**OD-Characterization of some orthogonal groups**

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**Abstract**

In this paper, it was shown that \(2D_n(2)\), where \(n = 2^m + 1 \geq 5\) and \(|\pi(2^{n-1} + 1)| = 1\), and \(2D_n(3)\), where \(n = 2^m + 1 \geq 9\) is not prime and \(|\pi(2^{n-1} + 1)| = 1\), are OD-characterizable.

**Keywords**: Simple groups; prime graph; degree of a vertex; degree pattern

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1. Introduction

Let \(G\) be a finite group, \(\pi(G)\) the set of all prime divisors of its order and let \(\omega(G)\) be the spectrum of \(G\), that is the set of its element orders. The prime graph \(\Gamma(G)\) of \(G\) is a simple graph with vertex set \(\pi(G)\) in which two distinct vertices \(p\) and \(q\) are joined by an edge (and written \(p \sim q\)) if and only if \(pq \in \omega(G)\). Denote by \(t(G)\) the number of connected components of \(\Gamma(G)\). The \(i\)-th connected component is denoted by \(\pi_i = \pi_i(G)\) for each \(i\). If \(2 \in \pi(G)\), then we assume that \(2 \in \pi_1\). For \(p \in \pi(G)\), \(\deg(p) = |\{q \in \pi(G) \mid p \sim q\}|\) is called the degree of \(p\). If \(\pi(G) = \{p_1, p_2, \ldots, p_k\}\) with \(p_1 < p_2 < \ldots < p_k\), we also define \(D(G) = (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))\) which is called the degree pattern of \(G\). It is clear that the order of \(G\) can be expressed as the product of the numbers \(m_1, m_2, \ldots, m_t(G)\), where \(\pi(m_i) = \pi_i, 1 \leq i \leq t(G)\). If the order of \(G\) is even and \(t(G) \geq 2\), according to our notation \(m_2, m_3, \ldots, m_t(G)\) are odd numbers. The positive integers \(m_1, m_2, \ldots, m_t(G)\) are called the order components of \(G\) and \(OC(G) = \{m_1, m_2, \ldots, m_t(G)\}\) is called the set of order components of \(G\). and \(T(G) = \{\pi_i(G) \mid i = 1, 2, \ldots, t(G)\}\) is called the set of connected components of \(G\).

\(\Omega_0(G) = \{p \in \pi(G) \mid \deg(p) = 0\}\) and \(\Omega_0(G) = \{p \in \pi(G) \mid \deg(p) \neq 0\}\). Clearly, \(\pi(G) = \Omega_0(G) \cup \Omega_0(G)\). Given a finite group \(M\), denote by \(h_{OD}(M)\) the number of isomorphism classes of finite groups \(G\) such that \(|G| = |M|\) and \(D(G) = D(M)\). A finite group \(M\) is called \(k\)-fold OD-characterizable if \(h_{OD}(M) = k\). Usually a 1-fold OD-characterizable group is simply called OD-characterizable [1]. Also in [1], Darafsheh et.al proved that the sporadic simple groups, alternating groups \(A_p\), where \(p \neq 2\) are primes, and some simple groups of Lie type are OD-characterizable, and \(S_6(3)\) and \(O_7(3)\) are 2-fold OD-characterizable groups. In [2], it has been proved that \(A_{10}\) is 2-fold OD-characterizable. In [3], it has been proved that all simple groups whose orders are less than \(10^6\) except \(A_{10}\) and \(U_6(2)\) are OD-characterizable. According to [4], \(B_n(3)\) and \(C_n(3)\), where \(r\) is an odd prime and \(|\pi(\frac{3^{p-1}}{2})| = 1\) and \(B_n(q), C_n(q)\), for certain \(n, q\), and the simple groups \(B_2(5)\) and \(C_2(5)\) are 2-fold OD-characterizable. In this paper, we prove that:

**Theorem A**. Let \(G\) be a finite group such that \(|G| = |2D_n(2)|\) and \(D(G) = D(2D_n(2))\), where \(n = 2^m + 1 \geq 5\) and \(|\pi(2^{n-1} + 1)| = 1\). Then \(G \cong 2D_n(2)\).

