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Existence of solutions for a class of functional integral equations of Volterra type in two variables via measure of noncompactness

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Abstract

This paper presents some results concerning the existence of solutions for a functional integral equation of Volterra type in two variables, via measure of noncompactness. Two examples are included to illustrate the main result.

Keywords*:* Measure of noncompactness; Darbo fixed point theorem; fixed point; Volterra integral equation

1. Introduction

Recently measures of noncompactness have been successfully applied to investigate the solvability and behavior of solutions of a large variety of integral equations (Aghajani et al., 2011; Banas et al, 2008; Benchohra, 2012; Darwish et al., 2011). Banas and Dhage (2008), Banas and Rzepka (2003), Hu and Yan (2006), Liu and Kang (2007) and Liu et al. (2012) studied the existence and behavior of solutions of integral equation of Volterra type on unbounded interval in one variable and Aghajani and Jalilian (2011) extended many of the above results by considering the following integral equation in general form

$$x(t) = f(t, x(\alpha(t)), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\,ds)$$

on $BC(R_+)$. Moreover, many authors studied the existence of solutions for systems of integral equations of Volterra type in one variable on unbounded intervals (Aghajani et al., 2011; Olszowy, 2009). Li, Gao and Peng in (2012) studied the existence of mild solutions for a class of semilinear fractional differential equations with nonlocal conditions in Banach spaces. Benchohra and Seba (2012) studied the existence of solutions for an integral equation of fractional order with multiple time delays in Banach spaces, and M. Mursaleen and A. Mohiuddine in (2012) applied the technique of measures of noncompactness to the theory of infinite system of differential equations in the Banach sequence spaces ℓ_p $(1 \le p \le \infty)$.

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In this paper, we study the existence of solutions for the following functional integral equation in two variables

$$\begin{aligned} x(t,s) = f\Big(&t, s, x(\xi_1(t), \xi_2(s)), \\ &\int_0^{\beta_2(s)}\int_0^{\beta_1(t)} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w)))\,dv\,dw, \\ &\int_0^{\beta_3(t)} g_2(t,s,v,x(\zeta_1(v),\zeta_2(s)))\,dv\Big). \end{aligned} \tag{1}$$

where $f, g_i, \xi_i, \eta_i, \zeta_i$ and β_i satisfy some certain conditions, using the technic of measure of noncom- pactness.

The first measure of noncompactness was defined by Kuratowski (1930). For a bounded subset S of a metric space X, the Kuratowski measure of noncom- pactness is defined as

$$\alpha(S) := \inf\left\{ \begin{array}{l} S = \bigcup_{i=1}^{n} S_i \; for\ some\ S_i\ with \\ diam(S_i) \le \delta \; for\ 1 \le i \le n < \infty \end{array} \right\}$$

where $diam(T)$ denotes the diameter of a set $T \subset X$, namely

$$diam(T) := \sup\{d(x,y) | x, y \in T\}.$$

Here, we recall some basic facts concerning measures of noncompactness from (Banas, 1980), which is defined axiomatically in terms of some natural conditions. Denote by R the set of real numbers and put $R_+ = [0, \infty)$. Let $(E, \|.\|)$ be a Banach space. The symbol $\overline{X}$, $ConvX$ will denote the closure and closed convex hull of a subset X of E, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets.

Definition 1.1. A mapping $\mu : \mathfrak{M}_E \to \mathrm{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions.

1° The family $Ker\,\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and Ker $\mu \subseteq \mathfrak{N}_E$.

2° $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$,

3° $\mu(\overline{X}) = \mu(X)$,

4° $\mu(ConvX) = \mu(X)$,

5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$,

6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subseteq X_n$ for $n = 1,2,...$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

In 1955, Darbo published the following fixed point theorem, using the concept of measures of noncompactness, which guarantees the existence of fixed point for condensing operators.

Theorem 1.1. (Darbo, 1955) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0,1)$ with the property

$$\mu(F(X)) \le k\,\mu(X) \tag{2}$$

for any $X \subset \Omega$. Then F has a fixed point in the set Ω.

Now let $BC(\mathrm{R}_+ \times \mathrm{R}_+)$ be the Banach space of all bounded and continuous functions on $\mathrm{R}_+ \times \mathrm{R}_+$ equi- pped with the standard norm

$$\|x\| = \sup\left\{|x(t,s)| : t,s \ge 0\right\}.$$

For any nonempty bounded subset X of $BC(\mathrm{R}_+ \times \mathrm{R}_+)$, $x \in X$, $L > 0$ and $\varepsilon \ge 0$, let

$$\omega^L(x,\varepsilon) = \sup\left\{\begin{matrix}|x(t,s) - x(u,v)| : \\ t,s,u,v \in [0,L], |t-u| \le \varepsilon, |v-s| \le \varepsilon\end{matrix}\right\},$$

$$\omega^L(X,\varepsilon) = \sup\left\{\omega^L(x,\varepsilon) : x \in X\right\},$$

$$\omega_0^L(X) = \lim_{\varepsilon\to 0}\omega^L(X,\varepsilon),$$

$$\omega_0(X) = \lim_{L\to\infty}\omega_0^L(X)$$

$$X(t,s) = \{x(t,s) : x \in X\},$$

and

$$\mu(X) = \omega_0(X) + \limsup_{t,s\to\infty} diamX(t,s), \tag{3}$$

where

$$\limsup_{t,s\to\infty} diamX(t,s) := \inf_{T>0}(\sup_{s,t>T} diamX(t,s))$$

Similar to (Banas et al.,2003) (cf. also (Banas et al, 2009)), it can be shown that the function μ is a measure of noncompactness in the space $BC(\mathrm{R}_+ \times \mathrm{R}_+)$ (in the sense of Definition 1.1).

