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OD-Characterization of some orthogonal groups

N. Ahanjideh, G. R. Rezaeezadeh* and Sh. Safari

Department of Mathematics, Faculty of Basic Sciences
University of Shahre-kord, P.O. Box: 115, Shahre-kord, Iran
E-mail: rezaeezadeh@sci.sku.ac.ir

Abstract

In this paper, it was shown that ${}^2D_n(2)$, where $n = 2^m + 1 \geq 5$ and $|\pi(2^{n-1} + 1)| = 1$, and ${}^2D_n(3)$, where $n = 2^m + 1 \geq 9$ is not prime and $|\pi(\frac{3^{n-1}+1}{2})| = 1$, are OD-characterizable.

***Keywords**:* Simple groups; prime graph; degree of a vertex; degree pattern

1. Introduction

Let $\mathbf{G}$ be a finite group, $\boldsymbol{\pi}(\mathbf{G})$ the set of all prime divisors of its order and let $\boldsymbol{\omega}(\mathbf{G})$ be the spectrum of $\mathbf{G}$, that is the set of its element orders. The prime graph $\mathbf{GK}(\mathbf{G})$ of $\mathbf{G}$ is a simple graph with vertex set $\boldsymbol{\pi}(\mathbf{G})$ in which two distinct vertices $\mathbf{p}$ and $\mathbf{q}$ are joined by an edge (and written $\mathbf{p}\sim\mathbf{q}$) if and only if $\mathbf{pq} \in \boldsymbol{\omega}(\mathbf{G})$. Denote by $\mathbf{t}(\mathbf{G})$ the number of connected components of $\mathbf{GK}(\mathbf{G})$. The $\mathbf{i}$-th connected component is denoted by $\boldsymbol{\pi}_\mathbf{i} = \boldsymbol{\pi}_\mathbf{i}(\mathbf{G})$ for each $\mathbf{i}$. If $\mathbf{2} \in \boldsymbol{\pi}(\mathbf{G})$, then we assume that $\mathbf{2} \in \boldsymbol{\pi}_\mathbf{1}$. For $\mathbf{p} \in \boldsymbol{\pi}(\mathbf{G})$, $\mathbf{deg}(\mathbf{p}) = |\{\mathbf{q} \in \boldsymbol{\pi}(\mathbf{G}) | \mathbf{p}\sim\mathbf{q}\}|$ is called the degree of $\mathbf{p}$. If $\boldsymbol{\pi}(\mathbf{G}) = \{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_k}\}$ with $\mathbf{p_1} < \mathbf{p_2} < \ldots < \mathbf{p_k}$, we also define $\mathbf{D}(\mathbf{G}) = (\mathbf{deg}(\mathbf{p_1}), \mathbf{deg}(\mathbf{p_2}), \ldots, \mathbf{deg}(\mathbf{p_k}))$ which is called the degree pattern of $\mathbf{G}$. It is clear that the order of $\mathbf{G}$ can be expressed as the product of the numbers $\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_{t(G)}}$, where $\boldsymbol{\pi}(\mathbf{m_i}) = \boldsymbol{\pi}_\mathbf{i}, \mathbf{1} \leq \mathbf{i} \leq \mathbf{t}(\mathbf{G})$. If the order of $\mathbf{G}$ is even and $\mathbf{t}(\mathbf{G}) \geq \mathbf{2}$, according to our notation $\mathbf{m_2}, \mathbf{m_3}, \ldots, \mathbf{m_{t(G)}}$ are odd numbers. The positive integers $\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_{t(G)}}$ are called the order components of $\mathbf{G}$ and $\mathbf{OC}(\mathbf{G}) = \{\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_{t(G)}}\}$ is called the set of order components of $\mathbf{G}$, and $\mathbf{T}(\mathbf{G}) = \{\boldsymbol{\pi}_\mathbf{i}(\mathbf{G}) | \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{t}(\mathbf{G})\}$ is called the set of connected components of $\mathbf{G}$. set $\boldsymbol{\Omega}_\mathbf{0}(\mathbf{G}) = \{\mathbf{p} \in \boldsymbol{\pi}(\mathbf{G}) | \mathbf{deg}(\mathbf{p}) = \mathbf{0}\}$ and $\boldsymbol{\Omega}_{\mathbf{0}'}(\mathbf{G}) = \{\mathbf{p} \in \boldsymbol{\pi}(\mathbf{G}) | \mathbf{deg}(\mathbf{p}) \neq \mathbf{0}\}$. Clearly, $\boldsymbol{\pi}(\mathbf{G}) = \boldsymbol{\Omega}_\mathbf{0}(\mathbf{G}) \cup \boldsymbol{\Omega}_{\mathbf{0}'}(\mathbf{G})$. Given a finite group $\mathbf{M}$, denote by $\mathbf{h_{OD}}(\mathbf{M})$ the number of isomorphism classes of finite groups $\mathbf{G}$ such that $|\mathbf{G}| = |\mathbf{M}|$ and $\mathbf{D}(\mathbf{G}) = \mathbf{D}(\mathbf{M})$. A finite group $\mathbf{M}$ is called $\mathbf{k}$-fold OD-characterizable if $\mathbf{h_{OD}}(\mathbf{M}) = \mathbf{k}$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable [1]. Also in [1], Darafsheh et.al proved that the sporadic simple groups, alternating groups $\mathbf{A_p}$, where $\mathbf{p}$ and $\mathbf{p - 2}$ are primes, and some simple groups of Lie type are OD-characterizable, and $\mathbf{S_6}(\mathbf{3})$ and $\mathbf{O_7}(\mathbf{3})$ are $\mathbf{2}$-fold OD-characterizable groups. In [2], it has been proved that $\mathbf{A_{10}}$ is $\mathbf{2}$-fold OD-characterizable. In [3], it has been proved that all simple groups whose orders are less than $\mathbf{10^8}$ except $\mathbf{A_{10}}$ and $\mathbf{U_4}(\mathbf{2})$ are OD-characterizable. According to [4], $\mathbf{B_r}(\mathbf{3})$ and $\mathbf{C_r}(\mathbf{3})$, where $\mathbf{r}$ is an odd prime and $|\boldsymbol{\pi}(\frac{\mathbf{3^r - 1}}{\mathbf{2}})| = \mathbf{1}$ and $\mathbf{B_n}(\mathbf{q}), \mathbf{C_n}(\mathbf{q})$, for certain $\mathbf{n}, \mathbf{q}$, and the simple groups $\mathbf{B_3}(\mathbf{5})$ and $\mathbf{C_3}(\mathbf{5})$ are $\mathbf{2}$-fold OD-characterizable. In this paper, we prove that:

