IJST (2014) 38A1: 61-68
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

Some inclusion results for lacunary statistical convergence in locally solid Riesz spaces

B. Hazarika[1], S. A. Mohiuddine[2]* and M. Mursaleen[3]

[1]*Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh 791112, Arunachal Pradesh, India*
[2]*Department of Mathematics, Faculty of Science, King Abdulaziz University,*
P.O. Box 80203, Jeddah 21589, Saudi Arabia
[3]*Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
E-mail: mohiuddine@gmail.com

Abstract

Recently, Mohiuddine and Alghamdi introduced the notion of lacunary statistical convergence in a locally solid Riesz space and established some results related to this concept. In this paper, some inclusion relations between the sets of statistically convergent and lacunary statistically convergent sequences are established and extensions of a decomposition theorem, a Tauberian theorem to the locally solid Riesz space setting are proved. Further, we introduce the concepts of θ-summable and statistically lacunary convergence in locally solid Riesz space and establish a relationship between them.

***Keywords**:* Lacunary density; lacunary sequence; statistical convergence; statistically Cauchy; locally solid Riesz space

1. Introduction

Actually the idea of statistical convergence was previously given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 (Zygmund, 1979). The concept was formally introduced by Fast (1951) and Steinhaus (1951) and later on reintroduced by Schoenberg (1959), and also independently by Buck (1953). Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later it was further investigated from various points of view, see (Belen and Mohiuddine, 2013; Çakalli and Khan, 2011; Çolak and Bektas, 2011; Fridy, 1985; Mohiuddine et al., 2013; Mursaleen and Mohiuddine, 2009; Mursaleen and Mohiuddine, 2010; Mursaleen and Mohiuddine, 2012; Šalát, 1980; Prullage, 1967). This notion has also been defined and studied in different setups, for example, in topological groups (Çakalli, 1996; Çakalli, 2009), topological spaces (Di Maio and Kočinac, 2008), function spaces (Caserta and Kočinac, 2012; Caserta et al., 2011), locally convex spaces (Maddox, 1988), intuitionistic fuzzy normed space (Mohiuddine and Lohani, 2009). Fridy and Orhan, (1993) introduced the concept of lacunary statistical convergence. In (Mursaleen and Mohiuddine, 2009), Mursaleen and Mohiuddine introduced the concept of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. For details related to these concepts, we refer to (Çakalli, 1995; Çakan et al., 2010; Gürdal and Acik, 2008; Hazarika and Mohiuddine, 2013; Li, 2000; Mohiuddine and Aiyub, 2012; Mohiuddine et al., 2012; Mohiuddine et al., 2010; Mohiuddine et al., 2012; Mohiuddine and Alotaibi, 2011; Mursaleen et al., 2010; Mursaleen and Edely, 2003; Mursaleen et al., 2010; Savas and Mohiuddine, 2012).

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by Riesz (1928). Riesz spaces have many applications in measure theory, operator theory and optimization. They also have some applications in economics (Aliprantis and Burkinshaw, 2003), and we refer to (Albayrak and Pehlivan, 2012; Kantorovich, 1937; Luxemburg and Zaanen, 1971; Mohiuddine et al. 2012; Zannen, 1997) for more details.

2. Background, notations and preliminaries

In this section, we recall some of the basic concepts related to the notions of statistical convergence and lacunary sequence.

Let $E \subseteq \mathbb{N}$. Then the natural density of E is denoted by $\delta(E)$ and is defined by

*Corresponding author
Received: 2 March 2013 / Accepted: 5 October 2013

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \in E : k \leq n\}| \text{ exists}$$

where the vertical bar denotes the cardinality of the respective set.

Definition 2.1. (Maio and Kočinac, 2008). A sequence $x = (x_k)$ in a topological space X is said to be *statistically convergent* to ℓ if for every neighborhood V of ℓ

$$\delta(\{k \in \mathbb{N} : x_k \notin V\}) = 0.$$

In this case, we write $S - \lim x = \ell$.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by q_r (see Freedman et al., 1978).

Definition 2.2. Let θ be a lacunary sequence and $I_r = \{k : k_{r-1} < k \leq k_r\}$. Let $K \subset \mathbb{N}$. The number $\delta_\theta(K)$ is called the *lacunary density* or *θ-density* of K if

$$\delta_\theta(K) = \lim_r \frac{1}{h_r} |\{i \in I_r : i \in K\}| \text{ exists}.$$

The generalized lacunary mean is defined by

$$t_r(x) = \frac{1}{h_r} \sum_{k \in I_r} x_k.$$

Definition 2.3. A sequence $x = (x_k)$ is said to be *θ-summable* to number ℓ if $t_r(x) \to \ell$ as $r \to \infty$. In this case we write ℓ is the θ-limit of x. If $\theta = (2^r)$, then θ-summable reduces to C_1-summable (see Freedman et al., 1978).

We now give the definition of lacunary statistical convergence in topological spaces.

Definition 2.4. Let θ be a lacunary sequence. A sequence $x = (x_k)$ in a topological space X is said to be *lacunary statistical convergent* or *S_θ-convergent* to ℓ provided that for each neighborhood V of zero, the set

$$K(V) = \{k \in \mathbb{N} : x_k - \ell \notin V\}$$

has θ-density zero. In this case we write S_θ-$\lim x = \ell$ or $(x_k) \overset{S_\theta}{\to} \ell$.

Let X be a real vector space and $\leq$ be a partial order on this space. Then X is said to be an *ordered vector space* if it satisfies the following properties:

(i) if $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$.

