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Vertex centered crossing number for maximal planar graph

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Abstract

The crossing number of a graph G is the minimum number of edge crossings over all possible drawings of G in a plane. The crossing number is an important measure of the non-planarity of a graph, with applications in discrete and computational geometry and VLSI circuit design. In this paper we introduce vertex centered crossing number and study the same for maximal planar graph.

Keywords: Maximal planar; crossing number; vertex-centered crossing number

1. Introduction

A *drawing* (Sergio Cabello and Bojan Mohar, 2010) of a graph G in the plane is a representation of G where vertices are represented by distinct points of $\mathbb{R}^2$, edges are represented by simple polygonal arcs in $\mathbb{R}^2$ joining points that correspond to their endvertices, and the interior of every arc representing an edge contains no points representing the vertices of G. A crossing of a drawing $\mathcal{D}$ is a pair $(\{e, e'\}, p)$, where e and e' are distinct edges and $p \in \mathbb{R}^2$ is a common point that belongs to the interior of both arcs representing e and e' in the drawing $\mathcal{D}$. A drawing $\mathcal{D}$ is said to be *good*, if it satisfies the following conditions:

1. no edge crosses itself
2. adjacent edges do not cross each other
3. non-adjacent edges cross each other at most once
4. atmost two edges crosses at a point
5. no two edges are tangential

The number of crossings of a good drawing $\mathcal{D}$ is denoted by $Cr(\mathcal{D})$ and is called crossing number of the drawing. The *crossing number* $Cr(G)$ of a graph G is the minimum $Cr(\mathcal{D})$ taken over all good drawings of G. A drawing of G with exactly $Cr(G)$ crossings is said to be an optimal drawing. A planar graph is a graph whose crossing number is 0. A drawing $\mathcal{D}$ with $Cr(\mathcal{D}) = 0$ is called an embedding of G (in the plane). The drawings considered in this paper are all good.

A plane graph G is called *maximal planar* if, for every pair u, v of nonadjacent vertices of G, the graph $G + uv$ is nonplanar. A triangulation is a planar graph in which every face is bounded by three edges (including its infinite face). In any embedding of a maximal planar graph G of order at least 3, the boundary of every region of G is a triangle and has precisely $3n - 6$ edges and $2n - 4$ faces. Thus, we can say that each maximal planar graph is a triangulation.

In this paper we consider only finite simple undirected graphs. Let the graph be $G = (V, E)$ with $|V| = n$ and $|E| = m$. We deal with the complete graphs K_n, which has n vertices and with all possible $\binom{n}{2}$ edges. A graph is planar if it has an embedding on the plane. The complete graphs K_1, K_2, K_3 and K_4 are planar. But K_n for $n \geq 5$ is non-planar. We construct planar graphs from K_n $(n \geq 5)$.

Usually, we exclude the edges $\{(v_k, v_l) \mid k = 3, 4, \dots, n-2; l = k+2, k+3, \dots, n\}$ from the edges $\{(v_i, v_j) \mid i < j, 1 \leq i, j \leq n\}$ of K_n, to obtain a new class of graphs $Pl_n: n \geq 5$ which are planar.

Definition 1.1. The graph (J. Baskar Babujee, 2003) $Pl_n = (V, E)$ with a vertex set $V = \{v_1, v_2, \dots, v_n\}$ and an edge set $E = E(K_n) \setminus \{(v_k, v_l) \mid k = 3, 4, \dots, n-2; l = k+2, k+3, \dots, n\}$ is a planar graph having maximum number of edges, with n vertices.

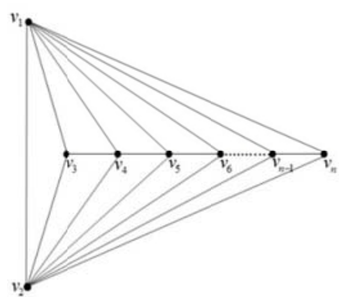

Fig 1. The Maximal Planar Graph, Pl_n

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The Pl_n-class can be valued as $P_2 + P_{n-2}$, where P_2 represents an edge v_1v_2 and P_{n-2} represents the path $v_3v_4v_5 \dots v_n$. The vertices $v_1, v_2 \in V(Pl_n)$ are the central vertices of Pl_n with maximum vertex degree $n-1$ and $v_3, v_n \in V(Pl_n)$ have the minimum vertex degree 3. All remaining vertices $v_i \in V(Pl_n): 4 \le i \le n-1$ holds the degree 4 in each Pl_n graph. Also, the distance between any two vertices in each Pl_n graph is atmost two.

Let (K. R. Parthasarathy, 1994) $G = (V, E)$ be a simple connected graph, with n vertices. The length of any shortest path between u and v of a connected graph G is called the distance between u and v and is denoted by $d(u, v)$. For any vertex v of G, the eccentricity of a vertex v is given by $e(v) = max\{d(u, v) \mid u \in V(G)\}$. Then the radius of G is $r = min\{e(v) \mid v \in V(G)\}$ and the centre of G is $C(G) = \{v \in V(G) \mid e(v) = r\}$. The degree of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v. Let $\Delta(G)$ denote the maximum vertex degree and $\delta(G)$ denote the minimum vertex degree of a graph G. We introduce a vertex centered crossing number and study the same for maximal planar graph in the following section.