Theorem A. Let $\mathbf{G}$ be a finite group such that $|\mathbf{G}| = |{}^\mathbf{2}\mathbf{D_n}(\mathbf{2})|$ and $\mathbf{D}(\mathbf{G}) = \mathbf{D}({}^\mathbf{2}\mathbf{D_n}(\mathbf{2}))$, where $\mathbf{n} = \mathbf{2^m} + \mathbf{1} \geq \mathbf{5}$ and $|\boldsymbol{\pi}(\mathbf{2^{n-1}} + \mathbf{1})| = \mathbf{1}$. Then $\mathbf{G} \cong {}^\mathbf{2}\mathbf{D_n}(\mathbf{2})$.

Theorem B. Let $\mathbf{G}$ be a finite group such that $|\mathbf{G}| = |{}^\mathbf{2}\mathbf{D_n}(\mathbf{3})|$ and $\mathbf{D}(\mathbf{G}) = \mathbf{D}({}^\mathbf{2}\mathbf{D_n}(\mathbf{3}))$, where $\mathbf{n} = \mathbf{2^m} + \mathbf{1} \geq \mathbf{9}$ is not prime and $|\boldsymbol{\pi}(\frac{\mathbf{3^{n-1}+1}}{\mathbf{2}})| = \mathbf{1}$. Then $\mathbf{G} \cong {}^\mathbf{2}\mathbf{D_n}(\mathbf{3})$.

2. Preliminary results

If $\mathbf{a}$ is a natural number, $\mathbf{r}$ is an odd prime and $(\mathbf{r}, \mathbf{a}) = \mathbf{1}$, then by $\mathbf{e}(\mathbf{r}, \mathbf{a})$ we denote the smallest natural number $\mathbf{n}$ with $\mathbf{a^n} \equiv \mathbf{1} (\mathbf{mod}\, \mathbf{r})$. A prime $\mathbf{r}$ with $\mathbf{e}(\mathbf{r}, \mathbf{a}) = \mathbf{n}$ is called a primitive prime divisor of $\mathbf{a^n} - \mathbf{1}$. We denote by $\mathbf{R_n}(\mathbf{a})$ the set of all the primitive prime divisors of $\mathbf{a^n} - \mathbf{1}$ and by $\mathbf{r_n}(\mathbf{a})$ every element of $\mathbf{R_n}(\mathbf{a})$, and $\mathbf{n_p}$ is $\mathbf{p}$-part of $\mathbf{n}$.

*Corresponding author
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Lemma 2.1. (Zsigmondy 's Theorem)[5]Let $\mathbf{a}$ and $\mathbf{n}$ be integers greater than 1. There exists a prime divisor $\mathbf{p}$ of $\mathbf{a^n-1}$ such that $\mathbf{p}$ does not divide $\mathbf{a^j-1}$ for all $\mathbf{1\le j<n}$, except exactly in the following cases:
(i) $\mathbf{n=2, a=2^s-1}$, where $\mathbf{s\ge 2}$;
(ii) $\mathbf{n=6, a=2}$.
By Zsigmondy's Theorem, $\mathbf{R_n(a)\neq\phi}$, unless $\mathbf{a=2, n=6}$ or $\mathbf{n=2}$ and $\mathbf{a=2^w-1}$ for some natural number $\mathbf{w}$. Obviously by Fermat's little theorem it follows that $\mathbf{e(r,a)|r-1}$. Also, if $\mathbf{a^m\equiv 1}$ (mod $\mathbf{r}$), then $\mathbf{e(r,a)|m}$. Also, for an integer $\mathbf{n}$, by $\mathbf{\eta(n)}$ we denote the following function:

$$\mathbf{\eta(n)}=\begin{cases}\mathbf{n} & \mathbf{n\ is\ odd}\\ \mathbf{\frac{n}{2}} & \mathbf{otherwise.}\end{cases}$$

Lemma 2.2. [6, 7] Let $\mathbf{G={}^2D_n(q)}$ be a finite simple group of Lie type over a field of characteristic $\mathbf{p}$. Let $\mathbf{r}$ and $\mathbf{s}$ be odd primes and $\mathbf{r,s\in\pi(G)\backslash\{p\}}$. Put $\mathbf{k=e(r,q)}$ and $\mathbf{l=e(s,q)}$. Then:
(i) $\mathbf{r}$ and $\mathbf{p}$ are non-adjacent if and only if $\mathbf{\eta(e(r,q))>n-2}$;
(ii) if $\mathbf{1\le\eta(k)\le\eta(l)}$, then $\mathbf{r}$ and $\mathbf{s}$ are non-adjacent if and only if $\mathbf{2\eta(k)+2\eta(l)>2n-(1+(-1)^{k+l})}$ and $\mathbf{\frac{l}{k}}$ is not an odd natural number;
(iii) if $\mathbf{p\neq 2}$, then $\mathbf{r}$ and $\mathbf{2}$ are non-adjacent if and only if one of the following holds:
1. $\mathbf{\eta(k)=n}$ and $\mathbf{(4,q^n+1)=(q^n+1)_2}$;
2. $\mathbf{\eta(k)=\frac{k}{2}=n-1}$, $\mathbf{n}$ is odd and $\mathbf{e(2,q)=2}$.

Lemma 2.3. Let $\mathrm{M}={}^2\mathbf{D_n(p)}$, where $\mathbf{p\in\{2,3\}}$, $\mathbf{n=2^m+1}$ and $\mathbf{|\pi(\frac{p^{n-1}+1}{(2,p-1)})|=1}$. Then $\mathbf{deg(p)=|\pi(M)|-(2+|R_{2n}(p)|)}$.