(ii) if $x, y \in X$ and $y \leq x$, then $ay \leq ax$ for each $a \geq 0$.

If, in addition, X is a lattice with respect to the partial ordered, then X is said to be a *Riesz space* (or a *vector lattice*)(Zannen, 1997).

For an element x of a Riesz space X, the *positive part* of x is defined by $x^+ = x \vee \overline{0} = \sup\{x, \overline{0}\}$, the *negative part* of x by $x^- = -x \vee \overline{0}$ and the *absolute value* of x by $|x| = x \vee (-x)$, where $\overline{0}$ is the zero element of X.

A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A *topological vector space* (X, τ) is a vector space X which has a topology (linear) τ, such that the algebraic operations of addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f : X \times X \to X$ defined by $f(x, y) = x + y$ is continuous on $X \times X$, and continuity of scalar multiplication means that the function $f : \mathbb{R} \times X \to X$ defined by $f(a, x) = ax$ is continuous on $\mathbb{R} \times X$.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\overline{\theta}$ satisfying the following properties:

(1) Each $Y \in N$ is a *balanced set*, that is, $ax \in Y$ holds for all $x \in Y$ and for every $a \in \mathbb{R}$ with $|a| \leq 1$.

(2) Each $Y \in N$ is an *absorbing set*, that is, for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.

(3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology τ on a Riesz space X is said to be *locally solid* (Roberts, 1952) if τ has a base at zero consisting of solid sets. A *locally solid Riesz space* (X, τ) is a Riesz space equipped with a locally solid topology τ.

Recall that a first countable space is a topological space satisfying the "first axiom of countability". Specifically, a space X is said to be first countable if each point has a countable neighborhood basis(local base). That is, for each point x in X there exists a sequence $V_1, V_2, \cdots$ of open neighborhoods of x such that for any open neighborhood V of x there exists an integer j with V_j contained in V.

The purpose of this article is to give certain characterizations of lacunary statistically convergent sequences in locally solid Riesz spaces and to obtain extensions of a decomposition theorem, a Tauberian theorem and some inclusion results related to the notions statistical convergence and lacunary statistical convergence in locally solid Riesz spaces.

Throughout the article, the symbol N_{sol} will denote any base at zero consisting of solid sets and satisfying the conditions (1), (2) and (3) in a locally solid topology.

3. Lacunary statistical topological convergence in locally solid Riesz spaces

Throughout the article X will denote the Hausdorff locally solid Riesz space, which satisfies the first axiom of countability.

Recently, Albayrak and Pehlivan (2012) introduced the notion of statistical convergence in locally solid Riesz spaces as follows.

Definition 3.1. (Albayrak and Pehlivan, 2012). Let (X,τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $S(\tau)$- *convergent* to an element x_0 of X if for each τ-neighborhood V of zero,

$$\delta(\{k \in \mathbb{N}: x_k - x_0 \notin V\}) = 0$$

i.e.,

$$\lim_m \frac{1}{m} |\{k \leq m: x_k - x_0 \notin V\}| = 0.$$

In this case, we write $S(\tau) - \lim_{k\to\infty} x_k = x_0 \; or \; (x_k) \overset{S(\tau)}{\to} x_0$.

Recently, Mohiuddine and Alghamdi (2012) introduced the notion of lacunary statistical convergence in locally solid Riesz spaces as follows.

Definition 3.2. (Mohiuddine and Alghamdi, 2012). Let (X,τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $S_\theta(\tau)$-*convergent* to an element x_0 of X if for each τ-neighborhood V of zero,

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V\}) = 0$$

i.e.,

$$\lim_r \frac{1}{h_r} |\{k \in I_r: x_k - x_0 \notin V\}| = 0.$$

In this case, we write $S_\theta(\tau) - \lim_{k\to\infty} x_k = x_0 \; or \; (x_k) \overset{S_\theta(\tau)}{\to} x_0$.

Example 3.1. Let us consider the locally solid Riesz space $(\mathbb{R}^2, ||.||)$ with the Euclidean norm $||.||$ and coordinate-wise ordering. In this space, let us define a sequence (x_k) by

$$x_k = \begin{cases} (1+\frac{1}{k+1}, 2+\frac{5}{k+1}), & if \; k \neq n^2; \\ (4,4), & if \; k = n^2, \end{cases}$$

for each $n \in \mathbb{N}$. Let $\theta = (2^r - 1)$. The family N_{sol} of all U_ε defined by

$$U_\varepsilon = \{x \in \mathbb{R}^2: ||x|| < \varepsilon\},$$

where $0 < \varepsilon \in \mathbb{R}$ constitutes a base at zero $(\overline{\theta} = (0,0))$. For $x_0 = (1,2)$, we have

$$x_k - x_0 = \begin{cases} (\frac{1}{k+1}, \frac{5}{k+1}), & if \;\; k \neq n^2; \\ (3,2), & if \;\; k = n^2. \end{cases}$$

For each τ-neighborhood V of zero, there exists some $U_\varepsilon \in N_{sol}$, $\varepsilon > 0$ such that $U_\varepsilon \subseteq V$ and $\{k \in \mathbb{N}: x_k - x_0 \notin U_\varepsilon\} = K \cup \{1,4,9,16,\cdots,n^2,\cdots\}$, where K is a finite set. Then, we have

$$\begin{gathered} \delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V\}) \\ \leq \delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin U_\varepsilon\}) \\ = \delta_\theta(K) + \delta_\theta(\{1,4,9,16,\cdots,n^2,\cdots\}) = 0. \end{gathered}$$

Hence $S_\theta(\tau) - \lim_k x_k = (1,2)$.

