IJST (2014) 38A1: 43-48
Iranian Journal of Science & Technology
http://ijsts.shirazu.ac.ir

A generalization of reversible rings

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Abstract

In this paper, we introduce a class of rings which is a generalization of reversible rings. Let R be a ring with identity. A ring R is called *central reversible* if for any $a,b \in R$, $ab=0$ implies ba belongs to the center of R. Since every reversible ring is central reversible, we study sufficient conditions for central reversible rings to be reversible. We prove that some results of reversible rings can be extended to central reversible rings for these general settings.

***Keywords**:* Reversible rings; central reversible rings; weakly reversible rings

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent elements, similarly a ring R is called *central reduced* (Agayev, et al., 2009) if every nilpotent element of R is central. According to Cohn (Cohn, 1999) a ring R is said to be *reversible* if for any $a,b \in R$, $ab=0$ implies $ba=0$. Recently, Baser, et al. studied extensions of reversible rings in (Baser et al., 2009) and (Baser et al., 2010). A ring R is called *semicommutative* if for any $a,b \in R$, $ab=0$ implies $aRb=0$. A ring R is *right (left) principally quasi-Baer* (Birkenmeier et al., 2001) if the right (left) annihilator of a principal right (left) ideal of R is generated by an idempotent. Finally, a ring R is called *right (left) principally projective* (Birkenmeier et al., 2001) if the right (left) annihilator of an element of R is generated by an idem-potent.

In this paper, we introduce central reversible rings as a generalization of reversible rings. Clearly, reversible rings are central reversible and central reversible rings are weakly reversible. We supply some examples to show that all central reversible rings need not be reversible and all weakly reversible rings need not be central reversible. Therefore, we show that the class of central reversible rings lies strictly between classes of reversible rings and weakly reversible rings. Among others we prove that central reversible rings are abelian and there exists an abelian ring, but it is not central reversible. It is shown that every central reversible ring is weakly reversible, 2-primal, abelian and so directly finite. For an Armendariz ring R, we prove that R is central reversible if and only if the polynomial ring $R[x]$ is central reversible if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is central reversible. Moreover, it is also proven that if R is central reversible, then the Dorroh extension of R is central reversible. Throughout this paper, Z denotes the ring of integers. We write R[x] and $R[x, x^{-1}]$ for the polynomial ring and the Laurent polynomial ring, respectively.

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Received: 16 May 2012 / Accepted: 31 July 2013

2. Central reversible rings

In this section we introduce and study a class of rings, called central reversible rings, which is a generalization of reversible rings. We prove that some results of reversible rings can be extended to central reversible rings for this general setting. We now give our main definition.

Definition 2.1. A ring R is called central reversible if for any $a,b \in R$, $ab=0$ implies ba is central in R.

Central reversible rings are abundant around. Commutative rings, reduced rings, central reduced rings, symmetric rings and reversible rings are central reversible. One may suspect that central reversible rings are reversible. But the following example erases the possibility.

Example 2.2. Let R be a commutative reduced ring and consider the ring

$$S=\left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,b,c,d \in R\right\}$$

We prove that S is central reversible but not reversible. Let

$$x=\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, y=\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S \text{ with } xy=0.$$

Then we have the following equations:

$$(1)\, a_1a_2=0,$$
$$(2)\, a_1b_2+b_1a_2=0,$$
$$(3)\, a_1c_2+b_1d_2+c_1a_2=0,$$
$$(4)\, a_1d_2+d_1a_2=0.$$

Since R is commutative, we have $a_2a_1=0$ Multiplying the equation (2) from the left by b_1a_2 then$(b_1\ a_2)^2=0$ and so $b_1a_2=0$ since R is reduced. Using a similar method we get $a_1d_2=0$ $=d_1\, a_2$. Multiplying the equation (3) from the right by a_1c_2, then $(a_1c_2)^2=0$ It follows that a_1c_2 $=c_2a_1=0$. Then we get $b_1d_2+c_1\, a_2=0$. Multiplying this equation from the right by b_1d_2, then $b_1d_2=d_2b_1=0$ and so $c_1a_2=a_2c_1=0$. Consequently

$$\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & b_2d_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is central and therefore S is central reversible. On the other hand, S is not reversible from (Kim and Lee, 2003; Example 1.5).

Recall that a ring R is *semiprime* if $aRa=0$ implies $a=0$ for $a \in R$. We now investigate under what conditions central reversible rings are reversible.

Proposition 2.3. If R is a reversible ring, then R is central reversible. The converse holds if R satisfies any of the following conditions.
(1) R is a semiprime ring.
(2) R is a right (left) principally projective ring.
(3) R is a right (left) principally quasi-Baer ring.