The rest of the paper is organized as follows: In Section 2, we present an extension of Darbo fixed point theorem and state our main results concerning the existence of solutions of the integral equation (1). In Section 3, we provide two examples to show the usefulness and applicability of main results.

2. Main results

In this section, we study the functional integral equation (1) with the following conditions:

i. $\xi_i, \eta_i, \beta_i, \zeta_i : \mathrm{R}_+ \to \mathrm{R}_+$ $(\mathrm{i} = 1,2,3)$ are continuous and $\xi_i(\mathrm{t}) \to \infty$ as $\mathrm{t} \to \infty$ $(\mathrm{i} = 1,2)$.

ii. $f : \mathrm{R}_+ \times \mathrm{R}_+ \times R \times R \times R \to R$ is continuous. Moreover, there exist a constant $k \in [0,1)$ and nondecreasing continuous functions $\Phi_1, \Phi_2 : \mathrm{R} \to R$ with $\Phi_i(0) = 0$ $(\mathrm{i} = 1,2)$ such that

$$\begin{aligned} &|f(\mathrm{t,s,x,y,v}) - \mathrm{f(t,s,u,z,w)}| \le \\ &k\,|\mathrm{x-u}| + \Phi_1(\mathrm{m}_1(\mathrm{t,s})\,|\mathrm{y-z}|) \\ &+\Phi_2(\mathrm{m}_2(\mathrm{t,s})\,|\mathrm{v}-w|) \end{aligned} \tag{4}$$

where $m_i : \mathrm{R}_+ \times \mathrm{R}_+ \to \mathrm{R}_+$ $(\mathrm{i} = 1,2)$ are continuous functions.

iii. $M := \sup\{|f(\mathrm{t,s,0,0,0})| : \mathrm{t,s} \in R_+\} < \infty$

iv. $\mathrm{g}_1 : \mathrm{R}_+ \times \mathrm{R}_+ \times \mathrm{R}_+ \times \mathrm{R}_+ \times R \to R$ is continuous and

$$D_1 := \sup\left\{\begin{matrix}\mathrm{m}_1(\mathrm{t,s})\left|\int_0^{\beta_2(\mathrm{s})}\int_0^{\beta_1(\mathrm{t})} \begin{matrix} g_1(\mathrm{t,s,v,w,} \\ \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w}))) \\ dvdw\end{matrix}\right| : \\ t,s \in R_+, x \in BC(\mathrm{R}_+ \times \mathrm{R}_+)\end{matrix}\right\} < \infty. \tag{5}$$

Further,

$$\lim_{t,s\to\infty} m_1(t,s)\left|\int_0^{\beta_2(s)}\int_0^{\beta_1(t)} \begin{array}{l} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w))) \\ -g_1(t,s,v,w,y(\eta_1(v),\eta_2(w))) \\ dvdw \end{array}\right| = 0$$

uniformly with respect to $x, y \in BC(R_+\times R_+)$.

v. $g_2 : R_+\times R_+\times R_+\times R \to R$ is continuous and

$$D_2 := \sup\left\{ \begin{array}{l} m_2(t,s)\left|\int_0^{\beta_3(t)} g_2(t,s,v, x(\zeta_1(v),\zeta_2(s)))dv\right| : \\ t,s \in R_+, x \in BC(R_+\times R_+) \end{array}\right\} < \infty. \quad (6)$$

Moreover,

$$\lim_{t,s\to\infty} m_2(t,s)\left|\int_0^{\beta_3(t)} \begin{array}{l} g_2(t,s,v,x(\zeta_1(v),\zeta_2(s))) \\ -g_2(t,s,v,y(\zeta_1(v),\zeta_2(s)))dv \end{array}\right| = 0$$

uniformly with respect to $x, y \in BC(R_+\times R_+)$.
Before we discuss the existence of solutions for the functional integral equation (1) and prove the main theorem, let us provide some auxiliary lemmas in this respect.

Lemma 2.1. Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : C \to E$ be an operator such that

$$\|F(x) - F(y)\| \le k\,\|x - y\| \quad (7)$$

where $k \in [0,1)$. Assume that $G_i : C \to E$ $(i = 1,2, \ldots, n)$ are compact and continuous operators and $T : C \to C$ is an operator such that

$$\|T(x) - T(y)\| \le \|F(x) - F(y)\| + \sum_{i=1}^{n}\Phi_i\left(\|G_i(x) - G_i(y)\|\right) \quad (8)$$

where $\Phi_i : R_+ \to R_+$ are nondecreasing continuous functions and $\Phi_i(0) = 0$ $(i = 1,2,\ldots,n)$. Then T has a fixed point.