2. Main Results

Definition 2.1. Let $G = (V, E)$ be a simple connected graph, with n vertices and $v \in V(G)$ be any arbitrary vertex with $d_G(v) = k < n-1$. Then G_v is a simple graph obtained from G, by adding $(n-k-1)$ edges that connect all the vertices of G to v which are not adjacent to v. So the vertex v becomes a central vertex of a graph G_v with $d_{G_v}(v) = n-1$. The minimum crossing number of a graph G_v is called a *vertex centered crossing number* $VCR(G_v)$ (or $VCR(G_{v,n})$) of a graph G of order n, with respect to a vertex $v \in V(G)$.

In this paper, we consider G as a maximal planar graph Pl_n. The construction of graphs $\{G_{v_i}: 3 \le i \le n\}$ from a maximal planar graph Pl_n and finding their corresponding minimum crossing number is our interest.

Theorem 2.2. If G is a maximal planar graph Pl_n, with odd order $n \ge 5$ and obtaining the simple graph G_{v_i} from G by joining $v_i \in V(G)$ to all its non-adjacent vertices in G, then the vertex centered crossing number of a graph G with respect to v_i, for each $i = 3, 4, \dots, n$ is given by,

$$VCR(G_{v_i}) = \left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right) + \left|\frac{n+3}{2} - i\right|,$$

for $3 \le i \le n$.

Proof: Let G be a maximal planar graph Pl_n, with odd order $n \ge 5$ and G_{v_i} be a simple graph obtained from G by including the edges $\{v_iv_j \mid 3 \le j \le i-2\} \cup \{v_iv_j \mid i+2 \le j \le n\}$ which are pair-wise non-crossing, in any of its good drawing.

In a Pl_n graph, v_1 and v_2 are the vertices with $\Delta(G) = n-1$; v_3 and v_n are the vertices with $\delta(G) = 3$; and the remaining vertices $v_i: 4 \le i \le n-1$, are with degree 4. Hence v_1 and v_2 are the central vertices of a graph G with maximum vertex degree $n-1$. Thus we obtain G_{v_i}, for $3 \le i \le n$.

We prove the theorem by the method of induction on i.

The basis step involves $i = 3$.

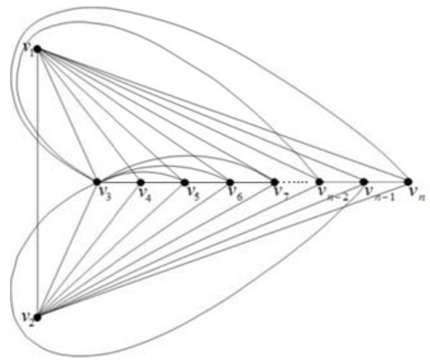

Fig 2. The Graph G_{v_3} obtained from a Maximal Planar Graph Pl_n

$E(G_{v_3}) = E(Pl_n) \cup \{v_3v_j \mid 5 \le j \le n\}$

Since G is a maximal planar graph, we have,
$Cr(G) = 0$;
$Cr(G + uv) > 0$, for any $u, v \in V(G)$ and $uv \notin E(G)$.
From the Fig. 2, we can observe that,
$Cr(G + v_3v_5) = Cr(G + v_3v_n) = 1, \forall\, n \ge 5$
$Cr(G + v_3v_6) = Cr(G + v_3v_{n-1}) = 2, \forall\, n \ge 7$
$Cr(G + v_3v_7) = Cr(G + v_3v_{n-2}) = 3, \forall\, n \ge 9$
$Cr(G + v_3v_j) = Cr(G + v_3v_{(n+5)-j}) = j - 4,$

$$\forall\, n \ge 2j-5 \text{ and } 5 \le j \le \tfrac{n+5}{2} \qquad (1)$$

Hence the maximum crossing edge v_3v_j, incident with a vertex v_3, falls on the median value from 5 to n, i.e., on $\frac{n+5}{2}$, for any finite odd value of $n \ge 5$.

Thus by substituting $j = \frac{n+5}{2}$, in equation (1), we get,

$$Cr\left(G + v_3v_{\frac{n+5}{2}}\right) = Cr\left(G + v_3v_{(n+5)-\frac{n+5}{2}}\right) = \frac{n+5}{2} - 4$$

$$\Rightarrow Cr\left(G + v_3v_{\frac{n+5}{2}}\right) = \frac{n-3}{2}$$

For $j = \frac{n+3}{2}$, the equation (1) becomes,

$$Cr\left(G+v_3v_{\frac{n+3}{2}}\right)=Cr\left(G+v_3v_{(n+5)-\frac{n+3}{2}}\right)$$
$$=\frac{n+3}{2}-4$$
$$\Rightarrow\ Cr\left(G+v_3v_{\frac{n+3}{2}}\right)=Cr\left(G+v_3v_{\frac{n+7}{2}}\right)$$
$$=\frac{n-5}{2}$$
$$\Rightarrow\ Cr\left(G+v_3v_{\frac{n+3}{2}}\right)=Cr\left(G+v_3v_{\frac{n+7}{2}}\right)$$
$$=\frac{n-3}{2}-1$$