Definition 3.3. (Mohiuddine and Alghamdi, 2012). Let (X,τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $S_\theta(\tau)$-*bounded* in X if for each τ-neighborhood V of zero, there is some $a > 0$,

$$\delta_\theta(\{k \in \mathbb{N}: a x_k \notin V\}) = 0.$$

Definition 3.4. (Mohiuddine and Alghamdi, 2012). Let (X,τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be $S_\theta(\tau)$-*Cauchy* in X if for each τ-neighborhood V of zero there is an integer $n \in \mathbb{N}$,

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_n \notin V\}) = 0.$$

Theorem 3.1. Let (X,τ) be a locally solid Riesz space. A sequence (x_k) is $S_\theta(\tau)$-convergent to x_0 in X if and only if for each τ-neighborhood V of zero there exists a subsequence $(x_{k'(m)})$ of (x_k) such that $\lim_{r\to\infty} x_{k'(r)} = x_0$ and

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_{k'(r)} \notin V\}) = 0.$$

Proof: Let $x = (x_k)$ be a sequence in X such that $S_\theta(\tau) - \lim_{k\to\infty} x_k = x_0$. Let V be an arbitrary τ-neighborhood of zero. Let $\{V_n\}$ be a sequence of nested base of τ-neighborhood of zero. We write

$$E^{(i)} = \{k \in \mathbb{N}: x_k - x_0 \notin V_i\},$$

for any positive integer i. Then for each i, we have $E^{(i+1)} \subset E^{(i)}$ and $\lim_r \frac{|E^{(i)} \cap I_r|}{h_r} = 1$. Choose $n(1)$ such that $r > n(1)$, then $|E^{(1)} \cap I_r| > 0$ i.e., $E^{(1)} \cap I_r \neq \phi$. Then for each positive integer r such that $n(1) \leq r < n(2)$, choose $k'(r) \in I_r$ such that $k'(r) \in E^{(i)} \cap I_r$, i.e. $x_{k'(r)} - x_0 \in V_1$. In general, choose $n(p+1) > n(p)$ such that $r > n(p+1)$, then $E^{(p+1)} \cap I_r \neq \phi$. Then for all r satisfying $n(p) \leq r < n(p+1)$, choose $k'(r) \in E^{(p)} \cap I_r$, i.e. $x_{k'(r)} - x_0 \in V_p$. Hence it follows that $\lim_r x_{k'(r)} = x_0$.

Since V is an arbitrary τ-neighborhood of zero, there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose $W \in N_{sol}$ such that $W + W \subseteq V$. Now we have

$$\{k \in \mathbb{N}: x_k - x_{k'(r)} \notin V\} \subseteq \{k \in \mathbb{N}: x_k - x_0 \notin W\} \cup \{r \in \mathbb{N}: x_{k'(r)} - x_0 \notin W\}.$$

Since $S_\theta(\tau) - \lim_{k\to\infty} x_k = x_0$ and $\lim_{r\to\infty} x_{k'(r)} = x_0$ implies that

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_{k'(r)} \notin V\}) = 0.$$

Next suppose for an arbitrary τ-neighborhood V of zero there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $\lim_{r\to\infty} x_{k'(r)} = x_0$ and

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_{k'(r)} \notin V\}) = 0.$$

Since V is any τ-neighborhood of zero, we choose $W \in N_{sol}$ such that $W + W \subseteq V$. Then we have

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V\}) \leq \delta_\theta(\{k \in \mathbb{N}: x_k - x_{k'(r)} \notin W\}) + \delta_\theta(\{r \in \mathbb{N}: x_{k'(r)} - x_0 \notin W\}).$$

Therefore

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V\}) = 0.$$

Theorem 3.2. Let (X, τ) be a locally solid Riesz space. If a sequence (x_k) is $S_\theta(\tau)$-convergent to x_0 in X, then there are sequences (y_k) and (z_k) such that $S_\theta(\tau) - \lim_{k\to\infty} y_k = x_0$ and $x_k = y_k + z_k$, for all $k \in \mathbb{N}$ and $\delta_\theta(\{k \in \mathbb{N}: x_k \neq y_k\}) = 0$ and (z_k) is a $S_\theta(\tau)$-null sequence.

Proof: Let $\{V_i\}$ be a nested base of τ-neighborhoods of zero. Take $n_0 = 0$ and choose an increasing sequence (n_i) of positive integers such that

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V_i\}) < \frac{1}{i} \text{ for } k > n_i.$$

Let us define the sequences (y_k) and (z_k) as follows:

$$y_k = x_k \text{ and } z_k = 0, if\ 0 < k \leq n_1$$

and suppose $n_i < n_{i+1}$, for $i \geq 1$,

$$y_k = x_k \text{ and } z_k = 0, if\ x_k - x_0 \in V_i$$

$$y_k = x_0 \text{ and } z_k = x_k - x_0, if\ x_k - x_0 \notin V_i.$$

To show that (i) $\lim_{k\to\infty} y_k = x_0$ (ii) (z_k) is a $S_\theta(\tau)$-null sequence.

(i) Let V be an arbitrary τ-neighborhood of zero. Since X is first countable, we may choose a positive integer i such that $V_i \subseteq V$. Then $y_k - x_0 = x_k - x_0 \in V_i$, for $k > n_i$. If $x_k - x_0 \notin V_i$, then $y_k - x_0 = x_0 - x_0 = 0 \in V$.Hence $\lim_{k\to\infty} y_k = x_0$.