Proof: Let X be an arbitrary subset of C. By the definition of Kuratowski measure of noncompactness, for every $\varepsilon > 0$ there exist $S_1, S_2, \ldots, S_m$ such that $X \subseteq \bigcup_{i=1}^{i=m} S_k$,

$$diam(F(S_k)) < \alpha(F(X)) + \varepsilon$$

and

$$diam(G_i(S_k)) < \varepsilon$$

for $i = 1,2,\ldots,n$. Let us fix arbitrarily $1 \le k \le m$. Then by (8) and properties of Φ_i we obtain

$$\begin{aligned} &diam(T(S_k)) \le \\ &diam(F(S_k)) + \sum_{i=1}^{n}\Phi_i(diam(G_i(S_k))) \le \\ &\alpha(F(X)) + \varepsilon + \sum_{i=1}^{n}\Phi_i(\varepsilon), \end{aligned}$$

and since ε is an arbitrarily positive number and Φ_i are nondecreasing continuous functions, it concludes that

$$\alpha(T(X)) \le \alpha(F(X)). \quad (9)$$

Now, we show that T satisfies (2). To do this, fix arbitrary $x, y \in X$ then we have

$$\begin{aligned} \|F(x) - F(y)\| &\le k\,\|x - y\| \\ &\le k\ diam\ X \end{aligned}$$

So

$$diam(F(X)) \le k\ diam\ X,$$

which gives

$$\alpha(F(X)) \le k\,\alpha(X). \quad (10)$$

From (7) and (10) we deduce that

$$\alpha(T(X)) \le k\,\alpha(X)$$

Also, from (8), T is a continuous operator and the application of Theorem 1.1 completes the proof.

Lemma 2.2. Assume that g_1 satisfies the hypothesis iv, then $G_1 : BC(R_+\times R_+) \to BC(R_+\times R_+)$ defined by

$$G_1(x)(t,s) = m_1(t,s)\int_0^{\beta_2(s)}\int_0^{\beta_1(t)} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w)))dvdw \quad (11)$$

is a compact and continuous operator.

Proof: Obviously, $G_1(x)(t,s)$ for any $x \in BC(R_+\times R_+)$ is continuous on $R_+\times R_+$ and by (5), G_1 is a self operator on

$BC(\mathrm{R}_+\times\mathrm{R}_+)$. Now we show that G_1 is continuous. To verify this, take $x\in BC(\mathrm{R}_+\times\mathrm{R}_+)$ and $\varepsilon>0$ arbitrarily. Moreover, take $y\in BC(\mathrm{R}_+\times\mathrm{R}_+)$ with $\|x-y\|<\varepsilon$. Then we have

$$|\mathrm{G}_1(\mathrm{x})(\mathrm{t,s})-\mathrm{G}_1(\mathrm{y})(\mathrm{t,s})|\le \left|\begin{array}{l}\mathrm{m}_1(\mathrm{t,s})\int_0^{\beta_2(\mathrm{s})}\int_0^{\beta_1(\mathrm{t})} g_1(\mathrm{t,s,v,w},\\ \qquad \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\\ -\mathrm{m}_1(\mathrm{t,s})\int_0^{\beta_2(\mathrm{s})}\int_0^{\beta_1(\mathrm{t})} g_1(\mathrm{t,s,v,w},\\ \qquad \mathrm{y}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\end{array}\right| \le \quad (12)$$

$$\left|\mathrm{m}_1(\mathrm{t,s})\int_0^{\beta_2(\mathrm{s})}\int_0^{\beta_1(\mathrm{t})} \begin{array}{l} g_1(\mathrm{t,s,v,w,x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))\\ -g_1(\mathrm{t,s,v,w,y}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\end{array}\right|$$

This result together condition (iii) imply that there exists $T>0$ such that for $t,s>T$ we have

$$|\mathrm{G}_1(\mathrm{x})(\mathrm{t,s})-\mathrm{G}_1(\mathrm{y})(\mathrm{t,s})|\le\varepsilon,$$

and if $t,s\in[0,\mathrm{T}]$, then the inequality in (12) follows that

$$|\mathrm{G}_1(\mathrm{x})(\mathrm{t,s})-\mathrm{G}_1(\mathrm{y})(\mathrm{t,s})|\le m_{1,T}\,\beta_T{}^2\vartheta_T(\varepsilon),$$

where

$$\beta_T=\sup\{\beta_i(\mathrm{t}):\mathrm{t}\in[0,\mathrm{T}],1\le\mathrm{i}\le3\},$$
$$\mathrm{m}_{1,T}=\sup\{\mathrm{m}_1(\mathrm{t,s}):\mathrm{t,s}\in[0,\mathrm{T}]\},$$
$$\vartheta_T(\varepsilon)=\sup\left\{\begin{array}{l}|\mathrm{g}_1(\mathrm{t,s,v,w,x})-\mathrm{g}_2(\mathrm{t,s,v,w,y})|:\\ \mathrm{t,s}\in[0,\mathrm{T}],\mathrm{v,w}\in[0,\beta_T],\\ \mathrm{x,y}\in[-\mathrm{b,b}],|\mathrm{x-y}|\le\varepsilon\end{array}\right\},$$

with $b=\|x\|+\varepsilon$. By using the continuity of g_1 on the compact set $[0,\mathrm{T}]\times[0,\mathrm{T}]\times[0,\beta_T]\times[0,\beta_T]\times[-\mathrm{b,b}]$, we have $\vartheta_T(\varepsilon)\to0$ as $\varepsilon\to0$. Thus, G_1 is a continuous function on $BC(\mathrm{R}_+\times\mathrm{R}_+)$. To complete the proof we need to verify that G_1 is a compact operator. Let X be a nonempty and bounded subset of $BC(\mathrm{R}_+\times\mathrm{R}_+)$, and assume that T > 0 and $\varepsilon>0$ are arbitrary constants. Then for $x\in X$ and $t_1,t_2,s_1,s_2\in[0,\mathrm{T}]$, with $|t_1-t_2|\le\varepsilon$ and $|\mathrm{s}_1-s_2|\le\varepsilon$ we have

$$|\mathrm{G}_1(\mathrm{x})(\mathrm{t}_1,\mathrm{s}_1)-\mathrm{G}_1(\mathrm{x})(\mathrm{t}_2,\mathrm{s}_2)|\le$$