**Theorem B**. Let \(G\) be a finite group such that \(|G| = |2D_n(3)|\) and \(D(G) = D(2D_n(3))\), where \(n = 2^m + 1 \geq 9\) is not prime and \(|\pi(\frac{3^{p-1}}{2})| = 1\). Then \(G \cong 2D_n(3)\).

2. Preliminary results

If \(a\) is a natural number, \(r\) is an odd prime and \((r, a) = 1\), then by \(e(r, a)\) we denote the smallest natural number \(n\) with \(a^n \equiv 1 \pmod r\). A prime \(r\) with \(e(r, a) = n\) is called a primitive prime divisor of \(a^n - 1\). We denote by \(R_n(a)\) the set of all the primitive prime divisors of \(a^n - 1\) and by \(r_n(a)\) every element of \(R_n(a)\), and \(n_p\) is \(p\)-part of \(n\).

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Lemma 2.1. (Zsigmondy’s Theorem)[5]Let a and n be integers greater than 1. There exists a prime divisor p of \(a^n - 1\) such that p does not divide \(a^j - 1\) for all \(1 \leq j < n\), except exactly in the following cases:

(i) \(n = 2, a = 2^2 - 1\), where \(s \geq 2\);

(ii) \(n = 6, a = 2\).

By Zsigmondy’s Theorem, \(R_n(a) \neq \phi\), unless \(a = 2, n = 6\) or \(n = 2\) and \(a = 2^n - 1\) for some natural number w. Obviously by Fermat’s little theorem it follows that \(e(r, a) | r - 1\). Also, if \(a^m \equiv 1 \pmod{r}\), then \(e(r, a) | m\). Also, for an integer \(n\), by \(\eta(n)\) we denote the following function:

\[
\eta(n) = \begin{cases} 
n & \text{n is odd} \\
\frac{n}{2} & \text{otherwise.} 
\end{cases}
\]

Lemma 2.2. [6, 7] Let \(G = \mathbb{G}_a(q)\) be a finite simple group of Lie type over a field of characteristic p. Let r and s be odd primes and \(r, s \in \pi(G) \setminus \{p\}\). Put \(k = e(r, q)\) and \(l = e(s, q)\).

Then:

(i) \(r, p\) and \(p\) are non-adjacent if and only if \(\eta(e(r, q)) > n - 2\);

(ii) if \(1 \leq \eta(k) \leq \eta(l)\), then \(r, s\) and \(s\) are non-adjacent if and only if \(2\eta(k) + 2\eta(l) > 2n - (1 + (-1)^k + 1)\) and \(\frac{1}{k}\) is not an odd natural number;

(iii) if \(p, f \neq 2\), then \(r, 2, 2\) are non-adjacent if and only if one of the following holds:

1. \(\eta(k) = n\) and \((4, q^n + 1) = (q^n + 1)_2\);

2. \(\eta(k) = \frac{k}{2} = n - 1, n\) is odd and \(e(2, q) = 2\).

Lemma 2.3. Let \(M = \mathbb{G}_a(p)\), where \(p \in \{2, 3\}\), \(n = 2^m + 1\) and \(|\pi_p(2n^2 - 1)\| = 1\). Then \(\deg(p) = |\pi(M) - (2 + |R_{2n^2}(p)|)|\).

Proof: First assume that \(p = 2\) and \(s \in \pi(G) \setminus \{2\}\). Then by Table 2, we have \(\pi(M) = \pi(2n^2 - 1 + 1)\) and by Lemma 2.2(i), \(s\) is non-adjacent to \(2\) if and only if \(s \in R_{2n^2(2)} \cup R_{2n^2(1)}\). But since \(R_{2n^2(1)} \subseteq \pi(2n^2 - 1)\), we have \(|R_{2n^2(2)}| = 1\). Therefore \(\deg(2) = |\pi(M) - (2 + |R_{2n^2}(2)|)|\). For \(p = 3,\) the same argument shows that \(\deg(3) = |\pi(M) - (2 + |R_{2n^2}(3)|)|\).