Proof: First assume that $\mathbf{p=2}$ and $\mathbf{s\in\pi(G)\backslash\{2\}}$. Then by Table $\mathbf{2}$, we have $\mathbf{\pi_2(M)=\pi(2^{n-1}+1)}$ and by Lemma 2.2(i), $\mathbf{s}$ is non-adjacent to $\mathbf{2}$ if and only if $\mathbf{s\in R_{2n}(2)\cup R_{2(n-1)}(2)}$. But since $\mathbf{R_{2(n-1)}(2)\subseteq\pi(2^{n-1}+1)}$, we have $\mathbf{|R_{2(n-1)}(2)|=1}$. Therefore $\mathbf{deg(2)=|\pi(M)|-(2+|R_{2n}(2)|)}$. For $\mathbf{p=3}$, the same argument shows that $\mathbf{deg(3)=|\pi(M)|-(2+|R_{2n}(3)|)}$.

Lemma 2.4. [8] Let $\boldsymbol{G}$ be either a Frobenius group or a $\mathbf{2}$-Frobenius group of even order. Then $\boldsymbol{t(G)=2}$.

Lemma 2.5. [1] Let $\boldsymbol{G}$ and $\boldsymbol{M}$ be finite groups such that $\boldsymbol{|G|=|M|}$ and $\boldsymbol{D(G)=D(M)}$. In addition, we suppose one of the following conditions holds:
(i) $\boldsymbol{|\Omega_{o'}(M)|=0}$;
(ii) $\boldsymbol{|\Omega_{o'}(M)|=2}$;
(iii) $\boldsymbol{|\Omega_{o'}(M)|\ge 3}$ and there exists a vertex $\boldsymbol{p\in\pi(M)}$ such that $\boldsymbol{deg(p)\ge|\Omega_{o'}(M)|-2}$.
Then $\boldsymbol{OC(G)=OC(M)}$.

Lemma 2.6. [8, 9] Let $\boldsymbol{G}$ be a finite group with $\boldsymbol{t(G)\ge 2}$. Then one of the following holds:
(i) $\boldsymbol{G}$ is a Frobenius or a $\mathbf{2}$-Frobenius group;
(ii) $\boldsymbol{G}$ has a normal series $\boldsymbol{1\trianglelefteq H\triangleright K\trianglelefteq G}$ such that $\boldsymbol{H}$ and $\boldsymbol{\frac{G}{K}}$ are $\boldsymbol{\pi_1}$-groups and $\boldsymbol{\frac{K}{H}}$ is a non-abelian finite simple group. Moreover, $\boldsymbol{H}$ is nilpotent, $\mathbf{|\frac{G}{K}|}$ divides $\boldsymbol{|Out(\frac{K}{H})|}$ and every odd order component of $\boldsymbol{G}$ is an odd order component of $\boldsymbol{\frac{K}{H}}$.

Lemma 2.7. [10] Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be primes and $\boldsymbol{m,n>1}$. Then:
(i) the only solution of the diophantine equation $\boldsymbol{p^m-q^n=1}$ is $\boldsymbol{(p^m,q^n)=(3^2,2^3)}$;
(ii) with the exceptions of the relations $\boldsymbol{(239)^2-2(13)^4=-1}$ and $\boldsymbol{3^5-2.11^2=1}$ every solution of $\boldsymbol{p^m-2q^n=\pm 1}$ has exponents of $\boldsymbol{m=n=2}$, i.e. it comes from a unit $\boldsymbol{p-q.2^{\frac{1}{2}}}$ of the quadratic field $\boldsymbol{Q(2^{\frac{1}{2}})}$.

Lemma 2.8. [11] Let $\boldsymbol{G}$ be a finite group with $\boldsymbol{t(G)\ge 2}$. If $\boldsymbol{H\trianglelefteq G}$ is a $\boldsymbol{\pi_i}$-group, then $\boldsymbol{(\prod_{1\le j\neq i\le t(G)} m_j)}$ divides $\boldsymbol{|H|-1}$.

Lemma 2.9. [12] Let $\boldsymbol{G=A_l(q)}$ be a finite simple group of Lie type over a field of characteristic $\boldsymbol{p}$, where $\boldsymbol{q=p^k}$. Then:
(i) if $\boldsymbol{l=1}$, then $\boldsymbol{|Out(G)|=gcd(2,q-1)k}$;
(ii) if $\boldsymbol{l\ge 2}$, then $\boldsymbol{|Out(G)|=2gcd(l+1,q-1)k}$.

The list of finite simple groups with disconnected prime graph and their odd order components is given in Table 1-3 [9, 13].

Table 1. Finite Simple Groups P with $t(P) > 3$

P	Restrictions on P	$\pi_1(P)$	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		$\{2\}$	3	5	7		
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	$\{2\}$	$q-1$	$q-\sqrt{2q}+1$	$q+\sqrt{2q}+1$		
${}^2E_6(2)$		$\{2,3,5,7,11\}$	13	17	19		
$E_8(q)$	$q \equiv 2,3(5)$	$\pi(q(q^8-1)(q^{12}-1)$ $(q^{14}-1)(q^{18}-1)(q^{20}-1))$	$\dfrac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1	$\dfrac{q^{10}-q^5+1}{q^2-q+1}$		
M_{22}		$\{2,3\}$	5	7	11		
J_1		$\{2,3,5\}$	7	11	19		
$O'N$		$\{2,3,5,7\}$	11	19	31		
LyS		$\{2,3,5,7,11\}$	31	37	67		
Fi'_{24}		$\{2,3,5,7,11,13\}$	17	23	29		
F_1		$\{2,3,5,7,11,13,17,19,23,2$	41	59	71		
$E_8(q)$	$q \equiv 0,1,4(5)$	$\pi(q(q^8-1)(q^{10}-1)$ $(q^{12}-1)(q^{14}-1)(q^{18}-1))$	$\dfrac{q^{10}+q^5+1}{q^2+q+1}$	$\dfrac{q^{10}-q^5+1}{q^2-q+1}$	q^8-q^4+1	$\dfrac{q^{10}+1}{q^2+1}$	
J_4		$\{2,3,5,7,11\}$	23	29	31	37	43

