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Some sequence spaces derived by Riesz mean in a real 2–normed space

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Abstract

In the present paper, we introduce some new sequence spaces derived by Riesz mean and the notions of almost and strongly almost convergence in a real 2–normed space. Some topological properties of these spaces are investigated. Further, new concepts of statistical convergence which will be called weighted almost statistical convergence, almost statistical convergence and $\left[\tilde{R}, p_n\right]$– statistical convergence in a real 2–normed space, are defined. Also, some relations between these concepts are investigated.

Keywords: Riesz mean; weighted statistical convergence; sequence space; almost convergence; $2-$norm

1. Introduction

Let (p_k) be a sequence of positive real numbers and $P_n=p_1+p_2+\ldots+p_n$ for n∈IN={1,2,3,…}. Then the Riesz transformation of $x=(x_k)$ is defined as:

$$t_n := \frac{1}{P_n}\sum_{k=1}^{n} p_k x_k. \tag{1}$$

If the sequence (t_n) has a finite limit ξ then the sequence x is said to be (R,p_n)– convergent to ξ. Let us note that if $P_n\to\infty$ as $n\to\infty$ then Riesz transformation is a regular summability method, that is it transforms every convergent sequence to a convergent sequence and preserves the limit. If $p_k=1$ for all k∈IN in the equation (1) then Riesz mean reduces to Cesaro mean C_1 of order one. Related articles can be seen in (Altay and Basar, 2002; Altay and Basar, 2006; Altay and Basar, 2007; Basarir and Kara, 2012; Basarir and Kara, 2011; Basarir and Ozturk, 2008; Basarir and Kayikçi, 2009; Moricz and Orhan, 2004; Polat et al., 2011).

The concept of statistical convergence was introduced by Fast (1951) and Schoenberg (1959), independently for the real sequences. The natural density of a subset E of IN is denoted by

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\left|\{k\in E : k\le n\}\right|,$$

where the vertical bars denote the cardinality of the enclosed set. The concept of statistical convergence plays a considerable role in the summability theory and has been studied as a summability method by many researchers (Connor, 1988; Fridy and Miller, 1991; Fridy and Orhan, 1993; Mursaleen, 2000; Moricz, 2002; Moricz, 2004; Mursaleen and Alotaibi, 2011; Mursaleen and Edely, 2009; Edely and Mursaleen, 2009; Savas, 1992; Basarir et al., 2013; Konca and Basarir, 2013).

In general, statistical convergence of weighted mean is studied as a regular matrix transformation. Karakaya and Chishti (2009) and Mursaleen et al. (2012) have generalized the concept of statistical convergence by using Riesz summability method and they have called this new concept as weighted statistical convergence.

Lorentz (1948) has proved that x is almost convergent to a number ξ if and only if $t_{km}(x)\to\xi$ as $k\to\infty$ uniformly in m, where

$$t_{km}(x)=\frac{x_m+x_{m+1}+\ldots+x_{m+k-1}}{k}, \tag{2}$$

for all k∈IN and $m\ge 0$. Several authors including Lorentz (1948), King (1966), Schafer (1969), Duran (1972) have studied almost convergent sequences.

Maddox (1978) has defined that x is strongly almost convergent to a number ξ if and only if $t_{km}\left(|x-\xi e|\right)\to 0$ as $k\to\infty$, uniformly in m, where $x-\xi e=(x_j-\xi)$ for all j and

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$e=(1,1,1,...)$. This topic has been widely studied (Lorentz, 1948; Maddox, 1978; Maddox, 1979; Maddox, 1967; Maddox, 1986; Basarir, 1992; Das and Mishra, 1983; Das and Patel, 1989; Das and Sahoo, 1992).

The concept of $2-$normed space has been initially introduced by Gähler (1963, 1965) as an interesting non-linear generalization of a normed linear space which has been subsequently studied by many authors (Dutta, 2010; Gunawan and Mashadi, 2001; Gozali and Gunawan, 2010; Raymond et al., 2001; Sahiner et al., 2007; White, 1969; Savas, 2010; Savas, 2011; Basarir et al., 2013; Konca and Basarir, 2013). Statistical convergence has been generalized to the concept of 2–normed space by Gurdal and Pehlivan (2009).