Proof: First statement is clear. Conversely, assume that R is a central reversible ring and $a, b \in R$ with $ab=0$. Then ba is central. Now consider the following cases.
(1) Let R be a semiprime ring. Since ba is central, $baRba=0$ and so $ba=0$. Thus R is reversible.
(2) Let R be a right principally projective ring. Then there exists an idempotent $e \in R$ such that $r_R(a)=eR$. Hence $b=eb$ and $ae=0$. Since ba is central, $ba=eba=bae=0$ and so R is reversible. A similar proof may be given for left principally projective rings.
(3) Same as the proof of (2).
The following is a consequence of Proposition 2.3.

Corollary 2.4. If R is a central reversible ring, then the following conditions are equivalent.
(1) R is a right principally projective ring.
(2) R is a left principally projective ring.
(3) R is a right principally quasi-Baer ring.
(4) R is a left principally quasi-Baer ring.

Next we prove that central reversible rings are closed under finite direct sums.

Proposition 2.5. Let $\{R_i\}_{i\in I}$ be a class of rings for a finite index set I. Then R_i is central reversible for all $i \in I$ if and only if $\bigoplus_{i\in I} R_i$ is central reversible.

Proof: The necessity follows from definitions. The sufficiency is clear since a subring of a central reversible ring is central reversible.

The following result is a direct consequence of Proposition 2.5.

Corollary 2.6. Let R be a ring. Then eR and $(1-e)R$ are central reversible for some central idempotent e in R if and only if R is central reversible.

Note that the homomorphic image of a central reversible ring need not be central reversible. Consider the following example.

Example 2.7. Let D be a division ring, $R=D[x, y]$ and $I=(xy)$, where $xy \neq yx$. Since R is a domain, R is central reversible. On the other hand, $(x+I)(y+I)$ is zero but $(y+I)(x+I)$ is not central in R/I. Hence R/I is not central reversible.

Our next endeavor is to determine conditions when the homomorphic image of a ring is central reversible. Recall that a ring R is called unit-central (Khurana et al., 2010), if all unit elements are central in R. It is proven that every unit-central ring is abelian (i.e., every idempotent of the ring is central).

Lemma 2.8. Let R be a unit-central ring. If I is a nil ideal of R, then R/I is central reversible.

Proof: Let $a, b \in R$ with $(a+I)(b+I)=0+I$. Then $ab \in I$ and so there exists a positive integer n such that $(ab)^n=0$. Hence $(ba)^{n+1}=0$. It follows that $1-ba$ is unit and so central by hypothesis. Thus $rba=bar$ for any $r \in R$. Therefore $(b+I)(a+I)$ is central in R/I.

The next example shows that for a ring R and an ideal I, if R/I is central reversible, then R need not be central reversible.

Example 2.9. Let $R=\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is any field. Consider the ideal $I=\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ of R. Then R/I is central reversible because of the commutativity property of R/I. For $A=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $AB=0$. Consider $C=\begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix} \in R$ with $c_1 \neq c_3$. It is clear that $CBA \neq BAC$. Therefore R is not central reversible.

Lemma 2.10. Let R be a ring. If R/I is a central reversible ring with a reduced ideal I, then R is central reversible.

Proof: Let R/I be a central reversible ring. Let $a,b \in R$ with $ab=0$. Since $(a+I)(b+I)=0+I$, $(b+I)(a+I)$ is central in R/I. It follows that $bar-rba \in I$ for any $r \in R$. Then $I(bar-rab)=0$. Hence we have $(bar-rba)^2=0$. Since I is reduced, $bar=rba$ and so R is central reversible.

Reversible rings are generalized by Liang and Gang (Liang and Gang, 2007), a ring R is said to be weakly reversible, if for all $a, b, r \in R$ such that $ab=0$, $Rbra$ is a nil left ideal of R. In the sequel, we show that the class of central reversible rings lies strictly between classes of reversible rings and weakly reversible rings.

Theorem 2.11. Let R be a ring. Consider the following conditions.
(1) R is reversible.
(2) R is central reversible.
(3) R is weakly reversible.
Then (1) ⇒ (2) ⇒ (3).

Proof: (1) ⇒ (2) Obvious.
(2) ⇒ (3) Let $a, b \in R$ with $ab=0$. Then for all $x \in R$, $abx=0$. Since R is central reversible, clearly bxa is central. Then we have $(rbxa)^2=(rbxa)(rbxa)=r(bxa)rbxa=rrbx(ab)xa=0$ for all $r, x \in R$. This implies that R is weakly reversible.

There are weakly reversible rings which are not central reversible.

Example 2.12. Let R be a weakly reversible ring and consider the ring

$$S=\left\{\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \mid a,b,c,d,e,f \in R\right\}$$

By (Liang and Gang, 2007; Example 2.6) S is weakly reversible. We now prove that S is not central reversible. For

$$x=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \in S$$

we have $xy=0$, but $yx=x$ is not central in S.