Table 2. Finite Simple Groups P with $t(P) = 2$

P	Restrictions on P	$\pi_1(P)$	m_2
A_n	$6 < n = p, p+1, p+2$ n or $n-2$ is not prime	$\pi((n-3)!)$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2),(3,4)$	$\pi(q\prod_{i=1}^{p-1}(q^i-1))$	$\dfrac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1)\|(p+1)$	$\pi(q(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-1))$	$\dfrac{q^p-1}{q-1}$
${}^2A_{p-1}(q)$		$\pi(q\prod_{i=1}^{p-1}(q^i-(-1)^i))$	$\dfrac{q^p+1}{(q+1)(p,q+1)}$
${}^2A_p(q)$	$(q+1)\|(p+1)$ $(p,q) \neq (3,3),(5,2)$	$\pi(q(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-(-1)^i))$	$\dfrac{q^p+1}{q+1}$
${}^2A_3(2)$		$\{2,3\}$	5
$B_n(q)$	$n = 2^m \geq 4$, q is odd	$\pi(q\prod_{i=1}^{n-1}(q^{2i}-1))$	$\dfrac{q^n+1}{2}$
$B_p(3)$		$\pi(3(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1))$	$\dfrac{3^p-1}{2}$
$C_n(q)$	$n = 2^m \geq 2$	$\pi(q\prod_{i=1}^{n-1}(q^{2i}-1))$	$\dfrac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q = 2,3$	$\pi(q(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1))$	$\dfrac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p \geq 5, q = 2,3,5$	$\pi(q\prod_{i=1}^{p-1}(q^{2i}-1))$	$\dfrac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q = 2,3$	$\pi(q(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1))$	$\dfrac{q^p-1}{(2,q-1)}$
${}^2D_n(q)$	$n = 2^m \geq 4$	$\pi(q\prod_{i=1}^{n-1}(q^{2i}-1))$	$\dfrac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$	$n = 2^m+1, m \geq 2$	$\pi(2(2^n+1)\prod_{i=1}^{n-2}(2^{2i}-1))$	$2^{n-1}+1$
${}^2D_p(3)$	$5 \leq p \neq 2^m+1$	$\pi(3\prod_{i=1}^{p-1}(3^{2i}-1))$	$\dfrac{3^p+1}{4}$
${}^2D_n(3)$	$n = 2^m+1 \neq p, m \geq 2$	$\pi(3(3^n+1)\prod_{i=1}^{n-2}(3^{2i}-1))$	$\dfrac{3^{n-1}+1}{2}$
$G_2(q)$	$2 < q \equiv \varepsilon(3), \varepsilon = \pm 1$	$\pi(q(q^2-1)(q^3-\varepsilon))$	$q^2-\varepsilon q+1$

${}^3D_4(q)$		$\pi(q(q^6-1))$	q^4-q^2+1
$F_4(q)$	q is odd	$\pi(q(q^6-1)(q^8-1))$	q^4-q^2+1
${}^2F'_4(2)$		$\{2,3,5\}$	13
$E_6(q)$		$\pi(q(q^5-1)(q^8-1)(q^{12}-1))$	$\frac{q^6+q^3+1}{(3,q-1)}$
${}^2E_6(q)$	$q>2$	$\pi(q(q^5+1)(q^8-1)(q^{12}-1))$	$\frac{q^6-q^3+1}{(3,q+1)}$
M_{12}		$\{2,3,5\}$	11
J_2		$\{2,3,5\}$	7
Ru		$\{2,3,5,7,13\}$	29
He		$\{2,3,5,7\}$	17
McL		$\{2,3,5,7\}$	11
Co_1		$\{2,3,5,7,11,13\}$	23
Co_3		$\{2,3,5,7,11\}$	23
Fi_{22}		$\{2,3,5,7,11\}$	13
F_5		$\{2,3,5,7,11\}$	19

Table 3. Finite Simple Groups P with $t(P)=3$

P	Restrictions on P	$\pi_1(P)$	m_2	m_3
A_n	$n>6$, $n=p,p-2$ $prime$	$\pi((n-3)!)$	p	$p-2$
$A_1(q)$	$3<q\equiv\varepsilon(4)$, $\varepsilon=\pm1$	$\pi(q-\varepsilon)$	$\pi(q)$	$\frac{q+\varepsilon}{2}$
$A_1(q)$	$q>2, q$ $even$	$\{2\}$	$q-1$	$q+1$
${}^2A_5(2)$		$\{2,3,5\}$	7	11
${}^2D_p(3)$	$p=2^m+1\geq 3$	$\pi(3(3^{p-1}-1)\prod_{i=1}^{p-2}(3^{2i}-1))$	$\frac{3^{p-1}+1}{2}$	$\frac{3^p+1}{4}$
$G_2(q)$	$q\equiv 0(3)$	$\pi(q(q^2-1))$	q^2-q+1	q^2+q+1
${}^2G_2(q)$	$q=3^{2m+1}>3$	$\pi(q(q^2-1))$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$F_4(q)$	$qeven$	$\pi(q(q^4-1)(q^6-1))$	q^4-q^2+1	q^4+1
${}^2F_4(q)$	$q=2^{2m+1}>2$	$\pi(q(q^3+1)(q^4-1))$	$q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$
$E_7(2)$		$\{2,3,5,7,11,13,17,19,31,43\}$	73	127
$E_7(3)$		$\{2,3,5,7,11,13,19,37,41,61,7$	757	1093
M_{11}		$\{2,3\}$	5	11
M_{23}		$\{2,3,5,7\}$	11	23
M_{24}		$\{2,3,5,7\}$	11	23
J_3		$\{2,3,5\}$	17	19
HiS		$\{2,3,5\}$	7	11
Suz		$\{2,3,5,7\}$	11	13
Co_2		$\{2,3,5,7\}$	11	23
Fi_{23}		$\{2,3,5,7,11,13\}$	17	23
F_3		$\{2,3,5,7,13\}$	19	31
F_2		$\{2,3,5,7,11,13,17,19,23\}$	31	47