In this paper, we introduce some new weighted almost convergent sequence spaces which are derived by Riesz mean in a real 2–normed space and investigate some topological properties of these spaces. We also define new concepts of statistical convergence which will be called weighted almost statistical convergence, $[\widetilde{R},p_n]-$statistical convergence and almost statistical convergence in a real 2–normed space. Further, we investigate some relations between these concepts.

2. Definitions and preliminaries

A 2–norm on a vector space X of d dimension, where d≥2, is a function $\|\cdot,\cdot\|: X\times X\to\mathbb{R}$ which satisfies the following conditions:

1. $\|x_1,x_2\|=0$ if and only if x_1,x_2 are linearly dependent,
2. $\|x_1,x_2\|=\|x_2,x_1\|$,
3. $\|\alpha x_1,x_2\|=|\alpha|\,\|x_1,x_2\|$ for any α∈IR,
4. $\|x+x',x_1\|\le\|x,x_1\|+\|x',x_1\|$.

The pair $(X,\|\cdot,\cdot\|)$ is then called a 2–normed space.

A trivial example of a 2–normed space is on IR^2 equipped with the following 2–norm:

$$\|x_1,x_2\|_E=abs\left(\begin{vmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{vmatrix}\right), \tag{3}$$

with $x_i=(x_{i1},x_{i2})\in IR^2$ (i=1,2) where the subscript E is for Euclidean. The 2–norm given by the equation (3) corresponds to $\|x_1,x_2\|_E=$: the area of the paralelogram spanned by the vectors x_1 and x_2. The standart 2–norm is exactly the same as the Euclidean 2–norm if X= IR^2 (Gozali and Gunawan, 2010).

A sequence $x=(x_j)$ in a linear 2–normed space X is called a convergent sequence if there exists a ξ∈X such that $\lim_{j\to\infty}\|x_j-\xi,z\|=0$ for every z∈X. If x converges to ξ then we write $x_j\to\xi$ as $j\to\infty$.

Let A and B be any sequence spaces. Throughout the paper, we use the notation $A_{reg}\subset B_{reg}$ to mean for each sequence x is convergent to the limit ξ in A, then the sequence x is convergent to the same limit in B.

The following well-known inequality will be used throughout the paper. Let $q=(q_k)$ be any sequence of positive real numbers with $0<h=\inf_k q_k\le q_k\le\sup_k q_k=H$, $D=\max\{1,2^{H-1}\}$. Then we have for all $a_k,b_k\in\mathbb{C}$ and for all $k\in\mathbb{N}$

$$|a_k+b_k|^{q_k}\le D\left(|a_k|^{q_k}+|b_k|^{q_k}\right), \tag{4}$$

and for $a\in\mathbb{C}$, $|a|^{q_k}\le\max\left\{|a|^h,|a|^H\right\}$.

3. Main results

Let $(X,\|.,.\|)$ be a 2–normed space and $w(\|.,.\|)$, $l^\infty(\|.,.\|)$ be the set of all sequences and all bounded sequences in 2–normed space, respectively. We define the set of all almost convergent sequences and strongly almost convergent sequences by F and [F], respectively in 2–normed space for every nonzero z∈X and for some ξ as follows:

$$F=\left\{\begin{array}{c}x\in l^\infty(\|.,.\|):\lim_{k\to\infty}\|t_{km}(x-\xi e),z\|=0,\\ \text{uniformly in } m.\end{array}\right\}$$

and

$$[F]=\left\{\begin{array}{c}x\in l^\infty(\|.,.\|):\lim_{k\to\infty}t_{km}\left(\|x-\xi e,z\|\right)=0,\\ \text{uniformly in } m.\end{array}\right\}$$

where $t_{km}(x)$ is defined as in the equation (2). We write F–lim x = ξ if x is almost convergent to ξ and [F]–lim x = ξ if x is strongly almost convergent to ξ. Taking advantage of (3) and (4) conditions of

2–norm, the inclusions $[F]\subset F\subset l^{\infty}(\|.,.\|)$ hold from the following inequality:

$$\begin{aligned}\left\|t_{km}(x-\xi e),z\right\| &= \left\|\frac{1}{k}\sum_{i=0}^{k-1}(x_{i+m}-\xi),z\right\|\\ &\le \frac{1}{k}\sum_{i=0}^{k-1}\left\|x_{i+m}-\xi,z\right\|\\ &= t_{km}\left(\|x-\xi e,z\|\right).\end{aligned}$$

We can give the following example to show that the inclusion $[F]\subset F$ is strict.