It is well known that every reversible ring is abelian. In addition to this fact, we have the following proposition when we deal with central case.

Lemma 2.13. If R is a central reversible ring, then it is abelian.
Proof. Let $e^2=e \in R$. For any $r \in R$, $(1-e)(er-ere)=0$ implies $(er-ere)(1-e)=er-ere$ is central. Commuting $er-ere$ by e we have $er-ere=0$. Similarly for any $r \in R$, $(re-ere)(1-e)=0$ implies $re-ere=0$. Therefore R is abelian.

The next example shows that the reverse implication of Lemma 2.13 is not true in general, i.e., there exists an abelian ring which is not reversible.

Example 2.14. Consider the ring

$$R=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \ (mod\,2), b \equiv c \equiv 0 \ (mod\,2)\right\}$$

Since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only idempotents of R, R is abelian. On the other hand, consider $x=\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, y=\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in R$ with $xy=$

0. But yx is not central for $z = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \in R$. Hence R is not central reversible.

Recall that a ring R is called *directly finite* whenever $a, b \in R$, $ab = 1$ implies $ba = 1$. Then we have following.

Corollary 2.15. Every central reversible ring is directly finite.

Proof: Clear from Lemma 2.13, since every abelian ring is directly finite.

A ring R is said to be *weakly semicom-mutative* (Liang et al., 2007) if for any $a, b \in R$, $ab=0$ implies arb is a nilpotent element for each $r \in R$. It is well known that every reversible ring is semicommutative. We prove that central reversible rings are weakly semicom-mutative.

Lemma 2.16. Every central reversible ring is weakly semicommutative.

Proof: Let $a, b \in R$ with $ab = 0$, then ba is central. Thus we have $(arb)^2 = (arb)(arb) = ar(ba)rb = a(ba)rrb = 0$, and so arb is nilpotent for all $r \in R$. This shows that R is weakly semicommutative.

Corollary 2.17. Let R be a right principally projective ring. If R is central reversible, then it is semicommutative.

Proof: It follows from the fact that every central reversible and right principally projective ring is reversible by Proposition 2.3.

It is well known that a ring is a domain if and only if it is prime and reversible. We have the following proposition when we deal with central reversible rings.

Lemma 2.18. Let R be a ring. Then R is a prime and central reversible ring if and only if it is a domain.

Proof: Let $a, b \in R$ with $ab = 0$. Then $abr = 0$ for any $r \in R$ and so bra is central. By commuting bra with b, we have $b^2ra=brab=0$. By hypothesis, bratb is central for any $t \in R$. Since R is prime, $a = 0$ or $b = 0$. The rest is
clear.

Let $P(R)$ denote the prime radical and $N(R)$ the set of all nilpotent elements of the ring R. The ring R is called 2-primal if $P(R)=N(R)$ (see namely (Hirano, 1978) and (Hwang et al., 2007)). In (Shin, 1973; Theorem 1.5) it is proved that every reversible ring is 2-primal. In this direction we prove

Theorem 2.19. If R is a central reversible ring, then it is 2-primal. The converse holds for semiprime rings.

Proof: Let R be a central reversible ring. We always have $P(R) \subseteq N(R)$, since $P(R)$ is a nil ideal of R. For the converse inclusion, let $a \in N(R)$ with $a^n = 0$ for some positive integer n. Assume that $a \notin Q$ for a prime ideal Q. Since R is central reversible, a is central. For any $r_{n-1}, r_{n-2}, \ldots, r_2, r_1 \in R$, we have $ar_{n-1}ar_{n-2}a \ldots ar_2ar_1a = r_{n-1}r_{n-2} \ldots r_2r_1a^n = 0$.

For all prime ideals P, we have $aR(ar_{n-2}a \ldots ar_2ar_1a) \subseteq P$. Since $a \notin Q$, $ar_{n-2}a \ldots ar_2ar_1a \in P$ for all prime ideals P and $r_{n-2}, \ldots, r_2, r_1 \in R$. Hence $aR(ar_{n-3}a \ldots ar_2ar_1a) \subseteq P$ for all prime ideals P and $r_{n-3}, \ldots, r_2, r_1 \in R$. Using a similar reasoning, since $a \in Q$, $aR(ar_{n-4}a \ldots ar_2ar_1a) \subseteq P$ for all prime ideals P and for all $r_{n-4}, \ldots, r_2, r_1 \in R$ implies $ar_{n-4}a \ldots ar_2ar_1a \in P$ for all prime ideals P and for all $r_{n-4}, \ldots, r_2, r_1 \in R$. By going downward induction,we may reach $aRa \subseteq P$ for all prime ideals P. Hence $a \in P$ for all prime ideals P. This is the required contradiction. Thus if a is nilpotent, then $a \in P(R)$ and so $N(R) \subseteq P(R)$

Conversely, let R be a semiprime and 2-primal ring. Then $P(R) = 0$ and so $N(R) = 0$. Hence R is reduced and so central reversible. This completes the proof.