3. Proof of main theorems

3.1. Proof of Theorem A

Let $M={}^2D_n(2)$, where $n=2^m+1\geq 5$. Assume that G is a finite group such that $|G|=|M|$ and $D(G)=D(M)$. Recall that $t(M)=2$ and $\pi(M)=\pi(2^{n(n-1)}(2^n+1)(2^{n-1}-1)\prod_{i=1}^{n-2}(2^{2i}-1))\cup\pi(2^{n-1}+1)$. By assumption, $|\pi(2^{n-1}+1)|=1$, so $\pi(2^{n-1}+1)\in T(G)-\{\pi_1(G)\}$. This shows that $t(G)\geq 2$. First, suppose that $t(G)\geq 3$. We are going to reach a contradiction under this assumption. Thus by Lemma 2.4, G is neither a Frobenius group nor a 2-Frobenius group and hence, by Lemma 2.6(ii), there is a normal series $1\trianglelefteq H\lhd K\trianglelefteq G$ for G such that $P=\frac{K}{H}$ is a non-abelian finite simple group and every odd order component of G is an odd order component of P and H is a nilpotent group. So $t(P) \geq 3$ and

$$2^{n-1}+1 \in OC(P)-\{m_1(P)\},\ where\ n = 2^m+1 \geq 5. \quad (1)$$

Thus the classification theorem of finite simple groups and Tables 1 and 3 show that one of the following possibilities holds for P:

Case 1.
$P \cong {}^2A_5(2), E_7(2), E_7(3), M_{11}, M_{23}, M_{24}, J_3, HiS, Suz, Co_2, F_2, F_3, Fi_{23}, A_2(4), M_{22}, J_1, O'N, LyS, F_1, J_4, {}^2E_6(2), Fi'_{24}$.
By (1), $(2^{n-1}+1) \in OC(P)-\{m_1(P)\}$. Since $n \geq 5$, $2^{n-1}+1 \geq 17$. Thus considering the odd order components of the finite simple groups mentioned above leads to $P \cong J_3, Fi_{23}, {}^2E_6(2)$ or Fi'_{24}. In these cases, we can see that $n = 5$ and $|P|$ does not divide $|G| = |{}^2D_5(2)|$, which is impossible.

Case 2.
$P \cong A_p$, where $p > 6$ and $p, p-2$ are prime. Then $OC(P)-\{m_1(P)\} = \{p, p-2\}$, so by (1), $2^{n-1}+1 \in \{p, p-2\}$. If $p = 2^{n-1}+1$, then for every $m \geq 3$, the largest power of 2 dividing $|A_p|$ is $\left(\left[\frac{p}{2}\right]+\left[\frac{p}{4}\right]+\cdots\right)-1 = \left(\left[\frac{2^{n-1}+1}{2}\right]+\left[\frac{2^{n-1}+1}{2^2}\right]+\cdots\right)-1 = 2^{n-2}+2^{n-3}+\ldots+2+1-1 = 2^{n-1}-2 > n(n-1)$.

But $|G|_2 = |M|_2 = 2^{n(n-1)}$, so $|P| \nmid |G|$, which is impossible. If $m = 2$, then $p = 17$, so $|P| \nmid |G|$. If $p-2 = 2^{n-1}+1$, then $p = 2^{n-1}+3$, so the same argument as above leads us to a contradiction.

Case 3.
$P \cong {}^2D_p(3)$, where $p = 2^{m'}+1 \geq 5$. Then $OC(P)-\{m_1(P)\} = \{\frac{3^{p-1}+1}{2}, \frac{3^p+1}{4}\}$. Thus (1) shows that either $3^{p-1}-2^n = 1$ or $3^p = 2^{n+1}+3$. If $3^{p-1}-2^n = 1$, then by Lemma 2.7(i), $p = 3$ and $n = 3$, contradiction with assumption on n. If $3^p-3 = 2^{n+1}$, then $3|2^{n+1}$, which is impossible.

Case 4.
$P \cong A_1(q)$ and $2 < q$ is even. Then $OC(P)-\{m_1(P)\} = \{q-1, q+1\}$. Thus (1) shows that $2^{n-1}+1 \in \{q-1, q+1\}$. If $q-1 = 2^{n-1}+1$, then $q = 2(2^{n-2}+1)$, so q is not a power of 2, a contradiction. If $q+1 = 2^{n-1}+1$, then $q = 2^{n-1}$. Set $|\frac{G}{K}| = t$, so $|G| = t|H||P|$. Of course by Lemma 2.6(ii) and Lemma 2.9(i), $t|2^m$, so

$$t|H| = \frac{|G|}{|P|} = \frac{|{}^2D_n(2)|}{|A_1(q)|} = 2^{(n-1)^2}(2^n+1)\prod_{i=1}^{n-2}(2^{2i}-1).$$

Thus for every $r \in R_{2n}(2)$, $r||H|$. If $S \in Syl_r(H)$, then the order of S is a divisor of 2^n+1 and by Lemma 2.8, $m_2m_3|(|S|-1)$, which is a contradiction.

Case 5.
$P \cong A_1(q)$, where $q \equiv -1(mod\ 4)$. Then $OC(P)-\{m_1\} = \{q, \frac{q-1}{2}\}$. Thus by (1), $2^{n-1}+1 \in \{q, \frac{q-1}{2}\}$. If $q = 2^{n-1}+1$, then $q \equiv 1(mod\ 4)$, which is a contradiction. Now we assume that $\frac{q-1}{2} = 2^{n-1}+1$, so $q = 2^n+3$. Since $n = 2^m+1 \geq 5$, an easy computation shows that $5|2^n+3$, so q is a power of 5, say $q = 5^f$. Thus $q \equiv 1(mod 4)$, which is a contradiction.

Case 6.
$P \cong A_1(q)$, where $q \equiv 1(mod\ 4)$. Then $OC(P)-\{m_1(P)\} = \{q, \frac{q+1}{2}\}$. Thus by (1), $2^{n-1}+1 \in \{q, \frac{q+1}{2}\}$. First assume that $q = 2^{n-1}+1$ and $q = p^\alpha$, where $\alpha \geq 1$. So we have the following subcases:
(i) if $\alpha > 1$, then by Lemma 2.7(i), $\alpha = 2$ and $n = 4$, which is not the case;
(ii) if $\alpha = 1$, we have $q = p = 2^{n-1}+1$. Now we set $|\frac{G}{K}| = t$, so $|G| = t|H||P|$. By Lemma 2.6(ii) and Lemma 2.9(i), $t|2$. Thus

$$t|H| = \frac{|G|}{|P|} = 2^{(n-1)^2}(2^n+1)(2^{n-1}-1)(2^{n-2}-1)\prod_{i=1}^{n-3}(2^{2i}-1),$$

so repeating the argument given for Case 4 leads us to a contradiction.