Similarly,

$$Cr\left(G+v_3v_{\frac{n+1}{2}}\right)=Cr\left(G+v_3v_{\frac{n+9}{2}}\right)=\frac{n-3}{2}-2$$
$$\dots$$
$$Cr(G+v_3v_5)=Cr(G+v_3v_n)=1$$
$$\Rightarrow Cr\left(G+v_3v_{\frac{n+3}{2}}\right)+Cr\left(G+v_3v_{\frac{n+1}{2}}\right)+\cdots+Cr(G+v_3v_5)$$
$$=Cr\left(G+v_3v_{\frac{n+7}{2}}\right)+Cr\left(G+v_3v_{\frac{n+9}{2}}\right)+\cdots+Cr(G+v_3v_n)$$
$$\Rightarrow\ \sum_{j=5}^{n+3/2}Cr(G+v_3v_j)=\sum_{j=n+7/2}^{n}Cr(G+v_3v_j)$$
$$\Rightarrow\ VCR(G_{v_3})=\sum_{j=5}^{n}Cr(G+v_3v_j)$$
$$=\sum_{j=5}^{n+3/2}Cr(G+v_3v_j)+Cr\left(G+v_3v_{\frac{n+5}{2}}\right)+\sum_{j=n+7/2}^{n}Cr(G+v_3v_j)$$
$$=2\sum_{j=5}^{n+3/2}Cr(G+v_3v_j)+Cr\left(G+v_3v_{\frac{n+5}{2}}\right)$$
$$=2\left(1+2+3+\cdots+\frac{n-5}{2}\right)+\frac{n-3}{2}$$
$$=2\left[\frac{1}{2}\left(\frac{n-5}{2}\right)\left(\frac{n-5}{2}+1\right)\right]+\frac{n-3}{2}$$
$$=\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+\left(\frac{n+3}{2}-3\right)$$
$$\Rightarrow VCR(G_{v_3})=\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+\left|\frac{n+3}{2}-3\right|$$

Thus, $VCR(G_{v_i})=\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+\left|\frac{n+3}{2}-i\right|$, for $i=3$. We inductively assume that the result holds for $i=n-1$. That is,

$$VCR(G_{v_{n-1}})=\sum_{j=3}^{n-3}Cr(G+v_{n-1}v_j)$$
$$=\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+\left|\frac{n+3}{2}-(n-1)\right|$$

Let us prove the result for $i=n$.

By observing Fig. 3 and Fig. 4, we have,

$$Cr(G+v_nv_j)=Cr(G+v_{n-1}v_{j-1}),4\le j\le n-2$$
$$E(G_{v_n})=E(Pl_n)\cup\{v_nv_j\ |3\le j\le n-2\}$$
$$VCR(G_{v_n})=\sum_{j=3}^{n-2}Cr(G+v_nv_j)$$
$$=Cr(G+v_nv_3)+Cr(G+v_nv_4)+\cdots+Cr(G+v_nv_{n-2})$$

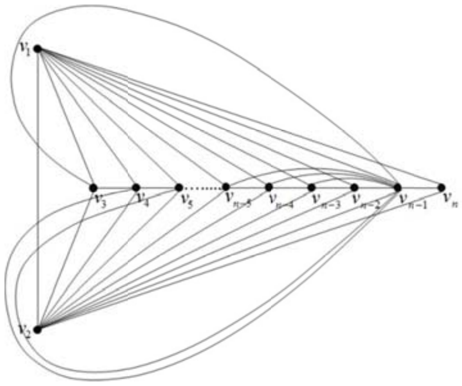

Fig 3. The Graph $G_{v_{n-1}}$ obtained from a Maximal Planar Graph Pl_n

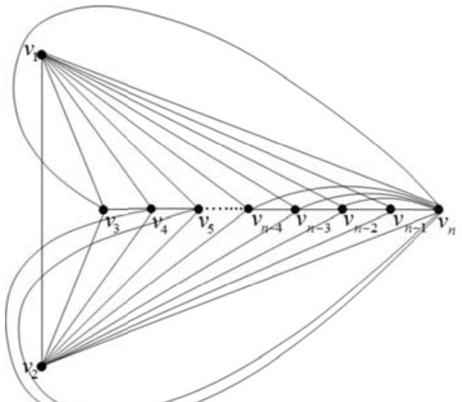

Fig 4. The Graph G_{v_n} obtained from a Maximal Planar Graph Pl_n

$$\Rightarrow VCR(G_{v_n})=Cr(G+v_nv_3)+Cr(G+v_{n-1}v_3)+\cdots+Cr(G+v_{n-1}v_{n-3})$$
$$=1+\sum_{j=3}^{n-3}Cr(G+v_{n-1}v_j)$$
$$=1+\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+\left|\frac{n+3}{2}-(n-1)\right|$$
$$=1+\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+(n-1)-\frac{n+3}{2}$$
$$=\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+n-\frac{n+3}{2}$$
$$VCR(G_{v_n})=\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)+\left|\frac{n+3}{2}-n\right|$$

Thus, the theorem is true for $i=n$.

Hence, the theorem holds for $\forall\ i=3,4,\dots,n$.

Corollary 2.3. For any odd value of $n\ge 5$,

$(i). VCR(G_{v_i}) = VCR\left(G_{v_{(n+3)-i}}\right), 3 \le i \le \frac{n+3}{2}$;
$(ii). VCR(G_{v_{i-1}}) = VCR(G_{v_i}) + 1, 4 \le i \le \frac{n+3}{2}$
and
$(iii). VCR(G_{v_{i+1}}) = VCR(G_{v_i}) + 1, \frac{n+3}{2} \le i \le n-1.$

Proof: It is easy to verify that, for any $G_{v_i}: 3 \le i \le n$ obtained from Pl_n,

$$VCr(G_{v_i}) = \sum_{j=3}^{i-2} Cr(G + v_i v_j) + \sum_{j=i+2}^{n} Cr(G + v_i v_j)$$

For any finite odd number $n \ge 5$, the minimum crossing number of a graph $G + v_i v_j$, corresponding to each pair v_i, v_j of non-adjacent vertices of Pl_n included in the set $\{(v_i, v_j) \mid i = 3, 4, \dots, n-2; j = i+2, i+3, \dots, n\}$ is given in Table 1:

Hence, by observing each column in Table 1, the results hold for any odd $n \ge 5$.