Lemma 2.4. [8] Let \(G\) be either a Frobenius group or a 2-Frobenius group of even order. Then \(t(G) = 2\).

Lemma 2.5. [1] Let \(G\) and \(M\) be finite groups such that \(|G| = |M|\) and \(D(G) = D(M)\). In addition, we suppose one of the following conditions holds:

(i) \(|\Omega_{\alpha}(M)| = 0\);

(ii) \(|\Omega_{\alpha}(M)| = 2\);
Table 1. Finite Simple Groups $P$ with $t(P) > 3$

<table>
<thead>
<tr>
<th>$P$</th>
<th>Restrictions on $P$</th>
<th>$\pi_1(P)$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$m_4$</th>
<th>$m_5$</th>
<th>$m_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$6 &lt; n = p, p+1, p+2$</td>
<td>$\pi((n-3)!)$</td>
<td>$p$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{p-1}(q)$</td>
<td>$(p, q) \neq (3, 2), (3, 4)$</td>
<td>$\pi(q \prod_{i=1}^{p-1} (q^i - 1))$</td>
<td>$q^p - 1$</td>
<td>$(q - 1)(p, q - 1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_p(q)$</td>
<td>$(q - 1)</td>
<td>(p + 1)$</td>
<td>$\pi(q(q^{p+1} - 1) \prod_{i=1}^{p-1} (q^i - 1))$</td>
<td>$q^p - 1$</td>
<td>$q - 1$</td>
<td>$q^p - 1$</td>
<td></td>
</tr>
<tr>
<td>$^2A_{p-1}(q)$</td>
<td>$(q + 1)</td>
<td>(p + 1)$</td>
<td>$\pi(q \prod_{i=1}^{p-1} (q^i - (-1)^i))$</td>
<td>$(q + 1)(p, q + 1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^2A_p(q)$</td>
<td>$(p, q) \neq (3, 3), (5, 2)$</td>
<td>$\pi(q(q^{p+1} - 1) \prod_{i=1}^{p-1} (q^i - (-1)^i))$</td>
<td>$q^p + 1$</td>
<td>$q + 1$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$B_n(q)$</td>
<td>$n = 2^m \geq 4, q$ is odd</td>
<td>$\pi(q \prod_{i=1}^{n-1} (q^{2^i} - 1))$</td>
<td>$q^n + 1$</td>
<td>$q^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_{p+1}(q)$</td>
<td>$q = 2, 3$</td>
<td>$\pi(q(q^p + 1) \prod_{i=1}^{p-1} (q^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
<td>$q^p - 1$</td>
<td>$q - 1$</td>
<td>$q^p - 1$</td>
<td></td>
</tr>
<tr>
<td>$D_{p+1}(q)$</td>
<td>$q = 2, 3$</td>
<td>$\pi(q(q^p + 1) \prod_{i=1}^{p-1} (q^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
<td>$q^p - 1$</td>
<td>$q - 1$</td>
<td>$q^p - 1$</td>
<td></td>
</tr>
<tr>
<td>$^2D_n(q)$</td>
<td>$n = 2^m \geq 4$</td>
<td>$\pi(q \prod_{i=1}^{n-1} (q^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
<td>$q^n + 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^2D_{n+2}(q)$</td>
<td>$n = 2^m + 1, m \geq 2$</td>
<td>$\pi(2(2^n + 1) \prod_{i=1}^{n-2} (2^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
<td>$q^n + 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^2D_{n+3}(q)$</td>
<td>$5 \leq p \neq 2^m + 1$</td>
<td>$\pi(2(3^n + 1) \prod_{i=1}^{n-2} (3^{2^i} - 1))$</td>
<td>$3^{p+1}$</td>
<td>$4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^2G_2(q)$</td>
<td>$2 &lt; q \equiv \varepsilon(3), \varepsilon = \pm 1$</td>
<td>$\pi(q(q^2 - 1)(q^2 - \varepsilon))$</td>
<td>$q^2 - \varepsilon q + 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Finite Simple Groups $P$ with $t(P) = 2$