$$\left|\begin{array}{l}\mathrm{m}_1(\mathrm{t}_2,\mathrm{s}_2)\int_0^{\beta_2(\mathrm{s}_2)}\int_0^{\beta_1(\mathrm{t}_2)} g_1(\mathrm{t}_2,\mathrm{s}_2,\mathrm{v,w},\\ \qquad \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\\ -\mathrm{m}_1(\mathrm{t}_2,\mathrm{s}_2)\int_0^{\beta_2(\mathrm{s}_2)}\int_0^{\beta_1(\mathrm{t}_2)} g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w},\\ \qquad \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\end{array}\right|$$
$$+\left|\begin{array}{l}\mathrm{m}_1(\mathrm{t}_2,\mathrm{s}_2)\int_0^{\beta_2(\mathrm{s}_2)}\int_0^{\beta_1(\mathrm{t}_2)} g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w},\\ \qquad \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\\ -\mathrm{m}_1(\mathrm{t}_2,\mathrm{s}_2)\int_0^{\beta_2(\mathrm{s}_2)}\int_0^{\beta_1(\mathrm{t}_1)} g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w},\\ \qquad \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\end{array}\right|$$
$$+\left|\begin{array}{l}\mathrm{m}_1(\mathrm{t}_2,\mathrm{s}_2)\int_0^{\beta_2(\mathrm{s}_2)}\int_0^{\beta_1(\mathrm{t}_1)} g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w},\\ \qquad \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\\ -\mathrm{m}_1(\mathrm{t}_1,\mathrm{s}_1)\int_0^{\beta_2(\mathrm{s}_1)}\int_0^{\beta_1(\mathrm{t}_1)} g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w},\\ \qquad \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw\end{array}\right|\le$$

$$m_{1,T}\left|\int_0^{\beta_2(\mathrm{s}_2)}\int_0^{\beta_1(\mathrm{t}_2)}\begin{array}{l}g_1(\mathrm{t}_2,\mathrm{s}_2,\mathrm{v,w,x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))\\ -g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w,x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))\\ dvdw\end{array}\right|$$
$$+\mathrm{m}_{1,T}\left|\int_0^{\beta_2(\mathrm{s}_2)}\int_{\beta_1(\mathrm{t}_1)}^{\beta_1(\mathrm{t}_2)} g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w,x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))\,dvdw\right|$$
$$+\mathrm{m}_{1,T}\left|\int_{\beta_1(\mathrm{s}_1)}^{\beta_2(\mathrm{s}_2)}\int_0^{\beta_1(\mathrm{t}_1)} g_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w,x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))\,dvdw\right|\le$$
$$m_{1,T}\,\beta_T\left(\begin{array}{l}\beta_T\,\omega_r{}^T(\mathrm{g}_1,\varepsilon)+\mathrm{U}_r{}^T\;\omega^T(\beta_1,\varepsilon)\\ +\mathrm{U}_r{}^T\;\omega^T(\beta_2,\varepsilon)\end{array}\right). \quad (13)$$

where

$$r=\sup\{\|\mathrm{x}\|:\mathrm{x}\in\mathrm{X}\},$$
$$\omega_r{}^T(\mathrm{g}_1,\varepsilon)=\sup\left\{\begin{array}{l}\left|\begin{array}{l}\mathrm{g}_1(\mathrm{t}_1,\mathrm{s}_1,\mathrm{v,w,x})\\ -\mathrm{g}_1(\mathrm{t}_2,\mathrm{s}_2,\mathrm{v,w,x})\end{array}\right|:\\ \mathrm{t}_1,\mathrm{t}_2,\mathrm{s}_1,\mathrm{s}_2\in[0,\mathrm{T}],\\ |\mathrm{t}_1-\mathrm{t}_2|\le\varepsilon,|\mathrm{s}_1-s_2|\le\varepsilon,\\ v,w\in[0,\beta_T],\mathrm{x}\in[-r,r]\end{array}\right\},$$
$$\omega^T(\beta_i,\varepsilon)=\sup\left\{\begin{array}{l}|\beta_i(\mathrm{a})-\beta_i(\mathrm{b})|:\\ \mathrm{a,b}\in[0,\mathrm{T}],|\mathrm{a-b}|\le\varepsilon\end{array}\right\},$$
$$\mathrm{U}_r{}^T=\sup\left\{\begin{array}{l}|\mathrm{g}_1(\mathrm{t,s,v,w,x})|:\mathrm{t,s}\in[0,\mathrm{T}],\\ v,w\in[0,\beta_T],\mathrm{x}\in[-r,r]\end{array}\right\}.$$

Since x was an arbitrary element of X in (13), we obtain

$$\omega^T(\mathrm{G}_1(\mathrm{X}),\varepsilon)\le m_{1,T}\,\beta_T\left(\begin{array}{l}\beta_T\,\omega_r{}^T(\mathrm{g}_1,\varepsilon)\\ +\mathrm{U}_r{}^T\;\omega^T(\beta_1,\varepsilon)\\ +\mathrm{U}_r{}^T\;\omega^T(\beta_2,\varepsilon)\end{array}\right).$$