If $\frac{q+1}{2} = 2^{n-1}+1$, then $q = 2^n+1$. Assume that $q = p^\alpha$ where $\alpha \geq 1$. So we have the following subcases:
(i) if $\alpha > 1$, then by Lemma 2.7(i), $\alpha = 2$ and $n = 3$, which is not the case;
(ii) if $\alpha = 1$, then $q = p = 2^n+1$ shows that p is a Fermat prime, and so n must be a power of 2, which is a contradiction because $n = 2^m+1$ is an odd prime.

Case 7.
$P \cong G_2(q)$, where $q \equiv 0(mod\ 3)$ or $P \cong {}^2G_2(q)$, where $q = 3^{2m+1} > 3$. If $P \cong G_2(q)$, then the same reasoning as above shows that $q^2+q+1 = 2^{n-1}+1$ or $q^2-q+1 = 2^{n-1}+1$. But $q^2+q+1, q^2-q+1 \equiv 1(mod\ 3)$ and $2^{n-1}+1 \equiv 2(mod\ 3)$ and hence, both cases are ruled out. If $P \cong {}^2G_2(q)$, then the same reasoning as above leads to a contradiction.

Case 8.
$P \cong {}^2F_4(q)$ or $P \cong {}^2B_2(q)$. Then the odd order components of P is a number of the form $2^if(2)+1$ such that $gcd(2, f(2)) = 1$. If $2^if(2)+1 =$

$2^{n-1}+1$, then we obtain $2^i f(2) = 2^{n-1}$, which is a contradiction.

Case 9.
$P \cong F_4(q)$, where q is even. Then $OC(P) - \{m_1(P)\} = \{q^4+1, q^4-q^2+1\}$, so by (1), $2^{n-1}+1 \in OC(P) - \{m_1\}$. If $2^{n-1}+1 = q^4 - q^2 + 1$, then $2^{n-1} = q^2(q^2-1)$, which is impossible. If $2^{n-1}+1 = q^4+1$, then $q^4 = 2^{n-1}$. If $\mathbf{n} = 5$, then $|P|_2 = 2^{24}$ which does not divide $|G|_2 = 2^{20}$. Thus $n \geq 6$ and hence, Zsigmondy's Theorem allows us to assume that r is a primitive prime divisor of $2^{\frac{3}{2}(n-1)} - 1$, but considering $|P|$ and $|G|$, we have

$$(2^{\frac{3}{2}(n-1)} - 1) | 2^{(n-6)(n-1)}(2^n+1) \prod_{i=1, i \neq i_0, i_1}^{n-2} (2^{2i}-1),$$

$where\ i_0 = \frac{n-1}{2}, i_1 = \frac{3}{4}(n-1).$ Thus:
(i) if $r | 2^{(n-1)(n-6)}$, then $r = 2$, which is a contradiction;
(ii) if $r|(2^n+1)$, then $r|(2^{2n}-1)$, so $r|(2^{2n}-1) - (2^{\frac{3}{2}(n-1)} - 1)$. Therefore $r | 2^{\frac{3}{2}(n-1)}(2^{2n-\frac{3}{2}(n-1)} - 1)$, hence $r|(2^{2n-\frac{3}{2}(n-1)} - 1)$ implies that $2n - \frac{3}{2}(n-1) \geq \frac{3}{2}(n-1)$ and so $n \leq 3$, contradicting;
(iii) if $r | \prod_{i=1, i\neq i_0, i_1}^{n-2} (2^{2i}-1)$, then for some j, $1 \leq j \leq n-2, j \notin \{i_0, i_1\}$, the same argument as above shows that $j > n-2$, which is a contradiction.

This shows that $|P| \nmid |G|$, which is contradiction.

Case 10.
$P \cong E_8(q)$. If $P \cong E_8(q)$ with $q \equiv 2, 3 (mod\ 5)$, then the odd order components of P are $q^8+q^7-q^5-q^4-q^3+q+1$, $q^8-q^7+q^5-q^4+q^3-q+1$ and $q^8-q^6+q^4-q^2+1$. If $2^{n-1}+1 = q^8+q^7-q^5-q^4-q^3+q+1$, then:
(i)if $q \equiv 2 (mod 5)$, then $q^8+q^7-q^5-q^4-q^3+q+1 \equiv 1 (mod 5)$, but $2^{n-1}+1 \equiv 2 (mod 5)$, which is a contradiction;
(ii) if $q \equiv 3 (mod 5)$, then we get a contradiction in a similar manner.

Therefore, $2^{n-1}+1 = q^8-q^7+q^5-q^4+q^3-q+1$ or $2^{n-1}+1 = q^8-q^6+q^4-q^2+1$, then the same reasoning as above leads to a contradiction. If $P \cong E_8(q)$, where $q \equiv 0, 1, 4 (mod\ 5)$, then we get a contradiction in a similar manner.

The above contradictions imply that $t(G) = 2$, so $\pi_1(G) = \pi_1(M)$ and $\pi_2(G) = \pi_2(M)$. Thus $OC(G) = OC(M)$, so the main theorem in [14] shows that $G \cong M$, as claimed.