Example 3.1. Let us take X=IR^2 and consider the 2–normed space as defined in the equation (3). Consider a bounded sequence $x=(x_j)=(\overline{-1},\overline{1},\overline{-1},\overline{1},..)$ where $\bar{l}=(l,l)$ for each $l=-1,1$. Let us consider a basis $\{e_1,e_2\}$ of $\mathbb{R}^2$ where $e_1=(1,0)$ and $e_2=(0,1)$. Since $\|t_{km}(x-0),e_2\|\to 0$ as k→∞ uniformly in m for ξ=0 then x belongs to F, but $t_{km}(\|(x-0),e_2\|)\to 1\neq 0$ as k→∞ uniformly in m. So, x does not belong to [F].

Now, we introduce some new sequence spaces derived by weighted mean and notions of almost and strongly almost convergence in a real 2–normed space for every nonzero z∈X as follows:

$$\left[\tilde{R},p_n,\|.,.\|\right]=\left\{\begin{array}{l}x:\lim\limits_{n\to\infty}\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{km}(x-\xi e),z\right\|=0,\\ \text{uniformly in m, for some }\xi.\end{array}\right\},$$

$$\left(\tilde{R},p_n,\|.,.\|\right)=\left\{\begin{array}{l}x:\lim\limits_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_k\left\|t_{km}(x-\xi e),z\right\|=0,\\ \text{uniformly in } m\text{, for some }\xi.\end{array}\right\},$$

$$\left|\tilde{R},p_n,\|.,.\|\right|=\left\{\begin{array}{l}x:\lim\limits_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{km}\left(\|x-\xi e,z\|\right)=0,\\ \text{uniformly in } m\text{, for some }\xi.\end{array}\right\}$$

If we take $m=0$ then the sequence spaces $\left[\tilde{R},p_n,\|.,.\|\right]$, $\left(\tilde{R},p_n,\|.,.\|\right)$, $\left|\tilde{R},p_n,\|.,.\|\right|$ are reduced to the sequence spaces $\left[C,1,\|.,.\|\right]$, $\left(C,1,\|.,.\|\right)$, $\left|C,1,\|.,.\|\right|$, respectively as follows

$$\left[C,1,\|.,.\|\right]=\left\{\begin{array}{l}x:\lim\limits_{n\to\infty}\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{k0}(x-\xi e),z\right\|=0,\\ \text{for some }\xi.\end{array}\right\},$$

$$\left(C,1,\|.,.\|\right)=\left\{\begin{array}{l}x:\lim\limits_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_k\left\|t_{k0}(x-\xi e),z\right\|=0,\\ \text{for some }\xi.\end{array}\right\},$$

$$\left|C,1,\|.,.\|\right|=\left\{\begin{array}{l}x:\lim\limits_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{k0}\left(\|x-\xi e,z\|\right)=0,\\ \text{for some }\xi.\end{array}\right\}.$$

Let Z be any sequence space. If x∈Z and $x_j\to\xi$ as j→∞, then x is said to be Z–convergent to ξ.

Now, we give the following theorem to demonstrate some inclusion relations among the sequence spaces above with the spaces F and [F].