Table 1. The Crossing Number of an Odd Order Graph $G + v_i v_j$ Corresponding to each Pair v_i, v_j of Non-adjacent Vertices of Pl_n

v_i / $v_i v_j$	Vertices in P_{n-2}								$Cr(G+v_iv_j)$
	v_3	…	$v_{(n+1)/2}$	$v_{(n+3)/2}$	$v_{(n+5)/2}$	…	v_{n-1}	v_n	
Edges incident with v_i	v_3v_5	…	$v_{(n+1)/2}v_{(n+5)/2}$	$v_{(n+3)/2}v_{(n+7)/2}$	$v_{(n+5)/2}v_{(n+9)/2}$	…	–	v_nv_3	1
	v_3v_6		$v_{(n+1)/2}v_{(n+7)/2}$	$v_{(n+3)/2}v_{(n+9)/2}$	$v_{(n+5)/2}v_{(n+11)/2}$		$v_{n-1}v_3$	v_nv_4	2
	v_3v_7		$v_{(n+1)/2}v_{(n+9)/2}$	$v_{(n+3)/2}v_{(n+11)/2}$	$v_{(n+5)/2}v_{(n+13)/2}$		$v_{n-1}v_3$	v_nv_5	3
	⋮		⋮	⋮	⋮		⋮	⋮	⋮
	$v_3v_{(n+1)/2}$		$v_{(n+1)/2}v_{n-2}$	$v_{(n+3)/2}v_{n-1}$	$v_{(n+5)/2}v_n$		$v_{n-1}v_{(n-5)/2}$	$v_nv_{(n-3)/2}$	$\frac{n-7}{2}$
	$v_3v_{(n+3)/2}$		$v_{(n+1)/2}v_{n-1}$	$v_{(n+3)/2}v_n$	–		$v_{n-1}v_{(n-3)/2}$	$v_nv_{(n-1)/2}$	$\frac{n-5}{2}$
	$v_3v_{(n+5)/2}$		$v_{(n+1)/2}v_n$	–	$v_{(n+5)/2}v_3$		$v_{n-1}v_{(n-1)/2}$	$v_nv_{(n+1)/2}$	$\frac{n-3}{2}$
	$v_3v_{(n+7)/2}$		–	$v_{(n+3)/2}v_3$	$v_{(n+5)/2}v_4$		$v_{n-1}v_{(n+1)/2}$	$v_nv_{(n+3)/2}$	$\frac{n-5}{2}$
	$v_3v_{(n+9)/2}$		$v_{(n+1)/2}v_3$	$v_{(n+3)/2}v_4$	$v_{(n+5)/2}v_5$		$v_{n-1}v_{(n+3)/2}$	$v_nv_{(n+5)/2}$	$\frac{n-7}{2}$
	⋮		⋮	⋮	⋮		⋮	⋮	⋮
	v_3v_{n-2}		$v_{(n+1)/2}v_{(n-7)/2}$	$v_{(n+3)/2}v_{(n-5)/2}$	$v_{(n+5)/2}v_{(n-3)/2}$		$v_{n-1}v_{n-5}$	v_nv_{n-4}	3
	v_3v_{n-1}		$v_{(n+1)/2}v_{(n-5)/2}$	$v_{(n+3)/2}v_{(n-3)/2}$	$v_{(n+5)/2}v_{(n-1)/2}$		$v_{n-1}v_{n-4}$	v_nv_{n-3}	2
	v_3v_n	…	$v_{(n+1)/2}v_{(n-3)/2}$	$v_{(n+3)/2}v_{(n-1)/2}$	$v_{(n+5)/2}v_{(n+1)/2}$	…	$v_{n-1}v_{n-3}$	v_nv_{n-2}	1

Table 2. The Crossing Number of an Even Order Graph $G + v_i v_j$ Corresponding to each Pair v_i, v_j of Non-adjacent Vertices of Pl_n