3.2. Proof of Theorem B

Let $M = {}^2D_n(3)$, where $n = 2^m + 1 \geq 9$ is not prime. Assume that G is a finite group such that $|G| = |M|$ and $D(G) = D(M)$. Recall that $t(M) = 2$ and $\pi(M) = \pi(\frac{1}{2}.3^{n(n-1)}(3^n+1)(3^{n-1}-1)\prod_{i=1}^{n-2}(3^{2i}-1)) \mathrm{U} \pi(\frac{3^{n-1}+1}{2})$. By assumption, $|\pi(\frac{3^{n-1}+1}{2})| = 1$, so $\pi(\frac{3^{n-1}+1}{2}) \in T(G) - \{\pi_1(G)\}$. This shows that $t(G) \geq 2$. First suppose that $t(G) \geq 3$. We will reach a contradiction under this assumption. Thus by Lemma 2.4, G is neither a Frobenius group nor a $\mathbf{2}$-Frobenius group and hence, by Lemma 2.6(ii), there is a normal series $1 \trianglelefteq H \lhd K \trianglelefteq G$ of G such that $P = \frac{K}{H}$ is a non-abelian finite simple group, H is a nilpotent group and every odd order component of G is an odd order component of P. So $t(P) \geq 3$ and

$$\frac{3^{n-1}+1}{2} \in OC(P) - \{m_1(P)\},\ where\ n = 2^m + 1 \geq 9\ is\ not\ prime. \quad (2)$$

Thus the classification theorem of finite simple groups and Tables $\mathbf{1}$ and $\mathbf{3}$ show that one of the following possibilities holds for P:

Case 1.
$P \cong {}^2A_5(2), E_7(2), E_7(3), M_{11}, M_{23}, M_{24}, J_3, HiS, Suz, Co_2, F_2, F_3, Fi_{23}, A_2(4), M_{22}, J_1, O'N, LyS, F_1, J_4, {}^2E_6(2), Fi'_{24}$. By (2), $\frac{3^{n-1}+1}{2} \in OC(P) - \{m_1\}$, so "$\frac{3^{n-1}+1}{2} \geq 3281$" implies that $\frac{3^{n-1}+1}{2}$ is larger than every odd order component of the above groups.

Case 2.
$P \cong A_p$, where $p > 6$ and $p, p-2$ are prime. Then $OC(P) - \{m_1(P)\} = \{p, p-2\}$ and hence, by (2), $\frac{3^{n-1}+1}{2} \in \{p, p-2\}$. If $p = \frac{3^{n-1}+1}{2}$, then $p - 2 = \frac{3(3^{n-2}-1)}{2}$. But $p-2$ is prime, so $3^{n-2} - 1 = 2$. Therefore $n = 3$, contradiction with assumption on n. If $p - 2 = \frac{3^{n-1}+1}{2}$, then $p = \frac{3^{n-1}+5}{2}$, so the largest power of $\mathbf{3}$ dividing $|A_p|$ is $[\frac{p}{3}] + [\frac{p}{9}] + \ldots > \frac{3^{n-2}-1}{2} > n(n-1)$. But $|G|_3 = |M|_3 = 3^{n(n-1)}$, so $|P| \nmid |G|$.

Case 3.
$P \cong {}^2D_p(3)$, where $p = 2^{m'} + 1 \geq 5$ is prime. Then $OC(P) - \{m_1(P)\} = \{\frac{3^{p-1}+1}{2}, \frac{3^p+1}{4}\}$. If $\frac{3^{n-1}+1}{2} = \frac{3^{p-1}+1}{2}$, then $p = n$, which is a contradiction, because assumption says that n is not prime. If $\frac{3^p+1}{4} = \frac{3^{n-1}+1}{2}$, then $3^p + 1 = 2(3^{n-1} +$

$\mathbf{1}$). We obtain that $3^p = 2.3^{n-1}+1 \equiv 1(mod\,3)$, which is a contradiction. Thus both cases are ruled out.

Case 4.
$P \cong A_1(q)$ and $2 < q$ is even. Then $OC(P) - \{m_1(P)\} = \{q-1, q+1\}$. Thus (2) shows that $\frac{3^{n-1}+1}{2} \in \{q-1, q+1\}$. If $q-1 = \frac{3^{n-1}+1}{2}$, then $3^{n-1}+3 = 2q$. So $3|q$, which is a contradiction. Therefore, $q+1 = \frac{3^{n-1}+1}{2}$, so $3^{n-1}-2q = 1$. Since q is a power of 2, then by Lemma 2.7(i), $n = 3$, which is not the case.

Case 5.
$P \cong A_1(q)$, where $q \equiv -1(mod\,4)$. Then $OC(P) - \{m_1(P)\} = \{q, \frac{q-1}{2}\}$. Thus by (2), $\frac{3^{n-1}+1}{2} \in \{q, \frac{q-1}{2}\}$. If $q = \frac{3^{n-1}+1}{2}$, then $2(q+1) = 3^{n-1}+3$. But $2(q+1) \equiv 0(mod\,8)$ and $3^{n-1}+3 \equiv 4(mod\,8)$, which is a contradiction. If $\frac{q-1}{2} = \frac{3^{n-1}+1}{2}$, then $q = 3^{n-1}+2$. This shows that $q-1 = 2\frac{3^{n-1}+1}{2}$. But by our assumption $\pi(\frac{3^{n-1}+1}{2}) = \{r\}$ and hence $q-1 = 2r^t$. Assume $q = p^\alpha$ where $\alpha \geq 1$ and obviously $p > 3$, because $q \equiv 1(mod\,2)$ and $q \equiv 2(mod\,3)$. So we have the following subcases:
(i) if $\alpha > 1$, then $p-1|q-1$, so $p-1|2r^t$. Hence $p-1|2$, which is a contradiction;
(ii) if $\alpha = 1$, we have $q = p = 3^{n-1}+2$. Now we set $|\frac{G}{K}| = t$, so $|G| = t|H||P|$. By Lemma 2.6(ii) and Lemma 2.9(i), $t|2$. Thus

$$3^n+1|t|H| = \frac{|G|}{|P|},$$

so for every $r \in R_{2n}(3)$, $r||H|$. If $S \in Syl_r(H)$, then the order of S is a divisor of 3^n+1 and by Lemma 2.8, $m_2m_3|(|S|-1)$, which is a contradiction.