Theorem 3.2. Let $P_n\to\infty$ as n→∞. Then the following statements are true:

1. $[F]\subset F\subset\left(\tilde{R},p_n,\|.,.\|\right)\subset\left[\tilde{R},p_n,\|.,.\|\right]\subset\left[C,1,\|.,.\|\right]$.
2. $[F]\subset\left|\tilde{R},p_n,\|.,.\|\right|\subset\left(\tilde{R},p_n,\|.,.\|\right)\subset\left[\tilde{R},p_n,\|.,.\|\right]\subset\left[C,1,\|.,.\|\right]$.
3. $[F]\subset\left|\tilde{R},p_n,\|.,.\|\right|\subset\left|C,1,\|.,.\|\right|\subset\left(C,1,\|.,.\|\right)\subset\left[C,1,\|.,.\|\right]$.

Proof: We give the proof only for (2). The proofs of (1) and (3) can be done, similarly. So we omit them.

Let $x\in[F]$ and [F]–lim x = ξ. Then $t_{km}\left(\|x-\xi e,z\|\right)\to 0$ as $k\to\infty$, uniformly in m. Since $P_n\to\infty$ as $n\to\infty$, then its weighted mean also converges to ξ as n→∞ uniformly in m. This proves that $x\in\left|\tilde{R},p_n,\|.,.\|\right|$ and $\left|\tilde{R},p_n,\|.,.\|\right|-\lim$ x = [F]–lim x = ξ. Also, since

$$\begin{aligned}&\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{km}(x-\xi e),z\right\|\\ &\le\frac{1}{P_n}\sum_{k=1}^{n}p_k\left\|t_{km}(x-\xi e),z\right\|\\ &\le\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{km}\left(\|x-\xi e,z\|\right),\end{aligned}$$

it follows that $[F]\subset\left|\tilde{R},p_n,\|.,.\|\right|\subset\left(\tilde{R},p_n,\|.,.\|\right)\subset\left[\tilde{R},p_n,\|.,.\|\right]$ and [F]–lim x=$\left(\tilde{R},p_n,\|.,.\|\right)-\lim$ x =

$\left[\tilde{R}, p_n, \|.,.\|\right] - \lim$ x=ξ. Since uniform convergence of $\left\| \frac{1}{P_n} \sum_{k=1}^{n} p_k t_{km}(x-\xi e), z \right\|$ with respect to m as n→∞ implies convergence for m=0 it follows that $\left[\tilde{R}, p_n, \|.,.\|\right] \subset \left[C,1,\|.,.\|\right]$ with $\left[\tilde{R}, p_n, \|.,.\|\right] - \lim x = \left[C,1,\|.,.\|\right] - \lim$ x=ξ. This completes the proof.

Let x be a sequence in 2–normed space $(X, \|.,.\|)$. The sequence x is said to be statistically convergent to ξ if for every $\varepsilon > 0$ the set $\{ j \in \mathbb{N}: \| x_j - \xi, z \| \geq \varepsilon \}$ has natural density zero for every nonzero z∈X, in other words, x is statistically convergent to ξ in 2–normed space $(X, \|.,.\|)$ if

$$\lim_{n\to\infty} \frac{1}{n} |\{ j \in \mathbb{N} : \| x_j - \xi, z \| \geq \varepsilon \}| = 0$$

for every nonzero z∈X.

Now, we define the new type of statistical convergence and investigate some inclusion relations.

Definition 3.3. A sequence x is said to be weighted almost statistically convergent to ξ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n} \left| \left\{ k \leq P_n : p_k \left\| t_{km}(x-\xi e), z \right\| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m, for every nonzero z∈X. By $\left(S_{\tilde{R}}, \|.,.\|\right)$, we denote the set of all weighted almost statistically convergent sequences in a 2–normed space.

In the definition above, if we take $p_k=1$ for all k∈IN then we obtain the definition of almost statistical convergence. That is, x is called almost statistically convergent to ξ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left\| t_{km}(x-\xi e), z \right\| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m, for every nonzero z∈X. We denote the set of all almost statistically convergent sequences in a 2–normed space by $\left(S, \|.,.\|\right)$.

Theorem 3.4. If the sequence x is $\left(\tilde{R}, p_n, \|.,.\|\right)-$ convergent to ξ then the sequence x is weighted almost statistically convergent to ξ.