(ii) It is enough to show that $\delta_\lambda(\{k \in \mathbb{N}: z_k \neq 0\}) = 0$. For any τ-neighborhood V of zero, we have

$$\delta_\theta(\{k \in \mathbb{N}: z_k \notin V\}) \leq \delta_\theta(\{k \in \mathbb{N}: z_k \neq 0\}).$$

If $n_p < k \leq n_{p+1}$, then

$$\{k \in \mathbb{N}: z_k \neq 0\} \subseteq \{k \in \mathbb{N}: x_k - x_0 \notin V_p\}.$$

If $p > i$ and $n_p < k \leq n_{p+1}$, then

$$\delta_\theta(\{k \in \mathbb{N}: z_k \neq 0\}) \leq \delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V_p\}) < \frac{1}{p} < \frac{1}{i} < \varepsilon.$$

This implies that $\delta_\theta(\{k \in \mathbb{N}: z_k \neq 0\}) = 0$. Hence (z_k) is a $S_\theta(\tau)$-null sequence.

Corollary 3.3. Any lacunary convergent sequence in a locally solid space has a convergent subsequence.

Proof: The proof of this result follows from the preceding theorem.

Theorem 3.4. Let (X, τ) be a locally solid Riesz space and let $x = (x_k)$ be a sequence in X. If there is a $S_\theta(\tau)$-convergent sequence $y = (y_k)$ in X such that $\delta_\theta(\{k \in \mathbb{N}: y_k \neq x_k \notin V\}) = 0$ then x is also $S_\theta(\tau)$-convergent.

Proof: Suppose that $\delta_\theta(\{k \in \mathbb{N}: y_k \neq x_k \notin V\}) = 0$ and $S_\theta(\tau) - \lim_k y_k = x_0$. Then for an arbitrary τ-neighborhood V of zero, we have

$$\delta_\theta(\{k \in \mathbb{N}: y_k - x_0 \notin V\}) = 0.$$

Now,

$$\{k \in \mathbb{N}: x_k - x_0 \notin V\} \subseteq \{k \in \mathbb{N}: y_k \neq x_k \notin V\} \cup \{k \in \mathbb{N}: y_k - x_0 \notin V\}$$
$$\Rightarrow \delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V\}) \leq \delta_\theta(\{k \in \mathbb{N}: y_k \neq x_k \notin V\}) + \delta_\theta(\{k \in \mathbb{N}: y_k - x_0 \notin V\}).$$

Therefore, we have

$$\delta_\theta(\{k \in \mathbb{N}: x_k - x_0 \notin V\}) = 0.$$

Now we give the definition of slowly oscillating in locally solid Riesz spaces.

Definition 3.5. A sequence (x_k) in a locally solid Riesz space X is called *slowly oscillating* if for each τ-neighborhood V of zero, there exists a positive integer m_0 and $\delta > 0$ such that if $m_0 \leq k \leq n \leq$

$(1+\delta)k$, then $(x_n - x_k) \in V$.
Now we give a Tauberian theorem.

Theorem 3.5. Let (X,τ) be a locally solid Riesz space. If (x_k) is statistically convergent and slowly oscillating, then it is convergent.

Proof: Let $S(\tau) - \lim x_k = x_0$. Then we have a subsequence (i_m) with $1 \le i_1 \le i_2 \le \ldots \le i_m \le \ldots$ of those indices n for which $y_n = x_n$. Since

$$\lim_{k\to\infty} \frac{1}{k} |\{n \le k : x_n \ne y_n\}| = 0.$$

Then, we have

$$\lim_{m\to\infty} \frac{1}{i_m} |\{n \le i_m : x_n = y_n\}| = \lim_{m\to\infty} \frac{m}{i_m} = 1.$$

Consequently, it follows that

$$\lim_{m\to\infty} \frac{i_{m+1}}{i_m} = \lim_{m\to\infty} \frac{i_{m+1}}{m+1} . \frac{m+1}{m} . \frac{m}{i_m} = 1. \quad (1)$$

By the definition of (i_m), we get

$$\lim_{m\to\infty} x_{i_m} = \lim_{m\to\infty} y_{i_m} = x_0. \quad (2)$$

By (1) and (2) we conclude that for each closed τ-neighborhood V of zero, there exists a positive integer n_0 such that if $m > n_0$ then $(x_k - x_{i_m}) \in V$ whenever $i_m < k < i_{m+1}$. Since V is arbitrary, it follows that

$$\lim_{m\to\infty} (x_m - x_{i_m}) = 0.$$

By (3.2), we have (x_m) is convergent to x_0. This completes the proof of the theorem.

4. Some inclusions relations in locally solid Riesz spaces

Theorem 4.1. Let (X,τ) be a locally solid Riesz space and $x = (x_k)$ be sequence in X. For any lacunary sequence $\theta = (k_r)$, $S(\tau) \subseteq S_\theta(\tau)$ if and only if $\liminf_r q_r > 1$.