Let R be a ring and M an (R,R)-bimodule. Recall that the trivial extension of R by M is defined to be ring $T(R,M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This ring is isomorphic to the ring

$$\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R, m \in M \right\}$$

with the usual matrix operations and isomorp-hic to $R[x]/(x^2)$, where (x^2) is the ideal generated by x^2. In (Kim and Lee, 2003), it is proved that if R is a reduced ring, then $T(R,R)$ is reversible. For central case we have the following.

Proposition 2.20. If R is a central reduced ring, then $T(R,R)$ is central reversible.

Proof. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R,R)$ with $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0$. Then $ac = ad + bc = 0$. By hypothesis, *R* is central reversible, then *ca* is central. Hence
$(ad)^3 = (-bc)(ad)(-bc) = b(ca)dbc = bdbcac = 0$
0, which implies *ad* is central, so *bc* is central. Hence $(da)^4 = 0$ and $(cb)^4 = 0$, which implies da, cb are central. Therefore $\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is central.

Let *S* denote a multiplicatively closed subset of a ring *R* consisting of central regular elements. Let $S^{-1}R$ be the localization of *R* at *S*. Then we have the following proposition.

Proposition 2.21. A ring *R* is central reversible if and only if $S^{-1}R$ is central reversible.

Proof: Let *R* be a central reversible ring and $a/r, b/s \in S^{-1}R$ where $a, b \in R, r, s \in S$ with $(a/r)(b/s) = 0$. Since $(a/r)(b/s) = ab/rs = 0$ we have $ab = 0$. By hypothesis ba is central, so $(b/s)(a/r)(c/t) = (c/t)(b/s)(a/r)$ for every $c/t \in S^{-1}R$, where $c \in R$ and $t \in S$. Therefore $S^{-1}R$ is central reversible. Con-versely, assume that $S^{-1}R$ is a central reversible ring. Since *R* may be embedded in $S^{-1}R$, the rest is clear.

Corollary 2.22. Let *R* be a ring. Then $R[x]$ is central reversible if and only if $R[x, x^{-1}]$ is central reversible.

Proof: Consider the subset $S = \{1, x, x^2, x^3, \ldots\}$ of $R[x]$ consisting of central regular elements. Then it follows from Proposition 2.21.

Let R be a ring and

$$f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$$

In (Rege and Chhawchharia, 1997), a ring R is called *Armendariz*, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all *i* and *j*. The name of the ring was given due to E. P. Armendariz (Armendariz, 1974) who proved that reduced rings satisfied this condition. The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring *R* and the annihilators of the polynomial ring $R[x]$. In this direction we have the following result.

Theorem 2.23. Let *R* be an Armendariz ring. Then the following statements are equivalent.

(1) R is central reversible.

(2) $R[x]$ is central reversible.

(3) $R[x, x^{-1}]$ is central reversible.

Proof: $(1) \Rightarrow (2)$ Let

$$f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$$

with $f(x)g(x) = 0$. Since *R* is Armendariz, $a_i b_j = 0$ for each *i* and *j*. But *R* is central reversible so $b_j a_i$ is central for each *i* and *j*.It follows that $g(x)f(x)$ is central in $R[x]$. Therefore $R[x]$ is central reversible.

$(2) \Rightarrow (1)$ Clear.

$(2) \Leftrightarrow (3)$ It follows from Corollary 2.22.

The *Dorroh extension* *D(R.Z)={(r,n): r ϵ R, n ϵ Z}* of a ring R is a ring with operations $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$ and $(r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)$. Obviously *R* is isomorphic to the ideal *{(r,0): r ϵ R}* of *D(R.Z)*. Then we have the following.

Proposition 2.24. A ring *R* is central reversible if and only if the Dorroh extension *D(R.Z)* of *R* is central reversible.

Proof: The sufficiency is clear. For necessity, let (r_1,n_1) $(r_2,n_2) \epsilon$ *D(R.Z)* with $(r_1, n_1)(r_2, n_2) = 0$. Then $n_1 n_2 = 0$ and assume that $n_1 = 0$. Since *R* is central reversible, $(r_2 + 1n_2)r_1$ is central in *R* and so $(r_2, n_2)(r_1, n_1)$ is central in *D(R.Z)*. Hence *D(R.Z)* is central reversible. A similar proof may be given for $n_2 = 0$.

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