Proof: Let the sequence x be $\left(\tilde{R}, p_n, \|.,.\|\right)$-convergent to ξ and

$$K_{nm}(\varepsilon) = \left\{ k \leq P_n : p_k \left\| t_{km}(x-\xi e), z \right\| \geq \varepsilon \right\}.$$

Then for a given $\varepsilon > 0$, we have

$$\frac{1}{P_n} \sum_{k=1}^{n} p_k \left\| t_{km}(x-\xi e), z \right\|$$

$$\geq \frac{1}{P_n} \sum_{\substack{k=1 \\ k\in K_{nm(\varepsilon)}}}^{n} p_k \left\| t_{km}(x-\xi e), z \right\|$$

$$\geq \varepsilon \frac{1}{P_n} \left| \left\{ k \leq P_n : p_k \left\| t_{km}(x-\xi e), z \right\| \geq \varepsilon \right\} \right|$$

for each $m \geq 0$ and for every nonzero z∈X. Hence we obtain that the sequence x is weighted almost statistically convergent to ξ by taking the limit as n→∞.

Now, we give a new definition which will be used in the next theorem:

Definition 3.5. A sequence x is said to be $[\tilde{R}, p_n]-$ statistically convergent to ξ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left\| \omega_{nm}(x-\xi e), z \right\| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m, for every nonzero z∈X, where

$$\omega_{nm}(x-\xi e) = \frac{1}{P_n} \sum_{k=1}^{n} p_k t_{km}(x-\xi e).$$

By $(S_{[\tilde{R}, p_n]})$, we denote the set of all $[\tilde{R}, p_n]-$ statistically convergent sequences in 2–normed space.

Theorem 3.6. Let $P_n \to \infty$ as n→∞ and $p_k \left\| t_{km}(x-\xi e), z \right\| \leq M$ for all k∈IN, for each $m \geq 0$ and for every nonzero z∈X. Then the following statements are true:

1. $\left(S_{\tilde{R}}, \|.,.\|\right)_{reg} \subset \left(\tilde{R}, p_n, \|.,.\|\right)_{reg}$.
2. $\left(S_{\tilde{R}}, \|.,.\|\right)_{reg} \subset \left(S_{[\tilde{R}, p_n]}\right)_{reg}$.

Proof: 1. Let x be convergent to ξ in $\left(S_{\tilde{R}},\|.,.\|\right)$ and let us take

$$K_{nm}(\varepsilon)=\left\{k\le P_n : p_k\left\|t_{km}(x-\xi e),z\right\|\ge\varepsilon\right\}.$$

Since $p_k\left\|t_{km}(x-\xi e),z\right\|\le M$ for all k∈IN, for each $m\ge 0$, for every nonzero z∈X and $P_n\to\infty$ as n→∞, then for a given $\varepsilon>0$ we have

$$\frac{1}{P_n}\sum_{k=1}^{n}p_k\left\|t_{km}(x-\xi e),z\right\|$$
$$=\frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{nm(\varepsilon)}}}^{n}p_k\left\|t_{km}(x-\xi e),z\right\|$$
$$+\frac{1}{P_n}\sum_{\substack{k=1\\ k\notin K_{nm(\varepsilon)}}}^{n}p_k\left\|t_{km}(x-\xi e),z\right\|$$
$$\le M\frac{1}{P_n}\left|\left\{k\le P_n : p_k\left\|t_{km}(x-\xi e),z\right\|\ge\varepsilon\right\}\right|+\frac{n}{P_n}\varepsilon$$
$$\le M\frac{1}{P_n}\left|\left\{k\le P_n : p_k\left\|t_{km}(x-\xi e),z\right\|\ge\varepsilon\right\}\right|+\varepsilon,$$

for each $m\ge 0$ and for every nonzero z∈X. Since ε is arbitrary, we have $x\in\left(\tilde{R},p_n,\|.,.\|\right)$ by taking the limit as n→∞.