On the other hand, by the uniform continuity of g_1 and β_i on the compact sets $[0,T]\times[0,T]\times[0,\beta_T]\times[0,\beta_T]\times[-r,r]$ and $[0,T]$, respectively, we have $\omega_r^T(g_1,\varepsilon)\to 0$ and $\omega^T(\beta i,\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, we obtain $\omega_0^T(G_1(X),\varepsilon)=0$, which gives

$$\omega_0(G_1(X))=0. \tag{14}$$

Moreover, for arbitrary $x,y\in X$ and $t,s\in R_+$ we have the following estimate

$$|G_1(x)(t,s)-G_1(y)(t,s)|\le m_1(t,s)\left|\int_0^{\beta_2(s)}\int_0^{\beta_1(t)} \begin{matrix} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w))) \\ -g_1(t,s,v,w,y(\eta_1(v),\eta_2(w))) \\ dvdw \end{matrix}\right|\le m_{1,T}\,\theta(t,s),$$

where

$$\theta(t,s)=\sup\left\{\left|\int_0^{\beta_2(s)}\int_0^{\beta_1(t)} \begin{matrix} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w))) \\ -g_1(t,s,v,w,y(\eta_1(v),\eta_2(w))) \\ dvdw \end{matrix}\right| : \begin{matrix} \\ x,y\in BC(R_+\times R_+) \end{matrix}\right\}$$

Thus, we obtain

$$diam\, G_1(X)(t,s)\le m_{1,T}\,\theta(t,s). \tag{15}$$

Taking $t,s\to\infty$ in the inequality (15), then using (iv) we arrive at

$$\limsup_{t,s\to\infty} diam\, G_1(X)(t,s)=0. \tag{16}$$

Further, combining (14) and (16) we get

$$\limsup_{t,s\to\infty} diam\, G_1(X)(t,s)+\omega_0(G_1(X))=0. \tag{17}$$

or, equivalently

$$\mu(G_1(X))=0.$$

So, it is a G_1 compact operator and the proof is complete.

Lemma 2.3. Assume that g_2 satisfies the hypothesis (v), then $G_2: BC(R_+\times R_+)\to BC(R_+\times R_+)$ defin- ed by

$$G_2(x)(t,s)=m_2(t,s)\int_0^{\beta_3(t)} g_2(t,s,v,x(\zeta_1(v),\zeta_2(s)))dv \tag{18}$$

is a compact and continuous operator.

Proof: Obviously, for any $x\in BC(R_+\times R_+)$, $G_2(x)(t,s)$ is a continuous function and by (6), G_2 is a self operator on $BC(R_+\times R_+)$. Similar to the proof of Lemma 2.2 we deduce that G_2 is continuous,

$$\omega^T(G_2(X),\varepsilon)\le m_{2,T}\left(\beta_T\,\omega_r^T(g_2,\varepsilon)+V_r^T\,\omega^T(\beta_3,\varepsilon)\right).$$

and

$$diam\, G_2(X)(t,s)\le m_{2,T}\,\varphi(t,s) \tag{19}$$

where

$$m_{2,T}=\sup\{m_2(t,s): t,s\in[0,T]\},$$

$$\omega_r^T(g_2,\varepsilon)=\sup\left\{\begin{matrix} |g_2(t_1,s_1,v,x)-g_2(t_2,s_2,v,x)|: \\ t_1,t_2,s_1,s_2\in[0,T], \\ |t_1-t_2|\le\varepsilon, |s_1-s_2|\le\varepsilon, \\ v\in[0,\beta_T], x\in[-r,r] \end{matrix}\right\}$$

$$V_r^T=\sup\left\{\begin{matrix} |g_2(t,s,v,x)|: t,s\in[0,T], \\ v\in[0,\beta_T], x\in[-r,r] \end{matrix}\right\},$$

$$\varphi(t,s)=\sup\left\{\left|\int_0^{\beta_3(t)} \begin{matrix} g_2(t,s,v,x(\zeta_1(v),\zeta_2(s))) \\ -g_2(t,s,v,y(\zeta_1(v),\zeta_2(s)))dv \end{matrix}\right| : \begin{matrix} \\ x,y\in BC(R_+\times R_+) \end{matrix}\right\}$$

and $r,\omega^T(\beta_3,\varepsilon)$ are as defined in the proof of Lemma 2.2. So by the uniform continuity of g_2 and β_3 on the compact sets $[0,T]\times[0,T]\times[0,\beta_T]\times[0,\beta_T]\times[-r,r]$ and $[0,T]$, respectively, we obtain $\omega_r^T(g_2,\varepsilon)\to 0$ and $\omega^T(\beta_3,\varepsilon)\to 0$ as $\varepsilon\to 0$, gives

$$\omega_0(G_2(X))=0.$$

Also, taking $t,s\to\infty$ in the inequality (18), then using (v) we arrive at

$$\limsup_{t,s\to\infty} diam\, G_2(X)(t,s)=0. \tag{20}$$

Thus,

$$\mu(G_2(X))=0.$$

So, it is a G_2 compact operator and the proof is complete.

Now we are in a position to present the main result of this paper.

Theorem 2.4. Under the assumptions (i)-(v), Eq. (1) has at least one solution in $BC(R_+\times R_+)$.

Proof: Define the operators $F,T:BC(R_+\times R_+)\to BC(R_+\times R_+)$ by the formulas

$$F(x)(t,s)=x(t,s)$$

and

$$T(x)(t,s)=f\begin{pmatrix} t,s,x(\xi_1(t),\xi_2(s)), \\ \int_0^{\beta_2(s)}\int_0^{\beta_1(t)} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w)))dvdw, \\ \int_0^{\beta_3(t)} g_2(t,s,v,x(\zeta_1(v),\zeta_2(s)))dv \end{pmatrix} \tag{21}$$