Proof: Suppose first that $\liminf_r q_r > 1$, $\liminf_r q_r = a$(say). Write $b = \frac{a-1}{2}$. Then there exists an integer $n_0 \in \mathbb{N}$ such that $q_r \ge 1 + b$ for $r \ge n_0$. Hence for $r \ge n_0$,

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} = 1 - \frac{1}{q_r} \ge 1 - \frac{1}{1+b} = \frac{b}{1+b}.$$

Suppose that $S(\tau) - \lim_k x_k = x_0$. We prove that $S_\theta(\tau) - \lim_k x_k = x_0$. Let V be an arbitrary τ-neighborhood of zero. Then for all $r \ge n_0$, we have

$$\begin{aligned}
&\frac{1}{k_r} |\{k \le k_r : x_k - x_0 \notin V\}| \\
&\ge \frac{1}{k_r} |\{k \in I_r : x_k - x_0 \notin V\}| \\
&= \frac{h_r}{k_r} \frac{1}{h_r} |\{k \in I_r : x_k - x_0 \notin V\}| \\
&\ge \frac{b}{1+b} \frac{1}{h_r} |\{k \in I_r : x_k - x_0 \notin V\}|.
\end{aligned}$$

Since $(x_k) \overset{S(\tau)}{\to} x_0$. Therefore this inequality implies that $(x_k) \overset{S_\theta(\tau)}{\to} x_0$. Hence $S(\tau) \subseteq S_\theta(\tau)$.

Next we suppose that $\liminf_r q_r = 1$. Then, we can choose a subsequence $(k_{r(i)})$ of the lacunary sequence θ such that

$$\frac{k_{r(i)}}{k_{r(i)-1}} < 1 + \frac{1}{i} \text{ and } \frac{k_{r(i)-1}}{k_{r(i-1)}} > i,$$

where $r(i) > r(i-1) + 2$. Take $a(\ne 0) \in X$. Now we define a sequence (x_k) by

$$x_k = \begin{cases} a, & \text{if } k \in I_{r(i)}, \text{for some } i = 1,2,3,\cdots \\ 0, & \text{otherwise.} \end{cases}$$

Then $S(\tau) - \lim_k x_k = 0$. To see this take V be an arbitrary τ-neighborhood of zero. We choose $W \in N_{sol}$ such that $W \subseteq V$ and $a \notin W$. On the other hand, for each m we can find a positive number i_m such that $k_{r(i_m)} < m \le k_{r(i_m+1)}$. Then

$$\begin{aligned}
&\frac{1}{m} |\{k \le m : x_k \notin V\}| \\
&\le \frac{1}{k_{r(i_m)}} |\{k \le m : x_k \notin W\}| \\
&\le \frac{1}{k_{r(i_m)}} \{|\{k \le k_{r(i_m)} : x_k \notin W\}| + |\{k_{r(i_m)} < k \le m : x_k \notin W\}|\} \\
&\le \frac{1}{k_{r(i_m)}} |\{k \le k_{r(i_m)} : x_k \notin W\}| + \frac{1}{k_{r(i_m)}} (k_{r(i_m+1)} - k_{r(i_m)}) \\
&< \frac{1}{i_m} + 1 + \frac{1}{i_m} - 1 < \frac{1}{i_m+1} + \frac{1}{i_m} \text{ for each } m.
\end{aligned}$$

Therefore $S(\tau) - \lim_k x_k = 0$. Now let us see that $(x_k) \notin S_\theta(\tau)$. Let V be a τ-neighborhood of zero such that $a \notin V$. Thus

$$\begin{aligned}
&\lim_{i\to\infty} \frac{1}{h_{r(i)}} |\{k_{r(i)-1} < k \le k_{r(i)} : x_k \notin V\}| \\
&= \lim_{i\to\infty} \frac{1}{h_{r(i)}} (k_{r(i)} - k_{r(i)-1}) \\
&= \lim_{i\to\infty} \frac{1}{h_{r(i)}} h_{r(i)} = 1
\end{aligned}$$

and for $r \ne r(i), i = 1,2,3, \ldots$

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k_{r-1}<k\le k_r: x_k-a\notin V\}|=1.$$

Hence neither a nor 0 can be lacunary statistical limit of (x_k). No other point of X can be lacunary statistical limit of the sequence (x_k) as well. Thus $(x_k)\notin S_\theta(\tau)$.

Theorem 4.2. Let (X,τ) be a locally solid Riesz space and let $x=(x_k)$ be sequence in X. For any lacunary sequence $\theta=(k_r)$, $S_\theta(\tau)\subseteq S(\tau)$ if and only if $\limsup_r q_r<\infty$.

Proof: Suppose that $\limsup_r q_r<\infty$. Then there exists an $H>0$ such that $q_r<H$ for all r. Let $S_\theta(\tau)-\lim_k x_k=x_0$. Let V be an arbitrary τ-neighborhood of zero. Let $\varepsilon>0$. We write

$$M_r=\{k\in I_r: x_k-x_0\notin V\}.$$

By the definition of lacunary statistical convergence, there is a positive number r_0 such that

$$\frac{M_r}{h_r}<\frac{\varepsilon}{2H}\text{ for all } r>r_0.$$

Let $M=\max\{M_r:1\le r\le r_0\}$ and let m be any integer satifying $k_{r-1}<m\le k_r$; then we can write

$$\frac{1}{m}|\{k\le m: x_k-x_0\notin V\}|\le r_0\frac{M}{k_{r-1}}+\varepsilon\frac{1}{2H}q_r.$$

Since $\lim_{r\to\infty}k_r=\infty$, there exists a positive integer $r_1\ge r_0$ such that

$$\frac{1}{k_{r-1}}<\frac{\varepsilon}{2r_0M}\text{ for } r>r_1.$$

Hence, for $r>r_1$

$$\frac{1}{m}|\{k\le m: x_k-x_0\notin V\}|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

It follows that $S(\tau)-\lim_k x_k=x_0$.