2. Let x be convergent to ξ in $\left(S_{\tilde{R}},\|.,.\|\right)$, then $lim_n\to\infty\frac{1}{P_n}\left|K_{nm}(\varepsilon)\right|=0$, where

$$K_{nm}(\varepsilon)=\left\{k\le P_n : p_k\left\|t_{km}(x-\xi e),z\right\|\ge\varepsilon\right\}.$$

Then for each $m\ge 0$ and for every nonzero z∈X we have

$$\left\|\omega_{nm}(x-\xi e),z\right\|$$
$$=\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{km}(x-\xi e),z\right\|$$
$$=\left\|\frac{1}{P_n}\left(\sum_{\substack{k=1\\ k\in K_{nm(\varepsilon)}}}^{n}+\sum_{\substack{k=1\\ k\notin K_{nm(\varepsilon)}}}^{n}\right)p_k t_{km}(x-\xi e),z\right\|$$
$$\le\frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{nm(\varepsilon)}}}^{n}p_k\left\|t_{km}(x-\xi e),z\right\|$$
$$+\frac{1}{P_n}\sum_{\substack{k=1\\ k\notin K_{nm(\varepsilon)}}}^{n}p_k\left\|t_{km}(x-\xi e),z\right\|$$
$$\le\frac{M}{P_n}\left|K_{nm(\varepsilon)}\right|+\frac{n}{P_n}\varepsilon$$

which leads us by taking limit as n→∞, uniformly in m so that x converges to ξ in $\left[\tilde{R},p_n,\|.,.\|\right]$. Hence, we can say that the sequence x is $[\tilde{R},p_n]-$statistically convergent to ξ. This completes the proof.

Theorem 3.7. The following statements are true:

1. If $p_k\le 1$ for all k∈IN then $\left(S,\|.,.\|\right)_{reg}\subseteq\left(S_{\tilde{R}},\|.,.\|\right)_{reg}$.

2. Let $p_k\ge 1$ for all k∈IN and $\left(\frac{P_n}{n}\right)$ be upper-bounded, then $\left(S_{\tilde{R}},\|.,.\|\right)_{reg}\subseteq\left(S,\|.,.\|\right)_{reg}$.

Proof: 1. If $p_k\le 1$ for all k∈IN, then $P_n\le n$ for all n∈IN. So, there exist M_1 and M_2 constants such that $0<M_1\le\frac{P_n}{n}\le M_2\le 1$ for all n∈IN. Let x be a sequence which converges to the limit ξ in $(S,\|.,.\|)$, so for an arbitrary $\varepsilon>0$ we have

$$\frac{1}{P_n}\left|\left\{k\le P_n : p_k\left\|t_{km}(x-\xi e),z\right\|\ge\varepsilon\right\}\right|$$
$$\le\frac{1}{nM_1}\left|\left\{k\le P_n\le nM_2\le n : \left\|t_{km}(x-\xi e),z\right\|\ge\varepsilon\right\}\right|$$
$$\le\frac{1}{M_1}.\frac{1}{n}\left|\left\{k\le n : \left\|t_{km}(x-\xi e),z\right\|\ge\varepsilon\right\}\right|,$$

for each $m\ge 0$ and for every nonzero z∈X. So we have the result by taking limit as n→∞.

2. Let $\left(\frac{P_n}{n}\right)$ be upper-bounded and $p_k \geq 1$ for all k∈IN, then $P_n \geq n$ for all n∈IN and there exist M_1 and M_2 constants such that $1 \leq M_1 \leq \frac{P_n}{n} \leq M_2 < \infty$ for all n∈IN. If x converges to the limit ξ in $(S_{\tilde{R}}, \|.,.\|)$, then we have

$$\frac{1}{n}\left|\left\{k \leq n : \left\|t_{km}\left(x-\xi e\right), z\right\| \geq \varepsilon\right\}\right|$$
$$\leq M_2 \frac{1}{P_n}\left|\left\{k \leq n \leq P_n : p_k \left\|t_{km}\left(x-\xi e\right), z\right\| \geq \varepsilon\right\}\right|$$
$$= M_2 \frac{1}{P_n}\left|\left\{k \leq P_n : p_k \left\|t_{km}\left(x-\xi e\right), z\right\| \geq \varepsilon\right\}\right|$$

for each $m \geq 0$ and for every nonzero z∈X. Hence, we have the result by taking the limit as n→∞.