Using conditions (i)-(iv), for arbitrarily fixed $t\in R_+$ we have

$$\begin{aligned}&|T(x)(t,s)|\le\\ &\left|f(t,s,x(t,s)A,B)-f(t,s,0,0,0)\right|+|f(t,s,0,0,0)|\le\\ &k|x(t,s)|+|f(t,s,0,0,0)|\\ &+\Phi_1\left(m_1(t,s)\left|\int_0^{\beta_2(s)}\int_0^{\beta_1(t)} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w)))dvdw\right|\right)\\ &+\Phi_2(m_1(t,s)\left|\int_0^{\beta_3(t)} g_2(t,s,v,x(\zeta_1(v),\zeta_2(s)))dv\right|)\end{aligned}$$

where

$$A=\int_0^{\beta_2(s)}\int_0^{\beta_1(t)} g_1(t,s,v,w,x(\eta_1(v),\eta_2(w)))dvdw$$

$$B=\int_0^{\beta_3(t)} g_2(t,s,v,x(\zeta_1(v),\zeta_2(s)))dv$$

Thus,

$$\|T(x)\|\le k\|x\|+M+\Phi_1(D_1)+\Phi_2(D_2). \tag{22}$$

and $T(x)\in BC(R_+\times R_+)$ for any $x\in BC(R_+\times R_+)$. Inequality (22) yields that T transforms the ball $\overline{B}_{r_0}$ into itself where $\frac{M+\Phi_1(D_1)+\Phi_2(D_2)}{1-k}$. Also, applying (4) and taking into account the definitions of G_1, G_2, F and T we obtain

$$\begin{aligned}&\left|T(x)(t,s)-T(y)(t,s)\right|\le\left|F(x)(t,s)-F(y)(t,s)\right|\\ &+\sum_{i=1}^{2}\Phi_i\left(\left|G_i(x)(t,s)-G_i(y)(t,s)\right|\right)\end{aligned}$$

Thus, T satisfies (8) and by Lemma 2.1, T has a fixed point.

By a similar reasoning, one can derive the following consequences of Lemmas 2.1 and 2.2.

Theorem 2.5. Assume that the following conditions are satisfied:

i. $\xi_i,\eta_i,\beta_i,\zeta_i:R_+\to R_+$ $(i=1,2)$ are continuous and $\xi_i(t)\to\infty$ as $t\to\infty$ for $i=1,2$.

ii. $f:R_+\times R_+\times R\times R\times R\to R$ is continuous. Mor- eover, there exist a constant $k\in[0,1)$ and nondecre- asing continuous functions $\Phi_1,\Phi_2:R\to R$ with $\Phi_i(0)=0$ for $i=1,2$ such that

$$\begin{aligned}&|f(t,s,x,y,v)-f(t,s,u,z,w)|\le k|x-u|\\ &+\Phi_1(m_1(t,s)|y-z|)+\Phi_2(m_2(t,s)|v-w|)\end{aligned}$$

where $m_i:R_+\times R_+\to R_+$ $(i=1,2)$ is a continuous function.

iii. $M:=\sup\{|f(t,s,0,0,0)|:t,s\in R_+\}<\infty$.

iv. $g_i:R_+\times R_+\times R_+\times R_+\times R\to R$ $(i=1,2)$ are continuous and

$$D:=\sup_i\left\{\begin{array}{l} m_i(t,s)\left|\int_0^{\beta_i(s)}\int_0^{\zeta_i(t)} g_i(t,s,v,w,x(\eta_1(v),\eta_2(w)))dvdw\right|: \\ t,s\in R_+,x\in BC(R_+\times R_+)\end{array}\right\}<\infty$$

Moreover,

$$\lim_{t,s\to\infty} m_i(t,s)\left|\int_0^{\beta_i(s)}\int_0^{\zeta_i(t)}\begin{array}{l} g_i(t,s,v,w,x(\eta_1(v),\eta_2(w))) \\ -g_i(t,s,v,w,y(\eta_1(v),\eta_2(w))) \end{array} dvdw\right| =0$$

uniformly with respect to $x,y\in BC(R_+\times R_+)$. Then the functional integral

$$x(\mathrm{t,s}) = (t, s, x(\xi_1(\mathrm{t}), \xi_2(\mathrm{s})), \int_0^{\beta_1(\mathrm{s})}\int_0^{\zeta_1(\mathrm{t})} g_1(\mathrm{t,s,v,w}, \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw, \int_0^{\beta_2(\mathrm{t})}\int_0^{\zeta_2(\mathrm{t})} g_2(\mathrm{t,s,v}, \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{s})))dvdw) \quad (22)$$

has at least one solution in the space $BC(\mathrm{R}_+\times\mathrm{R}_+)$.

Proof: Similar to the proof of Theorem 2.4, consider $F, \mathrm{G}_1, G_2, T : BC(\mathrm{R}_+\times\mathrm{R}_+) \to BC(\mathrm{R}_+\times\mathrm{R}_+)$ by the formulas

$$F(\mathrm{x})(\mathrm{t,s}) = \mathrm{x(t,s)},$$

$$\mathrm{G}_i(\mathrm{x})(\mathrm{t,s}) = \mathrm{m}_i(\mathrm{t,s})\int_0^{\beta_i(\mathrm{s})}\int_0^{\zeta_i(\mathrm{t})} g_i(\mathrm{t,s,v,w}, \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw,$$

$$T(x)(\mathrm{t,s}) = \begin{pmatrix} t, s, x(\xi_1(\mathrm{t}), \xi_2(\mathrm{s})), \\ \int_0^{\beta_1(\mathrm{s})}\int_0^{\zeta_1(\mathrm{t})} g_1(\mathrm{t,s,v,w}, \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{w})))dvdw, \\ \int_0^{\beta_2(\mathrm{t})}\int_0^{\zeta_2(\mathrm{t})} g_2(\mathrm{t,s,v}, \mathrm{x}(\eta_1(\mathrm{v}),\eta_2(\mathrm{s})))dvdw \end{pmatrix}$$

Then by applying Lemma 2.1, we see that (23) has at least one solution in the space $BC(\mathrm{R}_+\times\mathrm{R}_+)$.

