k-l}{\alpha} + 1, p\right) \delta(l - \alpha p) G(k-l), \quad (7)$$

where $B(\dots)$ is the Beta function.

Proof: By replacing $g(t, y(t)) = \sum_{k=0}^{\infty} G(k)t^{k/\alpha}$ in (6), one obtains

$$\begin{aligned} w(x) &= \int_0^x (x-t)^{p-1} g(t, y(t)) dt \\ &= \int_0^x (x-t)^{p-1} \sum_{k=0}^{\infty} G(k)t^{k/\alpha} dt \\ &= \sum_{k=0}^{\infty} G(k) \int_0^x (x-t)^{p-1} t^{k/\alpha} dt. \end{aligned}$$

Then, by changing of variables twice, we have

$$\int_0^x (x-t)^{p-1} t^{k/\alpha} dt = B\left(\frac{k}{\alpha} + 1, p\right) x^{p+k/\alpha}.$$

Now, by using Theorem 1, we obtain

$$\begin{aligned} w(x) &= x^p \sum_{k=0}^{\infty} B\left(\frac{k}{\alpha} + 1, p\right) G(k)x^{k/\alpha} \\ &= \sum_{l=0}^{\infty} \delta(l - \alpha p) x^{l/\alpha} \sum_{k=0}^{\infty} B\left(\frac{k}{\alpha} + 1, p\right) G(k)x^{k/\alpha} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^k B\left(\frac{k-l}{\alpha} + 1, p\right) \delta(l - \alpha p) G(k-l) x^{k/\alpha}.$$

Therefore,

$$W(k) = \sum_{l=0}^k B\left(\frac{k-l}{\alpha} + 1, p\right) \delta(l - \alpha p) G(k-l).$$

3. Description of the Method

Consider the Weakly Singular Volterra Integral Equation

$$y(x) = h(x) + \sum_{j=0}^m f_j(x) \int_0^x (x-t)^{-q} g_j(t, y(t)) dt, \quad 0 < q < 1. \quad (8)$$

Let us define $w_j(x) = \int_0^x (x-t)^{p-1} g_j(t, y(t)) dt$ where $p = 1 - q$. Suppose that $Y(k)$, $H(k)$, $G_j(k)$, $F_j(k)$ and $W_j(k)$, $j = 0, 1, 2, \dots, m$, are the fractional differential transforms of the functions $y(x)$, $h(x)$, $g_j(x, y(x))$, $f_j(x)$ and $w_j(x)$, respectively. Then, by choosing a suitable $\alpha \in \mathbb{Z}^+$ such that $\alpha p \in \mathbb{Z}^+$ and using Theorems 1 and 2, we obtain

$$Y(k) = H(k) + \sum_{j=0}^m \sum_{r=0}^k F_j(k-r) W_j(r) \quad (9)$$

$$= H(k) + \sum_{j=0}^m \sum_{r=0}^k \sum_{l=0}^r B\left(\frac{r-l}{\alpha} + 1, p\right) \delta(l - \alpha p) F_j(k-r) G_j(r-l),$$

with

$$Y(k) = \begin{cases} \frac{1}{\left(\frac{k}{\alpha}\right)!} \left[\frac{d^{\frac{k}{\alpha}} y(x)}{dx^{\frac{k}{\alpha}}} \right]_{x=x_0} & \text{if } \frac{k}{\alpha} \in \mathbb{Z}^+, \\ 0 & \text{if } \frac{k}{\alpha} \notin \mathbb{Z}^+, \end{cases} \quad (10)$$

for $k = 0, 1, 2, \dots, (n\alpha - 1)$, where α is the order of the considered fractional differential equation.

Therefore, the solution of WSVIE (8) is expressed by

$$y(x) = \sum_{k=0}^N Y(k) x^{k/\alpha}, \quad (11)$$

where N is decided by the convergence of natural frequency.

4. Applications and Results

In this section, we give some examples to clarify the advantages and the accuracy of FDTM for solving WSVIEs. In these examples, we first obtain

a recurrence relation and then solve it by programming in MATLAB environment.

Example 1. Consider the following linear Volterra integral equation with algebraic singularity (Hu, 1997)

$$y(x) = \frac{1}{2} \pi x + \sqrt{x} - \int_0^x (x-t)^{-1/2} y(t) dt, \quad (12)$$

with the exact solution $y(x) = \sqrt{x}$.

By choosing $\alpha = 2$ and applying fractional differential transform on the integral equation (12) and using Theorems 1 and 2, we obtain the following recurrence relation

$$Y(k) = \frac{\pi}{2} \delta(k-2) + \delta(k-1) - \sum_{l=0}^k B\left(\frac{1}{2}, \frac{k-l}{2} + 1\right) \delta(l-1) Y(k-l), \quad k \geq 1 \quad (13)$$

with $Y(0) = 0$. By solving (13) for $k \geq 1$, we obtain $Y(1) = 1$ and $Y(k) = 0$ for all $k \geq 2$. Therefore, using (9), the solution of the integral equation (12) is given by $y(x) = \sqrt{x}$ which is indeed the exact solution.