A paranormed space (X, g) is a topological linear space with the topology given by the paranorm g. It may be recalled that a paranorm g is a real subadditive function on X such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e. $\lambda_r \to \lambda$, $g(x^r - x) \to 0$ as r→∞ imply that $g(\lambda_r x^r - \lambda x) \to 0$ as r→∞ where λ_r, λ are scalars and x^r, x∈X.

Now, we introduce a new sequence space for some ξ and for every nonzero z∈X as follows:

$$\left(\tilde{R}, p_n, \|.,.\|, q_n\right) = \left\{ \begin{array}{l} x : \frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^{q_k} \to 0, \\ \text{as } n \to \infty, \text{uniformly in } m. \end{array} \right\}$$

where (q_k) is a bounded sequence of strictly positive real numbers with $h = \inf_k q_k$ and $H = \sup_k q_k$. If (q_k) is constant, then $(\tilde{R}, p_n, \|.,.\|, q)$ is reduced to $(\tilde{R}, p_n, \|.,.\|)_q$. If we take q_k=1 for all k∈IN then we get the sequence space $(\tilde{R}, p_n, \|.,.\|)$ which is defined in the beginning of this section.

Theorem 3.8. Let P_n→∞ as n→∞ and (q_k) be a bounded sequence of strictly positive real numbers with $h = \inf_k q_k$, $H = \sup_k q_k < \infty$ and $M = \max(1, H)$. Then $(\tilde{R}, p_n, \|.,.\|, q_n)$ is a linear topological space paranormed (need not be total) by

$$g(x) = \sup_{\substack{n\geq 1, m\geq 1\\ \theta\neq z\in X}} \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x\right), z\right\|^{q_k}\right)^{\frac{1}{M}}$$

and $(\tilde{R}, p_n, \|.,.\|, q)$ is a seminormed sequence space by $\|x\| = \sup_{\substack{n\geq 1, m\geq 1\\ \theta\neq z\in X}} \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x\right), z\right\|^{q}\right)^{\frac{1}{q}}$.

Proof: It is easy to see that $(\tilde{R}, p_n, \|.,.\|, q_n)$ is a linear space with coordinatewise addition and scalar multiplication. We will prove that g(x) is a paranorm on $(\tilde{R}, p_n, \|.,.\|, q_n)$. We omit the proof of the case $q_k = q \geq 1$ for all k∈IN in which $\|x\|$ is a seminorm.

Clearly $g(\theta) = 0$, $g(x) = g(-x)$ and g is subadditive. To prove the continuity of scalar multiplication, assume that (x^r) be any sequence of the points in $(\tilde{R}, p_n, \|.,.\|, q_n)$ such that $g(x^r - x) \to 0$ as r→∞ and (λ_r) be any sequence of scalars such that $\lambda_r \to \lambda$ as r→∞. Since the inequality

$$g(x^r) \leq g(x) + g(x^r - x)$$

holds by subadditivity of g , $g(x^r)$ is bounded. Thus, by using Minkowski's inequality for $q_k \geq 1$ we have

$$g(\lambda_r x^r - \lambda x)$$
$$= \sup_{\substack{n\geq 1, m\geq 1\\ \theta\neq z\in X}} \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(\lambda_r x^r - \lambda x\right), z\right\|^{q_k}\right)^{\frac{1}{M}}$$
$$\leq \left(\max\left\{\left|\lambda_r - \lambda\right|^h, \left|\lambda_r - \lambda\right|^H\right\}\right)^{\frac{1}{M}} g(x^r)$$
$$+ \max\left(\left|\lambda\right|^h, \left|\lambda\right|^H\right)^{\frac{1}{M}} g(x^r - x)$$

which tends to zero as r→∞. Moreover, the result holds for $0 < q_k < 1$ by using the equation (4). This proves the fact that g is a paranorm on $(\widetilde{R}, p_n, \|.,.\|, q_n)$.

Theorem 3.9. If the following conditions hold, then $\left(\tilde{R}, p_n, \|.,.\|, q\right)_{reg} \subset \left(S_{\tilde{R}}, \|.,.\|\right)_{reg}$.