3. Examples

In this section, we provide two examples to show the efficiency of the main results.

Example 3.1. Consider the following functional integral equation

$$x(\mathrm{t,s}) = \frac{t^2 s^2(1+\mathrm{x(t,s)})}{t^4 s^4+1} + \frac{\sin(\mathrm{ts})}{e^{ts}}\arctan\left(\int_0^t\int_0^s \frac{v^3\cos(\mathrm{x(v,w)}) + \mathrm{e}^w\sin(\mathrm{x}^4(\mathrm{v,w}))}{2+\sin(\mathrm{x(v,w)})}dvdw\right). \quad (24)$$

Eq. (23) is a special case of Eq. (1) with

$$\beta_1(\mathrm{t}) = \beta_2(\mathrm{t}) = \xi_1(\mathrm{t}) = \xi_2(\mathrm{t}) = \eta_1(\mathrm{t}) = \eta_2(\mathrm{t}) = \mathrm{t}$$

$$\mathrm{f(t,s,x,y,z)} = \frac{t^2s^2}{t^4s^4+1}(1+\mathrm{x}) + \frac{\sin(\mathrm{ts})}{e^{ts}}\arctan(y) + z$$

$$g_1(\mathrm{t,s,v,w,x}) = \frac{v^3\cos\mathrm{x} + \mathrm{e}^w\sin\mathrm{x}^4}{2+\sin\mathrm{x}}$$

$$g_2(\mathrm{t,s,v,w,x}) = 0$$

From the definitions of ξ_i, η_i, β_i and g_2, it is easy to see that conditions (i) and (v) of Theorem 2.4 are valid and taking $m(\mathrm{t,s}) = \frac{\sin(\mathrm{ts})}{e^{ts}}, k = \frac{1}{2}$ and $\Phi(\mathrm{t}) = \mathrm{t}$, we can find that f, m and Φ satisfy condition (ii) of Theorem 2.4. Also, g_1 is continuous on $\mathrm{R}_+\times\mathrm{R}_+\times\mathrm{R}_+\times\mathrm{R}_+\times\mathrm{R}$ and

$$D_1 = \sup\left\{ \left|\frac{\sin(\mathrm{ts})}{e^{ts}}\int_0^t\int_0^s \frac{v^3\cos(\mathrm{x(v,w)}) + \mathrm{e}^w\sin(\mathrm{x}^4(\mathrm{v,w}))}{2+\sin(\mathrm{x(v,w)})}dvdw\right| : t,s\in\mathrm{R}_+, x\in BC(\mathrm{R}_+\times\mathrm{R}_+)\right\} < \infty,$$

$$\lim_{t\to\infty}\frac{\sin(\mathrm{ts})}{e^{ts}}\int_0^t\int_0^s \left(\frac{v^3\cos(\mathrm{x(v,w)}) + \mathrm{e}^w\sin(\mathrm{x}^4(\mathrm{v,w}))}{2+\sin(\mathrm{x(v,w)})} - \frac{v^3\cos(\mathrm{y(v,w)}) + \mathrm{e}^w\sin(\mathrm{y}^4(\mathrm{v,w}))}{2+\sin(\mathrm{y(v,w)})}\right)dvdw = 0$$

for any $x, y \in BC(\mathrm{R}_+\times\mathrm{R}_+)$, which implies that condition (iv) is satisfied. Next we estimate

$$M := \sup\{|f(\mathrm{t,s,0,0,0})| : \mathrm{t,s}\in\mathrm{R}_+\} = \sup\{\frac{t^2s^2}{t^4s^4+1} : \mathrm{t,s}\in\mathrm{R}_+\} = \frac{1}{2}$$

and condition (iii) of Theorem 2.4 is valid. Then by Theorem 2.4, the integral equation (24) has at least one solution in $BC(\mathrm{R}_+\times\mathrm{R}_+)$.

Example 3.2. Consider the following functional integral equation

$$x(\mathrm{t,s}) = \frac{3ts\,\mathrm{x(t,s)}}{4ts+1} + \int_0^{t^2}\frac{u^3\cos(\mathrm{ux}(\sqrt{u},\mathrm{s})) + \mathrm{e}^u(2+\sin(\mathrm{x}^4(\sqrt{u},\mathrm{s})))}{e^{t^2}(2+\sin(\mathrm{x}^4(\sqrt{u},\mathrm{s})))}du \quad (25)$$

Eq. (25) is a special case of Eq. (1) with

$$\xi_1(t) = \xi_2(t) = \zeta_2(t) = t, \quad \beta_3(t) = t^2, \quad \zeta_1(t) = \sqrt{t}$$

$$f(t,s,x,y,z) = \frac{3ts}{4ts+1}x + y + z$$

$$g_1(t,s,v,w,x) = 0$$

$$g_2(t,s,u,x) = \frac{v^3\cos(ux) + e^u(2+\sin(x^4))}{e^{t^2}(2+\sin(x^4))}$$

Now we check all conditions of Theorem 2.4. By the definitions of ξ_i, ζ_i, β_3 and g_1, it is easily seen that conditions (i), (ii) and (iv) are satisfied with $k = \frac{3}{4}$ and also $f(t,s,0,0,0) = 0$ is bounded so condition (iii) of Theorem 2.4 is valid. Moreover, we have

$$\left|\frac{u^3\cos(ux(\sqrt{u},s)) + e^u(2+s\ in(x^4(\sqrt{u},s)))}{e^{t^2}(2+\sin(x^4(\sqrt{u},s)))}\right| \leq$$

$$\left|\frac{u^3+3e^u}{e^{t^2}}\right|$$

$$\left|\begin{array}{l}\dfrac{u^3\cos(ux(\sqrt{u},s)) + e^u(2+s\ in(x^4(\sqrt{u},s)))}{e^{t^2}(2+\sin(x^4(\sqrt{u},s)))} \\ -\dfrac{u^3\cos(uy(\sqrt{u},s)) + e^u(2+s\ in(y^4(\sqrt{u},s)))}{e^{t^2}(2+\sin(y^4(\sqrt{u},s)))}\end{array}\right| \leq \frac{2u^3}{e^{t^2}}$$