<table>
<thead>
<tr>
<th>$P$</th>
<th>Restrictions on $P$</th>
<th>$\pi_1(P)$</th>
<th>$m_2$</th>
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<tr>
<td>$A_n$</td>
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<td>$A_p(q)$</td>
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<td>(p + 1)$</td>
<td>$\pi(q(q^{p+1}) \prod_{i=1}^{p-1} (q^i - 1))$</td>
</tr>
<tr>
<td>$^2A_{p-1}(q)$</td>
<td>$(q + 1)</td>
<td>(p + 1)$</td>
<td>$\pi(q \prod_{i=1}^{p-1} (q^i - (-1)^i))$</td>
</tr>
<tr>
<td>$^2A_p(q)$</td>
<td>$(p, q) \neq (3, 3), (5, 2)$</td>
<td>$\pi(q(q^{p+1}) \prod_{i=1}^{p-1} (q^i - (-1)^i))$</td>
<td>$q^p + 1$</td>
</tr>
<tr>
<td>$B_n(q)$</td>
<td>$n = 2^m \geq 4, q$ is odd</td>
<td>$\pi(q \prod_{i=1}^{n-1} (q^{2^i} - 1))$</td>
<td>$q^n + 1$</td>
</tr>
<tr>
<td>$B_{p+1}(q)$</td>
<td>$q = 2, 3$</td>
<td>$\pi(q(q^p + 1) \prod_{i=1}^{p-1} (q^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
</tr>
<tr>
<td>$D_{p+1}(q)$</td>
<td>$q = 2, 3$</td>
<td>$\pi(q(q^p + 1) \prod_{i=1}^{p-1} (q^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
</tr>
<tr>
<td>$^2D_n(q)$</td>
<td>$n = 2^m \geq 4$</td>
<td>$\pi(q \prod_{i=1}^{n-1} (q^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
</tr>
<tr>
<td>$^2D_{n+2}(q)$</td>
<td>$n = 2^m + 1, m \geq 2$</td>
<td>$\pi(2(2^n + 1) \prod_{i=1}^{n-2} (2^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
</tr>
<tr>
<td>$^2D_{n+3}(q)$</td>
<td>$5 \leq p \neq 2^m + 1$</td>
<td>$\pi(2(3^n + 1) \prod_{i=1}^{n-2} (3^{2^i} - 1))$</td>
<td>$(2, q - 1)$</td>
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<td>$q^2 - \varepsilon q + 1$</td>
</tr>
</tbody>
</table>
3. Proof of main theorems

3.1. Proof of Theorem A

Let $M = 2D_2(n)(2)$, where $n = 2^m + 1 \geq 5$. Assume that $G$ is a finite group such that $|G| = |M|$ and $D(G) = D(M)$. Recall that $t(M) = 2$ and $\pi(M) = \pi(2^{n-1})(2^n + 1)(2^{n-1} - 1)[\prod_{i=1}^{n-2}(2^{2i} - 1)]\pi(2^{n-1} + 1)$. By assumption, $|\pi(2^{n-1} + 1)| = 1$, so $\pi(2^{n-1} + 1) \in T(G) - \pi_1(G)$. This shows that $t(G) \geq 2$. First, suppose that $t(G) \geq 3$. We are going to reach a contradiction under this assumption. Thus by Lemma 2.4, $G$ is not a Frobenius group nor a $2$-Frobenius group and hence, by Lemma 2.6(ii), there is a normal series $G = H \leq K \leq G$ such that $P = \frac{K}{H}$ is a non-abelian finite simple group and every odd order.
component of $G$ is an odd order component of $P$ and $H$ is a nilpotent group. So $t(P) \geq 3$ and

$$2^n + 1 \in OC(P) - \{m_4(P)\}, \text{ where } n = 2^m + 1 \geq 5.$$  \hfill (1)