Example 2. Now consider the following linear Volterra integral equations of the second kind with weakly singular kernels (Chena et al., 2009)

$$y(x) = b(x) - \int_0^x (x-t)^{-q} y(t) dt, \quad 0 \leq t \leq T, \quad (14)$$

with

$$b(x) = x^{n+\beta} + x^{n+1+\beta-q} B(n+1+\beta, 1-q),$$

where $0 \leq \beta \leq 1$, $B(\cdot, \cdot)$ is the Beta function. This problem has the unique solution $y(x) = x^{n+\beta}$.

Using Theorems 1 and 2, equation (14) can be transformed as follows:

$$Y(k) = \delta(k - \alpha(n+\beta)) + B(n+1+\beta, 1-q) \delta(k - \alpha(n+1+\beta-q)) - \sum_{l=0}^k B\left(\frac{k-l}{\alpha} + 1, 1-q\right) \delta(l - \alpha(1-q)) Y(k-l), \quad k \geq 1. \quad (15)$$

Taking $n = 3$, $q = 1/4$, $\beta = 3/4$ and $\alpha = 4$ and using (15), one obtains

$$Y(k) = \delta(k-15) + B(19/4, 3/4) \delta(k-18) - \sum_{l=0}^k B\left(\frac{k-l}{4} + 1, \frac{3}{4}\right) \delta(l-3) Y(k-l), \quad k \geq 3, \quad (16)$$

with $Y(0) = Y(1) = Y(2) = 0$. The recurrence relation (16) implies that the solution of the integral equation

$$y(x) = x^{15/4} + x^{9/2} B(19/4, 3/4) - \int_0^x (x-t)^{-1/4} y(t) dt,$$

is $y(x) = x^{15/4}$ which is again the exact solution.

Table 1. Absolute Error for Example 3

x	Abs.Err. ($N = 30$)	Abs.Err. ($N = 40$)	Abs.Err. ($N = 50$)
0.1	1.11022302e-	1.11022302e-	1.11022302e-
0.2	5.55111512e-	5.55111512e-	5.55111512e-
0.3	6.17284001e-	2.22044604e-	2.22044604e-
0.4	5.19645437e-	5.55111512e-	5.55111512e-
0.5	1.60972291e-	8.32667268e-	5.55111512e-
0.6	2.65634347e-	3.00870439e-	5.55111512e-
0.7	2.83893985e-	7.07156555e-	7.21644966e-
0.8	2.20796488e-	1.07536202e-	0
0.9	1.34721439e-	1.18434539e-	4.82947015e-
1.0	6.78849085e-	1.01214181e-	4.38538094e-

Example 3. Consider the following linear Volterra integral equations of the second kind with weakly singular kernels (Baratella et al., 2004)

$$y(x) = 1 - \int_0^x (x-t)^{-1/2} y(t) dt, \quad 0 \leq t \leq 1, \quad (17)$$

with the exact solution $y(x) = \exp(\pi x) \operatorname{erfc}(\sqrt{\pi x})$.

By choosing $\alpha = 2$ and applying the fractional differential transform method in the integral equation (20) and using Theorems 1 and 2, we obtain the following recurrence relation

$$Y(k) = \delta(k) - \sum_{l=0}^k B\left(\frac{1}{2}, \frac{k-l}{2} + 1\right) \delta(l-1) Y(k-l), \quad k \geq 1, \quad (18)$$

with $Y(0) = 1$. The absolute errors at some points for the cases $N = 30$, $N = 40$ and $N = 50$ are shown in Table 1, which shows a good accuracy of the method.

Example 4. We finally consider the following weakly singular nonlinear Volterra integral equation (Galperin et al., 2002)

$$y(x) = \sqrt{x} - \frac{1}{3} x^{3/2} - \frac{1}{4} \int_0^x (x-t)^{-1/2} y^2(t) dt, \quad (19)$$

with the exact solution $y(x) = \sqrt{x}$.

By choosing $\alpha = 2$ and applying the fractional differential transform method in the integral equation (19) and using Theorems 1 and 2, we obtain the following recurrence relation

$$Y(k) = \delta(k-1) - \frac{1}{3} \delta(k-3) + \frac{1}{4} \sum_{l=0}^k \sum_{m=0}^{k-l} B\left(\frac{1}{2}, \frac{k-l}{2} + 1\right) \delta(l-1) Y(m) Y(k-m-1), \quad k \geq 1. \quad (20)$$

With $Y(0) = 0$. Therefore, $Y(1) = 1$ and $Y(k) = 0$ for each $k \geq 2$ and, using (9), the solution of the integral equation (19) is $y(x) = \sqrt{x}$ with is the exact solution.

5. Conclusions

We successfully applied the fractional differential transform method to find the series solutions of weakly singular Volterra integral equations. The presented method reduces the computational difficulties exist in the other traditional methods and all the calculations can be done by simple manipulations. Several examples of WSVIEs were tested by applying FDTM and the results have revealed remarkable performances.

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