Thus,

$$D_2 = \sup\left\{\begin{array}{l} \left|\int_0^{t^2} \dfrac{u^3\cos(ux(\sqrt{u},s)) + e^u(2+\sin(x^4(\sqrt{u},s)))}{e^{t^2}(2+\sin(x^4(\sqrt{u},s)))}du\right| : \\ t,s \in R_+, x \in BC(R_+\times R_+)\end{array}\right\} < \infty,$$

$$\lim_{t\to\infty}\int_0^{t^2} \begin{array}{l}\dfrac{u^3\cos(ux(\sqrt{u},s)) + e^u(2+\sin(x^4(\sqrt{u},s)))}{e^{t^2}(2+\sin(x^4(\sqrt{u},s)))} \\ -\dfrac{u^3\cos(uy(\sqrt{u},s)) + e^u(2+\sin(y^4(\sqrt{u},s)))}{e^{t^2}(2+\sin(y^4(\sqrt{u},s)))}du\end{array}$$

$$= 0,$$

uniformly with respect to $x, y \in BC(R_+\times R_+)$ and thus condition (v) of Theorem 2.4 is satisfied. Then by Theorem 2.4, the integral equation (25) has at least one solution in $BC(R_+\times R_+)$.

References

Aghajani, A., Banas, J., Jalilian, Y. (2011). Existence of solutions for a class of nonlinear Volterra singular integral equations. Comput. Math. Appl. 62, 1215–1227.

Aghajani, A., Jalilian, Y. (2010). Existence and global attractivity of solutions of a nonlinear functional integral equation. Commun. Nonlin. Sci. Numer. Simul. 15, 3306–3312.

Aghajani, A., Jalilian, Y. (2011). Existence of nondecreasing positive solutions for a system of singular integral equations. Medit. J. Math. 8, 563–576.

Banas, J., Dhage, B. C. (2008). Global asymptotic stability of solutions of a functional integral equation. Nonlin. Anal. 69, 1945–1952

Banas, J., & Goebel, K. (1980). Measures of noncompactness in Banach spaces, New York: Lect. Notes Pure Appl. Math., vol. 60.

Banas, J., Rzepka, R. (2003). An application of a measure of noncompactness in the study of asymptotic stability. Appl. Math. Lett. 16, 1–6.

Banas, J., O'Regan, D., Sadarangani, K. (2009). On solutions of a quadratic hammerstein integral equation on an unbounded interval. Dynamic Systems and Applications, 18, 251–264.

Benchohra, M., Seba, D. (2012). Integral equations of fractional order with multiple time delays in Banach spaces. Electronic Journal of Differential Equations, 1–8.

Darbo, G. (1955). Punti uniti in trasformazioni a codominio non compatto. Rend. Sem. Mat. Univ. Padova, 24, 84–92

Darwish, M. A., Henderson, J, O'Regan, J. (2011). Existence and asymptotic stability of solutions of a perturbed fractional functional-integral equation with linear modification of the argument. Korean Math. Soc. 48, 539–553.

Dhage, B.C., Bellale, S. S. (2008). Local asymptotic stability for nonlinear quadratic functional integral equations. Electr. J. Qual. Th. Differ. Equations, 1–13.

Djebali, S., O'Regan, D., Sahnoun, Z. (2011). On the solvability of some operator equations and inclusions in Banach spaces with the weak topology. Applied Analysis, 15, 125–140.

Garcia-Falset, J. (2009). Existence of fixed points and measures of weak noncompactness. Nonlinear Anal. 71, 2625–2633.

Hu, X., Yan, J. (2006). The global attractivity and asymptotic stability of solution of a nonlinear integral equation. J. Math. Anal. Appl. 321, 147–156.

Kuratowski, K. (1930). sur les espaces complets. Fund. Math. 15, 301–309.

Liu, Z., Kang, S. M. (2007). Existence and asymptotic stability of solutions to a functional-integral equation. Taiwanese J. Math., 187–196.

Liu, L., Guo, F., Wu, C., Wu, Y. (2005). Existence theorems of global solutions for nonlinear volterra type integral equations in Banach spaces. J. Math. Anal. Appl. 309, 638–649.

Li, K., Gao, J., Peng, J. (2012). Existence results for semilinear fractional differential equations via Kuratowski measure of noncompactness. Fractional Calculus and Applied Analysis 15, 591–610.

Mursaleen, M., & Alotaibi, A. (2012). Infinite system of differential equations in some BK spaces, Abstract Appl. Anal., Article ID 863483, 20 pages.

Mursaleen, M., Mohiuddine, S. A. (2012). Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces. Nonlinear Analysis (TMA), 75, 2111–2115.

Olszowy, L. (2009). On some measures of noncompactness in the Frechet spaces of continuous functions. Nonlinear Anal. 71, 5157–5163.