Case 6.
$P \cong A_1(q)$, where $q \equiv 1(mod\,4)$. Then $OC(P) - \{m_1(P)\} = \{q, \frac{q+1}{2}\}$, so by (2), $\frac{3^{n-1}+1}{2} \in \{q, \frac{q+1}{2}\}$. First assume that $\frac{q+1}{2} = \frac{3^{n-1}+1}{2}$, so $q = 3^{n-1}$. Now we set $|\frac{G}{K}| = t$, so $|G| = t|H||P|$, of course by Lemma 2.6(ii) and Lemma 2.9(i), $t|2^{m+1}$. Thus

$$t|H| = \frac{|G|}{|P|} = \frac{1}{2}3^{(n-1)^2}(3^n+1)\prod_{i=1}^{n-2}(3^{2i}-1).$$

This implies that for every $r \in R_{2n}(3)$, $r||H|$. If $S \in Syl_r(H)$, then the order of S is a divisor of 3^n+1 and by Lemma 2.8, $m_2m_3|(|S|-1)$, which is a contradiction. This leads to $q = \frac{3^{n-1}+1}{2}$ and $q = p^\alpha$, so by Lemma 2.7(ii), $\alpha = 1$ and the same reasoning as above leads to get a contradiction.

Case 7.
$P \cong G_2(q)$, where $q \equiv 0(mod\,3)$ or $P \cong {}^2G_2(q)$, where $q = 3^{2m+1} > 3$. If $P \cong G_2(q)$, then the same reasoning as above shows that $q^2+q+1 = \frac{3^{n-1}+1}{2}$ or $q^2-q+1 = \frac{3^{n-1}+1}{2}$. But $2q^2+2q+1, 2q^2-2q+1 \equiv 1(mod\,3)$ and $3^{n-1} \equiv 0(mod\,3)$ and hence, both cases are ruled out. If $P \cong {}^2G_2(q)$, then the same reasoning as above leads to get a contradiction.

Case 8.
$P \cong {}^2F_4(q)$, where $q = 2^{2m'+1} > 2$. Then $OC(P) - \{m_1(P)\} = \{q^2+\sqrt{2q^3}+q+\sqrt{2q}+1, q^2-\sqrt{2q^3}+q-\sqrt{2q}+1\}$, so by (2), $\frac{3^{n-1}+1}{2} \in OC(P) - \{m_1\}$. If $\frac{3^{n-1}+1}{2} = q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$, then $3^{n-1} = 2^{4m'+3}+2^{3m'+3}+2^{2m'+2}+2^{m'+2}+1$. But $3^{n-1} \equiv 0(mod\,3)$ and $2^{4m'+3}+2^{3m'+3}+2^{2m'+2}+2^{m'+2}+1 \equiv 1(mod 3)$, which is a contradiction. If $\frac{3^{n-1}+1}{2} = q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$, then we get a contradiction in a similar manner.

Case 9.
$P \cong F_4(q)$, where q is even. Then $OC(P) - \{m_1\} = \{q^4+1, q^4-q^2+1\}$, so by (2), $\frac{3^{n-1}+1}{2} \in OC(P) - \{m_1(P)\}$. If $\frac{3^{n-1}+1}{2} = q^4+1$, then $3^{n-1}-2q^4 = 1$. Hence by Lemma 2.7(ii), $n = 3$, contraction with our assumption. Therefore $\frac{3^{n-1}+1}{2} = q^4-q^2+1$, so $3^{n-1} = 2q^4-2q^2+1$. Since q is a power of 2, an easy computation shows that $2q^4-2q^2+1 \equiv 1(mod\,3)$, which is a contradiction.

Case 10.
$P \cong {}^2B_2(q)$, where $q = 2^{2m'+1} > 2$. Then $OC(P) - \{m_1\} = \{q+\sqrt{2q}+1, q-\sqrt{2q}+1, q-1\}$, so by (2), $\frac{3^{n-1}+1}{2} \in OC(P) - \{m_1\}$. If $\frac{3^{n-1}+1}{2} = q-1$, then $3^{n-1}+3 = 2q$, therefore $3|q$, which is a contradiction. If $\frac{3^{n-1}+1}{2} = q+\sqrt{2q}+1$, then $3^{2^m} = 2^{2(m'+1)}+2.2^{m'+1}+1$ and hence, $(3^{2^{m-1}})^2 = (2^{m'+1}+1)^2$. Thus $3^{2^{m-1}} = 2^{m'+1}+1$, so by Lemma 2.7(i), $m = 2$, which is

impossible. If $\frac{3^{n-1}+1}{2}=q-\sqrt{2q}+1$, then $3^{n-1}=2^{2m'+2}-2^{m'+2}+1$. Thus:
(i) if m' is odd, then $1\equiv 3^{n-1}=2^{2m'+2}-2^{m'+2}+1\not\equiv 1(mod 5)$, which is a contradiction;
(ii) if m' is even, then $0\equiv 3^{n-1}=2^{2m'+2}-2^{m'+2}+1\equiv 1(mod 3)$, which is a contraction.

Case11.
$P\cong E_8(q)$. If $P\cong E_8(q)$ with $q\equiv 2,3(mod\ 5)$, then the odd order components of P are $q^8+q^7-q^5-q^4-q^3+q+1$, $q^8-q^7+q^5-q^4+q^3-q+1$ and $q^8-q^6+q^4-q^2+1$. If $\frac{3^{n-1}+1}{2}=q^8+q^7-q^5-q^4-q^3+q+1$, then:
(i) if $q\equiv o(mod 3)$, then $2(q^8+q^7-q^5-q^4-q^3+q+1)\equiv 2(mod\ 3)$ and hence, $3^{n-1}+1\equiv 2(mod 3)$, which is a contradiction;
(ii) if $q\equiv 1,2(mod\ 3)$, then we get a contradiction in a similar manner.

Therefore $\frac{3^{n-1}+1}{2}=q^8-q^7+q^5-q^4+q^3-q+1$ or $\frac{3^{n-1}+1}{2}=q^8-q^6+q^4-q^2+1$, then the same reasoning as above leads to get a contradiction. If $P\cong E_8(q)$, where $q\equiv 0,1,4(mod\ 5)$, then we get a contradiction in a similar manner.

The above contradictions imply that $t(G)=2$, so $\pi_1(G)=\pi_1(M)$ and $\pi_2(G)=\pi_2(M)$. Thus $OC(G)=OC(M)$ and hence, the main theorem in [15] shows that $G\cong M$, as claimed.

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