v_i / $v_i v_j$	Vertices in P_{n-2}								$Cr(G + v_i v_j)$
	v_3	v_4	…	$v_{(n-2)/2}$	$v_{n/2}$	$v_{(n+2)/2}$	…	v_n	
Edges incident with v_i	v_3v_5	v_4v_6	…	$v_{(n-2)/2}v_{(n+2)/2}$	$v_{n/2}v_{(n+4)/2}$	$v_{(n+2)/2}v_{(n+6)/2}$	…	v_nv_3	1
	v_3v_6	v_4v_7		$v_{(n-2)/2}v_{(n+4)/2}$	$v_{n/2}v_{(n+6)/2}$	$v_{(n+2)/2}v_{(n+8)/2}$		v_nv_4	2
	v_3v_7	v_4v_8		$v_{(n-2)/2}v_{(n+6)/2}$	$v_{n/2}v_{(n+8)/2}$	$v_{(n+2)/2}v_{(n+10)/2}$		v_nv_5	3
	⋮	⋮		⋮	⋮	⋮		⋮	⋮
	$v_3v_{n/2}$	$v_4v_{(n+2)/2}$		$v_{(n-2)/2}v_{n-4}$	$v_{n/2}v_{n-3}$	$v_{(n+2)/2}v_{n-2}$		$v_nv_{(n-4)/2}$	$\frac{n-8}{2}$
	$v_3v_{(n+2)/2}$	$v_4v_{(n+4)/2}$		$v_{(n-2)/2}v_{n-3}$	$v_{n/2}v_{n-2}$	$v_{(n+2)/2}v_{n-1}$		$v_nv_{(n-2)/2}$	$\frac{n-6}{2}$
	$v_3v_{(n+4)/2}$	$v_4v_{(n+6)/2}$		$v_{(n-2)/2}v_{n-2}$	$v_{n/2}v_{n-1}$	$v_{(n+2)/2}v_n$		$v_nv_{n/2}$	$\frac{n-4}{2}$
	$v_3v_{(n+6)/2}$	$v_4v_{(n+8)/2}$		$v_{(n-2)/2}v_{n-1}$	$v_{n/2}v_n$	–		$v_nv_{(n+2)/2}$	$\frac{n-4}{2}$
	$v_3v_{(n+8)/2}$	$v_4v_{(n+10)/2}$		$v_{(n-2)/2}v_n$	–	$v_{(n+2)/2}v_3$		$v_nv_{(n+4)/2}$	$\frac{n-6}{2}$
	$v_3v_{(n+10)/2}$	$v_4v_{(n+12)/2}$		–	$v_{n/2}v_3$	$v_{(n+2)/2}v_4$		$v_nv_{(n+6)/2}$	$\frac{n-8}{2}$
	⋮	⋮		⋮	⋮	⋮		⋮	⋮
	v_3v_{n-2}	v_4v_{n-1}		$v_{(n-2)/2}v_{(n-10)/2}$	$v_{n/2}v_{(n-8)/2}$	$v_{(n+2)/2}v_{(n-6)/2}$		v_nv_{n-4}	3
	v_3v_{n-1}	v_4v_n		$v_{(n-2)/2}v_{(n-8)/2}$	$v_{n/2}v_{(n-6)/2}$	$v_{(n+2)/2}v_{(n-4)/2}$		v_nv_{n-3}	2
	v_3v_n	–	…	$v_{(n-2)/2}v_{(n-6)/2}$	$v_{n/2}v_{(n-4)/2}$	$v_{(n+2)/2}v_{(n-2)/2}$	…	v_nv_{n-2}	1

Table 3. The crossing number of a simple graph $G_{v_i,n}: 3 \leq i \leq n$

$VCR(G_{v_i,n})$ \ v_i		Vertices in P_{n-2}												
		v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	…
Vertex centered crossing number of G with respect to v_i, for each $i = 3, 4, \dots, n$	$VCR(G_{v_i,5})$	1	0	1										
	$VCR(G_{v_i,6})$	2	1	1	2									
	$VCR(G_{v_i,7})$	4	3	2	3	4								
	$VCR(G_{v_i,8})$	6	5	4	4	5	6							
	$VCR(G_{v_i,9})$	9	8	7	6	7	8	9						
	$VCR(G_{v_i,10})$	12	11	10	9	9	10	11	12					
	$VCR(G_{v_i,11})$	16	15	14	13	12	13	14	15	16				
	$VCR(G_{v_i,12})$	20	19	18	17	16	16	17	18	19	20			
	$VCR(G_{v_i,13})$	25	24	23	22	21	20	21	22	23	24	25		
	$VCR(G_{v_i,14})$	30	29	28	27	26	25	25	26	27	28	29	30	
	⋮													

Theorem 2.4. Let G be a maximal planar graph Pl_n, with even order $n \geq 6$. If G_{v_i} is a simple graph obtained by including the edges $\{v_i v_j \mid 3 \leq j \leq i-2\} \cup \{v_i v_j \mid i+2 \leq j \leq n\}$ to the graph G, then $VCR(G_{v_i}) = VCR\left(G_{v_{(n+3)-i}}\right), 3 \leq i \leq \frac{n+2}{2}$.

Proof: Let $n \geq 6$ be any arbitrary even integer.
We prove the result by induction on i.
In basis, we prove the result for $i = 3$.
From Table 2, it can be observed that,

$$Cr(G + v_3 v_j) = Cr(G + v_n v_{j-2}), 5 \leq j \leq n \qquad (1)$$

$$\Rightarrow VCR(G_{v_3}) = \sum_{j=5}^{n} Cr(G + v_3 v_j)$$

$$= Cr(G + v_3 v_5) + Cr(G + v_3 v_6) + \cdots + Cr(G + v_3 v_n)$$

$$= Cr(G + v_n v_3) + Cr(G + v_n v_4) + \cdots + Cr(G + v_n v_{n-2}) \qquad (from\ (1))$$

$$= \sum_{j=3}^{n-2} Cr(G + v_n v_j)$$

$$= VCR(G_{v_n})$$

Hence, $VCR(G_{v_i}) = VCR\left(G_{v_{(n+3)-i}}\right)$, for $i = 3$.