1. $0 < q < 1$ and $0 \le \left\|t_{km}\left(x-\xi e\right), z\right\| < 1$.
2. $1 \le q < \infty$ and $1 \le \left\|t_{km}\left(x-\xi e\right), z\right\| < \infty$.

Proof: Let a sequence x be $\left(\tilde{R}, p_n, \|.,.\|, q\right)-$ convergent to the limit ξ. Since $p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q \ge p_k \left\|t_{km}\left(x-\xi e\right), z\right\|$ for case (1) and (2), then we have

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q$$
$$\ge \frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|$$
$$\ge \frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|$$
$$\ge \varepsilon \frac{1}{P_n}\left|K_{nm}(\varepsilon)\right|$$

for each $m \ge 0$ and for every nonzero z∈X. We get the result if we take the limit as n→∞. That is,

$\lim_{n\to\infty} \frac{1}{P_n}\left|K_{nm}(\varepsilon)\right| = 0$, where

$$K_{nm}\left(\varepsilon\right) = \left\{k \le P_n : p_k \left\|t_{km}\left(x-\xi e\right), z\right\| \ge \varepsilon\right\}.$$

Hence x converges to ξ in $\left(S_{\tilde{R}}, \|.,.\|\right)$. This completes the proof.

Theorem 3.10. Let $p_k \left\|t_{km}\left(x-\xi e\right), z\right\| \le M$ for all k∈IN, for each $m \ge 0$, for every nonzero z∈X and P_n→∞ as n→∞. If the following conditions hold then $\left(S_{\tilde{R}}, \|.,.\|\right)_{reg} \subset \left(\tilde{R}, p_n, \|.,.\|, q\right)_{reg}$.

1. $0 < q < 1$ and $1 \le \left\|t_{km}\left(x-\xi e\right), z\right\| < \infty$.
2. $1 \le q < \infty$ and $0 \le \left\|t_{km}\left(x-\xi e\right), z\right\| < 1$.

Proof: Assume that x converges to ξ in $\left(S_{\tilde{R}}, \|.,.\|\right)$ and P_n→∞ as n→∞. Then for $\varepsilon > 0$, we have $\delta(K_{nm}(\varepsilon)) = 0$ where

$K_{nm}(\varepsilon) = \left\{k \le P_n : p_k \left\|t_{km}\left(x-\xi e\right), z\right\| \ge \varepsilon\right\}$.

Since $p_k \left\|t_{km}\left(x-\xi e\right), z\right\| \le M$ for all k∈IN for each $m \ge 0$ and for every nonzero z∈X then we have

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q$$
$$\le \frac{1}{P_n}\sum_{\substack{k=1\\ k\notin K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q$$
$$+\frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q$$
$$= T_n + T_n'$$

for each $m \ge 0$ and for every nonzero z∈X where

$$T_n = \frac{1}{P_n}\sum_{\substack{k=1\\ k\notin K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q$$

and

$$T'_n = \frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q.$$

For $k \notin K_{nm}(\varepsilon)$, we have

$$T_n = \frac{1}{P_n}\sum_{\substack{k=1\\ k\notin K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}\left(x-\xi e\right), z\right\|^q$$
$$< \frac{n}{P_n}\varepsilon \le \varepsilon$$

for each $m \ge 0$ and for every nonzero z∈X. If $k \in K_{nm}(\varepsilon)$, then

$$T'_n = \frac{1}{P_n} \sum_{\substack{k=1 \\ k \in K_{nm}(\varepsilon)}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q$$

$$\leq \frac{1}{P_n} \sum_{\substack{k=1 \\ k \in K_{nm}(\varepsilon)}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|$$

$$\leq \frac{M}{P_n} \left| K_{nm}\left(\varepsilon\right) \right|$$

for each $m \geq 0$ and for every nonzero z∈X. If we take the limit as $n \to \infty$, since $\delta(K_{nm}(\varepsilon)) = 0$ x converges to ξ in $(\widetilde{R}, p_n, \|.,.\|)_q$. This completes the proof.

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