Now we suppose that $\limsup_r q_r=\infty$. Take an element $a\ne 0\in X$. Let $(k_{r(i)})$ be a subsequence of the lacunary sequence $\theta=(k_r)$ such that $q_{r(i)}>i, k_{r(i)}>i+3$. Define a sequence (x_k) by

$$x_k=\begin{cases}a, & \text{if } k_{r(i)-1}<k\le 2k_{r(i)-1},\\ 0, & \text{otherwise,}\end{cases}$$

for some $i=1,2,3,\cdots$. Let V be a τ-neighborhood of zero such that $a\notin V$. Then for $i>1$

$$\frac{1}{h_{r(i)}}|\{k\le k_{r(i)}: x_k\notin V\}|<\frac{k_{r(i)-1}}{h_{r(i)}}=\frac{k_{r(i)-1}}{k_{r(i)}-k_{r(i)-1}}<\frac{1}{i-1}.$$

Hence $(x_k)\in S_\theta(\tau)$. But $(x_k)\notin S(\tau)$, because

$$\frac{1}{2k_{r(i)-1}}|\{k\le 2k_{r(i)-1}: x_k\notin V\}|=\frac{1}{2k_{r(i)-1}}[k_{r(1)-1}+k_{r(2)-1}+\ldots+k_{r(i)-1}]>\frac{1}{2}.$$

Corollary 4.3. Let (X,τ) be a locally solid Riesz space and let $x=(x_k)$ be sequence in X. For any lacunary sequence $\theta=(k_r)$, $S_\theta(\tau)=S(\tau)$ if and only if $1<\liminf_r q_r\le\limsup_r q_r<\infty$.

Theorem 4.4. Let (X,τ) be a locally solid Riesz space and let $x=(x_k)$ be sequence in X. For any lacunary sequence $\theta=(k_r)$, if $x=(x_k)\in S_\theta(\tau)\cap S(\tau)$ then $S(\tau)-\lim_{k\to\infty}x_k=S_\theta(\tau)-\lim_{k\to\infty}x_k$.

Proof. Let $x=(x_k)\in S_\theta(\tau)\cap S(\tau)$ and $S(\tau)-\lim_{k\to\infty}x_k=x_0$, and $S_\theta(\tau)-\lim_{k\to\infty}x_k=y_0$. Suppose that $x_0\ne y_0$. Since X is a Hausdorff, then there exists a τ-neighborhood V of zero such that $x_0-y_0\notin V$. We choose $W\in N_{sol}$ such that $W+W\subseteq V$. Then, we have

$$\frac{1}{k_m}|\{k\le k_m: x_0-y_0\notin V\}|\le\frac{1}{k_m}|\{k\le k_m: x_k-x_0\notin W\}|+\frac{1}{k_m}|\{k\le k_m: y_0-x_k\notin W\}|. \quad (3)$$

It follows from this inequality that

$$1\le\frac{1}{k_m}|\{k\le k_m: x_k-x_0\notin W\}|+\frac{1}{k_m}|\{k\le k_m: y_0-x_k\notin W\}|.$$

We write

$$\frac{1}{k_m}|\{k\le k_m: y_0-x_k\notin W\}|=\frac{1}{k_m}|\{k\in\bigcup_{r=1}^{m}I_r: y_0-x_k\notin W\}|=\frac{1}{k_m}\sum_{r=1}^{m}|\{k\in I_r: y_0-x_k\notin W\}|=\left(\sum_{r=1}^{m}h_r\right)^{-1}\left(\sum_{r=1}^{m}h_r.T_r\right),$$

where

$$T_r=\frac{1}{h_r}|\{k\in I_r: y_0-x_k\notin W\}|.$$

Since $S_\theta(\tau)-\lim_{k\to\infty}x_k=y_0$, we have $\lim_{r\to\infty}T_r=0$. Therefore the regular weighted mean transform of (T_r) also tends to 0, i.e.,

$$\lim_{m\to\infty}\frac{1}{k_m}|\{k\le k_m: y_0-x_k\notin W\}|=0. \quad (4)$$

Also, since $S(\tau)-\lim_{k\to\infty}x_k=x_0$, we have

$$\lim_{m\to\infty}\frac{1}{k_m}|\{k\le k_m: x_k-x_0\notin W\}|=0. \qquad (5)$$

From (3), (4) and (5), we have

$$\frac{1}{k_m}|\{k\le k_m: x_0-y_0\notin V\}|=0.$$

5. Statistically lacunary τ-convergence in locally solid Riesz spaces

Definition 5.1. Let (X,τ) be a locally solid Riesz space. A sequence $x=(x_k)$ is said to be θ-*summable* in (X,τ) or simply $\theta_\tau - summable$ to an element x_0 of X if for each τ-neighborhood V of zero such that $t_r(x)-x_0\in V$. In this case we write $\theta_\tau - \lim x=x_0$ and x_0 is called the θ-limit of the sequence $x=(x_k)$.

Definition 5.2. Let (X,τ) be a locally solid Riesz space. A sequence (x_k) of points in X is said to be statistically lacunary τ-convergent or $S_{\delta_\theta}(\tau)$-convergent to an element x_0 of X if for each τ-neighborhood V of zero, the set $K(\theta)=\{r\in\mathbb{N}: t_r(x)-x_0\notin V\}$ has natural density zero, i.e., $\delta(K(\theta))=0$,
or

$$\lim_n\frac{1}{n}|\{r\le n: t_r(x)-x_0\notin V\}|=0.$$

In this case, we write $S_{\delta_\theta}(\tau)-\lim_{k\to\infty}x_k=x_0\ or\ (x_k)\overset{S_{\delta_\theta}(\tau)}{\to}x_0$.