Thus the classification theorem of finite simple groups and Tables 1 and 3 show that one of the following possibilities holds for $P$:

**Case 1.**

$P \cong A_5(2)$, $E_7(2)$, $E_7(3)$, $M_{11}$, $M_{22}$, $M_{24}$, $J_3$. His, Suz, Co$_2$J, F$_4$, FL$_{25}$, FL$_{24}$, FL$_g$, $H_1$, $O'N$, LyS, $F_4$, $J_4$, $E_6(2)$, FL$_{24}$.

By (1), $(2^n + 1) \in OC(P) - \{m_1(P)\}$. Since $n \geq 5$, $2^n + 1 \geq 17$. Thus considering the odd order components of the finite simple groups mentioned above leads to $P \cong J_3$, FL$_{23}$ or $E_6(2)$ or $FL_{24}$. In these cases, we can see that $n = 5$ and $|P|$ does not divide $|G| = |2D_2(5)|$, which is impossible.

**Case 2.**

$P \cong A_7(p)$, where $p > 6$ and $p - 2$ are prime. Then $OC(P) - \{m_4(P)\} = \{p, p - 2\}$, so by (1), $2^n + 1 \in \{p, p - 2\}$. If $p = 2^n + 1$, then for every $m \geq 3$, the largest power of dividing $|A_p|$ is $\left(\frac{2^m + 1}{p} \right)\cdots 1 = \left(\frac{2^n - 1}{2} + \frac{2^n - 1}{2} + \ldots + 1\right) - 1 = 2^n - 2 + 2^n - 3 + \ldots + 2 + 1 - 1 = 2^n - 1 > n(n - 1)$.

But $|G| = |M| = 2n(n-1)$, so $|P| \nmid |G|$, which is impossible. If $m = 2$, then $p = 17$, so $|P| \nmid |G|$. If $p - 2 = 2^n + 1$, then $p = 2^n + 3$, so the same argument as above leads us to a contradiction.

**Case 3.**

$P \cong 2D_2(3)$, where $p = 2^m + 1 \geq 5$. Then $OC(P) - \{m_4(P)\} = \{3p^m - 1, 3p^m + 1\}$. Thus (1) shows that either $3p^m - 1 - 2 = 0$ or $3p = 2^n - 1$. If $3p^m - 1 = 2n - 1$, then $p = 3$, contradiction with assumption on $n$. If $3p - 3 = 2n - 1$, then $3|2n + 1$, which is impossible.

**Case 4.**

$P \cong A_1(q)$ and $2 < q$ is even. Then $OC(P) - \{m_4(P)\} = \{q - 1, q, q + 1\}$. Thus (1) shows that $2^n + 1 \in \{q - 1, q, q + 1\}$. If $q - 1 = 2^n - 1$, then $q = 2^n + 1$, so $q$ is not a power of $2$, a contradiction. If $q + 1 = 2^n + 1$, then $q = 2^n - 1$. Set $|G| = t|H|$, so $|G| = t|H| |P|$. Of course by Lemma 2.6(ii) and Lemma 2.9(i), $t|2^n| = \frac{|G|}{|P|} = \frac{|2D_2(3)|}{|A_1(q)|} = 2^{2n-1} \left(\frac{2^n - 1}{2}\right)$. Thus for every $r \in R_{2n}(2)$, $|r| |H|$. If $S \in Sy_l(H)$, then the order of $S$ is a divisor of $2^n + 1$ and by Lemma 2.8, $m_2m_3 | (|S| - 1)$, which is a contradiction.