Let us assume that the result is true for $i = \frac{n}{2}$. That is,

$$VCR\left(G_{v_{\frac{n}{2}}}\right) = VCR\left(G_{v_{(n+3)-\frac{n}{2}}}\right) = VCR\left(G_{v_{\frac{n+6}{2}}}\right) \quad (2)$$

Now, we prove the result for $i = \frac{n+2}{2}$.
From Table 2, we can observe that,

$$\sum_{j={n+4}/2}^{n-1} Cr\left(G + v_{\frac{n}{2}} v_j\right) = \sum_{j={n+6}/2}^{n} Cr\left(G + v_{\frac{n+2}{2}} v_j\right);$$

$$\sum_{j=3}^{{n-4}/2} Cr\left(G + v_{\frac{n}{2}} v_j\right) = \sum_{j=4}^{{n-2}/2} Cr\left(G + v_{\frac{n+2}{2}} v_j\right)$$

$$\Rightarrow VCr\left(G_{v_{\frac{n}{2}}}\right) = \sum_{j=3}^{n-4/2} Cr\left(G + v_{\frac{n}{2}}v_j\right) + \sum_{j=n+4/2}^{n} Cr\left(G + v_{\frac{n}{2}}v_j\right)$$

$$= \sum_{j=4}^{n-2/2} Cr\left(G + v_{\frac{n+2}{2}}v_j\right) + \sum_{j=n+6/2}^{n} Cr\left(G + v_{\frac{n+2}{2}}v_j\right) + Cr\left(G + v_{\frac{n}{2}}v_n\right)$$

$$= VCR\left(G_{v_{\frac{n+2}{2}}}\right) - Cr\left(G + v_{\frac{n+2}{2}}v_3\right) + \frac{n-4}{2}$$

$$= VCR\left(G_{v_{\frac{n+2}{2}}}\right) - \frac{n-6}{2} + \frac{n-4}{2}$$

$$\Rightarrow VCR\left(G_{v_{\frac{n}{2}}}\right) = VCR\left(G_{v_{\frac{n+2}{2}}}\right) + 1 \quad (3)$$

Similarly, $VCR\left(G_{v_{\frac{n+6}{2}}}\right) = VCR\left(G_{v_{\frac{n+4}{2}}}\right) + 1 \quad (4)$

By substituting the equations (3) and (4) in equation (2), we obtain,

$$VCR\left(G_{v_{\frac{n+2}{2}}}\right) + 1 = VCR\left(G_{v_{\frac{n+4}{2}}}\right) + 1$$

$$\Rightarrow VCR\left(G_{v_{\frac{n+2}{2}}}\right) = VCR\left(G_{v_{\frac{n+4}{2}}}\right)$$

Therefore, $VCR(G_{v_i}) = VCR(G_{v_{(n+3)-i}})$, for $i = \frac{n+2}{2}$.
Thus, the theorem holds for $\forall\, i = 3, 4, \dots, \frac{n+2}{2}$.

Theorem 2.5. If G is a maximal planar graph Pl_n, with even order $n \geq 6$, and the simple graph G_{v_i} is obtained from G by joining all its vertices to a vertex $v_i \in G$ that are not adjacent to $v_i \in G$, then the vertex centered crossing number of a graph G with respect to v_i, for each $i = 3, 4, \dots, n$ is given by,

$$VCR(G_{v_i}) = \begin{cases} \left(\frac{n-4}{2}\right)^2 + \left(\frac{n+2}{2} - i\right), & 3 \leq i \leq \frac{n}{2}; \\ \left(\frac{n-4}{2}\right)^2, & i = \frac{n+2}{2}, \frac{n+4}{2}; \\ \left(\frac{n-4}{2}\right)^2 + \left(i - \frac{n+4}{2}\right), & \frac{n+6}{2} \leq i \leq n. \end{cases}$$

Proof: Let G be a maximal planar graph Pl_n, with even order $n \geq 6$.
Claim: $VCR(G_{v_{i+1}}) = VCR(G_{v_i}) - 1, 3 \leq i \leq \frac{n}{2}$

We prove the result by induction on i.
The inductive basis, $i = 3$.
From Table 2, we have,

$$\sum_{j=5}^{n-1} Cr(G + v_3v_j) = \sum_{j=6}^{n} Cr(G + v_4v_j)$$

$$\Rightarrow VCR(G_{v_3}) = \sum_{j=5}^{n-1} Cr(G + v_3v_j) + Cr(G + v_3v_n)$$

$$= \sum_{j=6}^{n} Cr(G + v_4v_j) + 1 = VCR(G_{v_4}) + 1$$

$$\Rightarrow VCR(G_{v_4}) = VCR(G_{v_3}) - 1$$

Hence the claim is verified for $i = 3$.
We inductively assume that, the result holds for $i = \frac{n-2}{2}$.

That is, $VCR\left(G_{v_{\frac{n}{2}}}\right) = VCR\left(G_{v_{\frac{n-2}{2}}}\right) - 1 \quad (1)$