Theorem 5.1. Let (X,τ) be a locally solid Riesz space. A sequence $x=(x_k)$ in X is $S_{\delta_\theta}(\tau)$-convergent to x_0 if and only if there exists a set $K=\{r_1<r_2<\cdots<r_n<\cdots\}\subseteq\mathbb{N}$ such that $\delta(K)=1$ and $\theta_\tau - \lim_{r\in K}x_r=x_0$.

Proof: Let V be an arbitrary τ-neighborhood of zero. Suppose that $\theta_\tau-\lim_{r\in K}x_r=x_0$, there exists a set $K=\{r_1<r_2<\cdots<r_n<\cdots\}\subseteq\mathbb{N}$ with $\delta(K)=1$ and $N=N(V)$ such that $(t_r(x)-x_0)\in V$ for $r>N$. Write $K_V=\{r\in\mathbb{N}: t_r(x)-x_0\notin V\}$ and $K_1=\{k_{N+1},k_{N+2},\ldots\}$. Then $\delta(K_1)=1$ and $K_V\subseteq\mathbb{N}-K_1$ which implies that $\delta(K_V)=0$. Hence $x=(x_k)$ is $S_{\delta_\theta}(\tau)$-convergent to x_0.
Conversely suppose that $x=(x_k)$ is $S_{\delta_\theta}(\tau)$-convergent to x_0. Fix a countable local base $V_1\supset V_2\supset\cdots$ at x_0. For each $i\in\mathbb{N}$, put

$$K_i=\{r\in\mathbb{N}: t_r(x)-x_0\notin V_i\}.$$

By hypothesis $\delta(K_i)=0$ for each i. Since the ideal $\mathcal{I}$ of all subsets of $\mathbb{N}$ having density zero is a P-ideal (see for instance Farah, 2000), then there exists a sequence of sets $(J_i)_i$ such that the symmetric difference $K_i\Delta J_i$ is a finite set for any $i\in\mathbb{N}$ and $J:=\cup_{i=1}^\infty J_i\in\mathcal{I}$.

Let $K=\mathbb{N}\setminus J$, then $\delta(K)=1$. In order to prove the theorem, it is enough to check that $\lim_{r\in K}t_r(x)=x_0$.

Let $i\in\mathbb{N}$. Since $K_i\Delta J_i$ is a finite, there is $r_i\in\mathbb{N}$, without loss of generality with $r_i\in K$, $r_i>i$, such that

$$(\mathbb{N}\setminus J_i)\cap\{r\in\mathbb{N}: r\ge r_i\}=(\mathbb{N}\setminus K_i)\cap\{r\in\mathbb{N}: r\ge r_i\}. \qquad (6)$$

If $r\in K$ and $r\ge r_i$, then $r\notin J_i$, and by (6) $r\notin K_i$. Thus $t_r(x)-x_0\in V_i$. So we have proved that for all $i\in\mathbb{N}$ there is $r_i\in K$, $r_i>i$, with $t_r(x)-x_0\in V_i$ for every $r\ge r_i$: without loss of generality, we can suppose $r_{i+1}>r_i$ for every $i\in\mathbb{N}$. The assertion follows taking into account that the V_i's form a countable local base at x_0.

References

Albayrak, H., Pehlivan, S. (2012). Statistical convergence and statistical continuity on locally solid Riesz spaces. Topology Appl., 159, 1887–1893.

Aliprantis, C. D., Burkinshaw, O. (2003). Locally solid Riesz spaces with applications to economics. second ed., Amer. Math. Soc.

Belen, C., Mohiuddine, S. A. (2013). Generalized weighted statistical convergence and application. Applied Math. Comput., 219, 9821–9826.

Buck, R. C. (1953). Generalized asymptotic density. Amer. J. Math., 75, 335–346.

Çakalli, H. (1995). Lacunary statistical convergence in topological groups. Indian J. Pure Appl. Math., 26(2), 113–119.

Çakalli, H. (1996). On statistical convergence in topological groups. Pure Appl. Math. Sci., 43, 27–31.

Çakalli, H., Khan, M. K. (2011). Summability in Topological Spaces. Appl. Math. Letters, 24, 348–352.

Çakalli, H. (2009). A study on statistical convergence. Funct. Anal. Approx. Comput., 1(2), 19–24.

Çakan, C., Altay, B., Çoskun, H. (2010). Double lacunary density and lacunary statistical convergence of double sequences. Studia Sci. Math. Hung., 47(1), 35–45.

Caserta, A., Maio, G. Di, Kočinac, Lj. D. R. (2011). Statistical convergence in function spaces. Abstr. Appl. Anal. Vol., 2011, Article ID 420419, 11 pages.

Caserta, A., Kočinac, Lj. D. R. (2012). On statistical exhaustiveness. Appl. Math. Letters, 25, 1447–1451.

Çolak, R., Bektas, C. A. (2011). λ-statistical convergence of order α. Acta Math. Scientia, 31B(3), 953–959.

Farah, I. (2000). Analytic quotients: Theory of liftings for quotients over analytic ideals on the integers. Mem. Amer. Math. Soc., 148.

Fast, H. (1951). Sur la convergence statistique. Colloq. Math., 2, 241–244.

Freedman, A. R., Sember, J. J., Raphael, M. (1978). Some Ces`aro-type summability spaces. Proc. London Math. Soc., 37(3), 508–520.

Fridy, J. A. (1985). On statistical convergence, Analysis,

5, 301-313.
Fridy, J. A., Orhan, C. (1993). Lacunary statistical convergence, Pacific J. Math., 160, 43–51.
Gürdal, M., Acik, I. (2008). On I-Cauchy sequences in 2-normed spaces. Math. Inequal. Appl., 11(2), 349–354.
Hazarika, B., Mohiuddine, S. A. (2013). Ideal convergence for random variables. J. Function Spaces Appl., Volume 2013, Article ID 148249, 7 pages.
Kantorovich, L.V. (1937). Lineare halbgeordnete Raume. Rec. Math., 2, 121–168.
Li, J. (2000). Lacunary statistical convergence and inclusion properties between lacunary methods. Internat. J. Math. Math. Sci., 23(3), 175–180.
Luxemburg, W. A. J., Zaanen, A. C.(1971). Riesz Spaces I. North-Holland, Amsterdam.
Maddox, I. J. (1988). Statistical convergence in a locally convex spaces. Math. Proc. Cambridge Philos. Soc., 104(1), 141–145.
Maio, G. Di., Kočinac, Lj. D. R. (2008). Statistical convergence in topology. Topology Appl., 156, 28–45.
Mohiuddine, S. A., Aiyub, M. (2012). Lacunary statistical convergence in random 2-normed spaces. Applied Math. Inform. Sciences, 6(3), 581–585.
Mohiuddine, S. A., Alghamdi, M. A. (2012). Statistical summability through a lacunary sequence in locally solid Riesz spaces. Jour. Inequa. Appl., Vol. 2012, Article 225.
Mohiuddine, S. A., Alotaibi, A., Mursaleen, M. (2012). Statistical convergence of double sequences in locally solid Riesz spaces. Abstr. Appl. Anal., Vol. 2012, Article ID 719729, 9 pages.
Mohuiddine, S. A., Alotaibi, A., Alsulami, S. M. (2012). Ideal convergence of double sequences in random 2-normed spaces. Adv. Difference Equ., Vol. 2012, Article 149.
Mohiuddine, S. A., Alotaibi, A., Mursaleen, M. (2013). Statistical convergence through de la Vallée-Poussin mean in locally solid Riesz spaces. Adv. Difference Equ., Vol. 2013, Article 66.
Mohiuddine, S. A., Alotaibi, A., Mursaleen, M. (2013). A new variant of statistical convergence. J. Inequal. Appl., Vol. 2013, Article 309.
Mohiuddine, S. A., Sevli, H. & Cancan, M. (2010). Statistical convergence in fuzzy 2-normed space. J. Comput. Anal. Appl., 12(4), 787–798.
Mohiuddine, S. A., Sevli, H., Cancan, M. (2012). Statistical convergence of double sequences in fuzzy normed spaces. Filomat, 26(4), 673–681.
Mohiuddine, S. A., Lohani, Q. M. D. (2009). On generalized statistical convergence in intuitionistic fuzzy normed space. Chaos, Solitons Fract., 42, 1731–1737.
Mursaleen, M., Alotaibi, A. (2011). On I-convergence in random 2-normed spaces. Math. Slovaca, 61(6), 933–940.
Mursaleen, M., Çakan,C., Mohiuddine, S. A., Savas, E. (2010). Generalized statistical convergence and statistical core of double sequences. Acta Math. Sinica (Engl. Ser.), 26, 2131–2144.
Mursaleen, M., Edely, O. H. H. (2003). Statistical convergence of double sequences. J. Math. Anal. Appl., 288, 223–231.
Mursaleen, M., Mohiuddine, S. A. (2009). Statistical convergence of double sequences in intuitionistic fuzzy normed spaces. Chaos Solitons Fract., 41, 2414–2421.
Mursaleen, M., Mohiuddine, S. A. (2010). On ideal convergence of double sequences in probabilistic normed spaces. Math. Reports, 12(62)(4), 359–371.
Mursaleen, M., Mohiuddine, S. A. (2012). On ideal convergence in probabilistic normed spaces. Math. Slovaca, 62, 49–62.
Mursaleen, M., Mohiuddine, S. A. (2009). On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. J. Comput. Appl. Math., 233(2), 142–149.
Mursaleen, M., Mohiuddine, S. A., Edely, O. H. H. (2010). On ideal convergence of double sequences in intuitionistic fuzzy normed spaces. Comput. Math. Appl., 59, 603–611.
Prullage, D. L. (1967). Summability in topological groups. Math. Z., 96, 259–279.
Roberts, G. T. (1952). Topologies in vector lattices, Math. Proc. Camb. Phil. Soc., 48, 533–546.
Riesz, F. (1928). Sur la décomposition des opérations fonctionelles linéaires, in: Atti del Congr. Internaz. dei Mat., 3, Bologna, Zanichelli, 1930, pp. 143–148.
Šalát, T. (1980). On statistical convergence of real numbers. Math. Slovaca, 30, 139–150.
Savas, E., Mohiuddine, S. A. (2012). $\bar{\lambda}$-statistically convergent double sequences in probabilistic normed spaces. Math. Slovaca, 62(1), 99–108.
Schoenberg, I. J. (1959). The integrability of certain functions and related summability methods. Amer. Math. Monthly, 66, 361–375.
Steinhaus, H. (1951). Sur la convergence ordinate et la convergence asymptotique. Colloq. Math., 2, 73–84.
Zannen, A. C. (1997). Introduction to Operator Theory in Riesz Spaces. Springer-Verlag, 1997.
Zygmund, A. (1979). Trigonometric Series. Cambridge Univ. Press, Cambridge, UK.