Let us prove the result for $i = \frac{n}{2}$.
From Table 2, we observe that,

$$\sum_{j=n+2/2}^{n-2} Cr\left(G + v_{\frac{n-2}{2}}v_j\right) = \sum_{j=n+6/2}^{n} Cr\left(G + v_{\frac{n+2}{2}}v_j\right);$$

$$\sum_{j=3}^{n-6/2} Cr\left(G + v_{\frac{n-2}{2}}v_j\right) = \sum_{j=5}^{n-2/2} Cr\left(G + v_{\frac{n+2}{2}}v_j\right)$$

$$\Rightarrow VCR\left(G_{v_{\frac{n-2}{2}}}\right) = \sum_{j=3}^{n-6/2} Cr\left(G + v_{\frac{n-2}{2}}v_j\right) + \sum_{j=n+2/2}^{n} Cr\left(G + v_{\frac{n-2}{2}}v_j\right)$$

$$= \sum_{j=5}^{n-2/2} Cr\left(G + v_{\frac{n+2}{2}}v_j\right) + \sum_{j=n+6/2}^{n} Cr\left(G + v_{\frac{n+2}{2}}v_j\right) + Cr\left(G + v_{\frac{n-2}{2}}v_{n-1}\right) + Cr\left(G + v_{\frac{n-2}{2}}v_n\right)$$

$$= \sum_{j=5}^{n-2/2} Cr\left(G + v_{\frac{n+2}{2}}v_j\right) + \sum_{j=n+6/2}^{n} Cr\left(G + v_{\frac{n+2}{2}}v_j\right) + \frac{n-4}{2} + Cr\left(G + v_{\frac{n+2}{2}}v_3\right)$$

$$= \sum_{j=3}^{n-2/2} Cr\left(G + v_{\frac{n+2}{2}} v_j\right) + \sum_{j=n+6/2}^{n} Cr\left(G + v_{\frac{n+2}{2}} v_j\right) + \frac{n-4}{2} - Cr\left(G + v_{\frac{n+2}{2}} v_4\right)$$

$$= VCR\left(G_{v_{\frac{n+2}{2}}}\right) + \frac{n-4}{2} - \frac{n-8}{2}$$

$$\Rightarrow VCR\left(G_{v_{\frac{n-2}{2}}}\right) = VCR\left(G_{v_{\frac{n+2}{2}}}\right) + 2 \qquad (2)$$

By substituting (2) in (1), we get,

$$VCR\left(G_{v_{\frac{n}{2}}}\right) = VCR\left(G_{v_{\frac{n+2}{2}}}\right) + 2 - 1$$

$$\Rightarrow VCR\left(G_{v_{\frac{n}{2}}}\right) = VCR\left(G_{v_{\frac{n+2}{2}}}\right) + 1$$

$$\Rightarrow VCR\left(G_{v_{\frac{n+2}{2}}}\right) = VCR\left(G_{v_{\frac{n}{2}}}\right) - 1$$

$\Rightarrow VCR(G_{v_{i+1}}) = VCR(G_{v_i}) - 1$, for $i = \frac{n}{2}$.

Thus, $VCR(G_{v_{i+1}}) = VCR(G_{v_i}) - 1$, holds for $3 \le i \le \frac{n}{2}$ (3)

Similarly, we can prove that,

$$VCR(G_{v_{i+1}}) = VCR(G_{v_i}) + 1, \frac{n+4}{2} \le i \le n-1 \qquad (4)$$

To find $VCR(G_{v_3})$:

$$VCR(G_{v_3}) = \sum_{j=5}^{n} Cr(G + v_3 v_j)$$

$$= \sum_{j=5}^{n+4/2} Cr(G + v_3 v_j) + \sum_{j=n+6/2}^{n} Cr(G + v_3 v_j)$$

$$= 2\left(1 + 2 + 3 + \cdots + \frac{n-4}{2}\right)$$

$$= 2\left[\frac{1}{2}\left(\frac{n-4}{2}\right)\left(\frac{n-4}{2} + 1\right)\right]$$

$$= \left(\frac{n-4}{2}\right)^2 + \frac{n-4}{2}$$

$$\Rightarrow VCR(G_{v_3}) = \left(\frac{n-4}{2}\right)^2 + \left(\frac{n+2}{2} - 3\right)$$

From equation (3), we obtain,

$$\left.\begin{aligned} VCR(G_{v_4}) &= \left(\tfrac{n-4}{2}\right)^2 + \left(\tfrac{n+2}{2} - 4\right) \\ VCR(G_{v_5}) &= \left(\tfrac{n-4}{2}\right)^2 + \left(\tfrac{n+2}{2} - 5\right) \\ &\cdots \\ VCR\left(G_{v_{\frac{n}{2}}}\right) &= \left(\tfrac{n-4}{2}\right)^2 + \left(\tfrac{n+2}{2} - \tfrac{n}{2}\right) \\ VCR\left(G_{v_{\frac{n+2}{2}}}\right) &= \left(\tfrac{n-4}{2}\right)^2 \end{aligned}\right\} \qquad (5)$$

Also, by the previous theorem, we have,

$$VCR(G_{v_i}) = VCR\left(G_{v_{(n+3)-i}}\right), 3 \le i \le \frac{n+2}{2} \qquad (6)$$

By substituting $i = \frac{n+2}{2}$ in Equation (6) we can arrive,

$$VCR\left(G_{v_{\frac{n+2}{2}}}\right) = VCR\left(G_{v_{(n+3)-\frac{n+2}{2}}}\right)$$

$$VCR\left(G_{v_{\frac{n+2}{2}}}\right) = VCR\left(G_{v_{\frac{n+4}{2}}}\right) = \left(\frac{n-4}{2}\right)^2 \qquad (7)$$

Also, from Equation (4), we get,

$$\left.\begin{aligned} VCR\left(G_{v_{\frac{n+6}{2}}}\right) &= VCR\left(G_{v_{\frac{n+4}{2}}}\right) + 1 \\ = \left(\tfrac{n-4}{2}\right)^2 + 1 &= \left(\tfrac{n-4}{2}\right)^2 + \left(\tfrac{n+6}{2} - \tfrac{n+4}{2}\right) \\ VCR\left(G_{v_{\frac{n+8}{2}}}\right) &= VCR\left(G_{v_{\frac{n+6}{2}}}\right) + 1 \\ &= \left(\tfrac{n-4}{2}\right)^2 + \left(\tfrac{n+8}{2} - \tfrac{n+4}{2}\right) \\ VCR\left(G_{v_{\frac{n+10}{2}}}\right) &= VCR\left(G_{v_{\frac{n+8}{2}}}\right) + 1 \\ &= \left(\tfrac{n-4}{2}\right)^2 + \left(\tfrac{n+10}{2} - \tfrac{n+4}{2}\right) \\ &\cdots \\ VCR(G_{v_n}) &= VCR(G_{v_{n-1}}) + 1 \\ &= \left(\tfrac{n-4}{2}\right)^2 + \left(n - \tfrac{n+4}{2}\right) \end{aligned}\right\} \qquad (8)$$

Thus, from all the above equations included in (5), (7) and (8), we conclude that, for any arbitrary even number $n \ge 6$, the vertex centered crossing number of a simple graph G with respect to v_i, for each $i = 3, 4, \ldots, n$ is given by,

$$VCR(G_{v_i}) = \begin{cases} \left(\frac{n-4}{2}\right)^2 + \left(\frac{n+2}{2} - i\right), 3 \le i \le \frac{n}{2}; \\ \left(\frac{n-4}{2}\right)^2, \quad i = \frac{n+2}{2}, \frac{n+4}{2}; \\ \left(\frac{n-4}{2}\right)^2 + \left(i - \frac{n+4}{2}\right), \frac{n+6}{2} \le i \le n. \end{cases}$$

Theorem 2.6. $VCR(G_{v_3,n-1}) = VCR(G_{v_i,n})$, for

$$i = \begin{cases} \frac{n+2}{2}, \frac{n+4}{2}, & when\ n\ is\ even; \\ \frac{n+3}{2}, & when\ n\ is\ odd. \end{cases}$$

Proof: Let $n \geq 5$ be any arbitrary integer.
Let $G_{v_3,n-1}$ be a simple graph obtained from a maximal planar graph Pl_{n-1}, by adding pair-wise non-crossing edges $\{v_3 v_j \mid 5 \leq j \leq n-1\}$, that are not incident with a vertex $v_3 \in V(Pl_{n-1})$.

Let $G_{v_i,n}$ be a simple graph arrived from a maximal planar graph Pl_n, by including the edges $\{v_i v_j \mid 3 \leq j \leq i-2\} \cup \{v_i v_j \mid i+2 \leq j \leq n\}$ which are pair-wise non-crossing, in any of its good drawing.

Case (i): n is even.

Thus, from previous theorem, for $i = \frac{n+2}{2}, \frac{n+4}{2}$, we have,

$$VCR\left(G_{v_{\frac{n+2}{2}},n}\right) = VCR\left(G_{v_{\frac{n+4}{2}},n}\right) = \left(\frac{n-4}{2}\right)^2$$

Clearly, $n-1$ is odd.
Thus by Theorem 2.1, we get,

$$VCR(G_{v_{3,n-1}}) = \left(\frac{(n-1)-3}{2}\right)\left(\frac{(n-1)-5}{2}\right) + \left(\frac{(n-1)+3}{2} - 3\right)$$
$$= \left(\frac{n-4}{2}\right)\left(\frac{n-6}{2}\right) + \left(\frac{n-4}{2}\right)$$
$$= \left(\frac{n-4}{2}\right)\left(\frac{n-6}{2} + 1\right)$$
$$= \left(\frac{n-4}{2}\right)^2$$

Thus for any even n,
$VCR(G_{v_{3,n-1}}) = VCR(G_{v_{i,n}})$, for $i = \frac{n+2}{2}, \frac{n+4}{2}$.

Case (ii): n is odd.

Then, the vertex centered crossing number of a graph G of order n with respect to a vertex $v_{\frac{n+3}{2}} \in V(G)$ will be,

$$VCR\left(G_{v_{\frac{n+3}{2}},n}\right) = \left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right) + \left|\frac{n+3}{2} - \frac{n+3}{2}\right|$$
$$= \left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)$$

Here, n is odd $\Rightarrow n-1$ is even.

Then by the previous theorem, we have,

$$VCR(G_{v_{3,n-1}}) = \left(\frac{(n-1)-4}{2}\right)^2 + \left(\frac{(n-1)+2}{2} - 3\right)$$
$$= \left(\frac{n-5}{2}\right)^2 + \left(\frac{n-5}{2}\right)$$
$$= \left(\frac{n-5}{2}\right)\left(\frac{n-5}{2} + 1\right)$$
$$= \left(\frac{n-5}{2}\right)\left(\frac{n-3}{2}\right)$$

Hence, for any odd n, we have,
$VCR(G_{v_{3,n-1}}) = VCR\left(G_{v_{\frac{n+3}{2}},n}\right)$.

Thus the proof obtained can be tabulated as follows:

Hence, the vertex centered crossing number of Pl_{n-1} with respect to the vertices $v_3, v_{n-1} \in V(Pl_{n-1})$ are identical with that of Pl_n with respect to $v_i \in V(Pl_n)$, where i takes the median value from 3 to n.

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