**Case 5.**

$P \cong A_1(q)$, where $q \equiv -1 \pmod{4}$. Then $OC(P) - \{m_4(P)\} = \{q, q + 1\}$. Thus by (1), $2^n + 1 \in \{q, \frac{q + 1}{2}\}$. If $q = 2^n - 1$, then $q \equiv 1 \pmod{4}$, which is a contradiction. Now we assume that $q \equiv 2^n - 1 \pmod{4}$, so $q = 2^n + 3$. Since $n = 2^m + 1 \geq 5$, an easy computation shows that $2^n + 3$, so $q$ is a power of $5$, say $q = 5^f$. Thus $q \equiv 1 \pmod{4}$, which is a contradiction.

**Case 6.**

$P \cong A_1(q)$, where $q \equiv 1 \pmod{4}$. Then $OC(P) - \{m_4(P)\} = \{q, \frac{q + 1}{2}\}$. Thus by (1), $2^n + 1 \in \{q, \frac{q + 1}{2}\}$. First assume that $q = 2^n - 1 + 1$, and $q = p^e$, where $\alpha \geq 1$. So we have the following subcases:

(i) if $\alpha > 1$, then by Lemma 2.7(i), $\alpha = 2$ and $n = 4$, which is not the case;

(ii) if $\alpha = 1$, we have $q - p = 2^n - 1 + 1$. Now we set $|G| = t|H| |P|$. By Lemma 2.6(ii) and Lemma 2.9(i), $t|2|$. Thus $t|2| = \frac{|G|}{|P|} = 2^{n-1}|(2^n - 1)(2^{n-1} - 1)| 2^{n-2} - 1$. Thus repeating the argument given for Case 4 leads us to a contradiction.

If $q = 2^n - 1 + 1$, then $q = 2^n + 1$. Assume that $q = p^e$, where $\alpha \geq 1$. So we have the following subcases:

(i) if $\alpha > 1$, then by Lemma 2.7(i), $\alpha = 2$ and $n = 3$, which is not the case;

(ii) if $\alpha = 1$, then $q = p = 2^n - 1$ shows that $\alpha$ is a Fermat prime, and so $n$ must be a power of $2$, which is a contradiction because $n = 2^m + 1$ is an odd prime.

**Case 7.**

$P \cong G_2(q)$, where $q \equiv 0 \pmod{3}$ or $P \cong G_2(q)$, where $q = 2^{2m + 1} > 3$. If $P \cong G_2(q)$, then the same reasoning as above shows that $q^2 - q + 1 = 2^n - 1 + 1$ or $q^2 - q + 1 = 2^n + 1$. But $q^2 + q + 1, q^2 - q + 1 \equiv 1 \pmod{3}$ and $2^n - 1 + 1 \equiv 2 \pmod{3}$ and hence, both cases are ruled out. If $P \cong G_2(q)$, then the same reasoning as above leads to a contradiction.

**Case 8.**

$P \cong F_4(q)$ or $P \cong B_2(q)$. Then the odd order components of $P$ is a number of the form $2f(2) + 1$ such that $gcd(2, f(2)) = 1$. If $2f(2) + 1 = 2^n + 1 \geq 5$. Then $2^n + 1 \in OC(P) - \{m_4(P)\}$, which is a contradiction.
Let $M = D_n(3)$, where $n = 2^m + 1 \geq 9$ is not prime. Assume that $G$ is a finite group such that $|G| = |M|$ and $D(G) = D(M)$. Recall that $t(M) = 2$ and $\pi(M) = \pi(2^{3(n-1)}(3^n + 1)(3^{n-1} - 1))$. By assumption, $\pi(2^{3(n-1)})(3^n + 1)(3^{n-1} - 1)$ is not a prime divisor of $2^n - 1$, which is a contradiction.}

3.2. Proof of Theorem B

Case 9.

**P** ≠ $E_0(q)$, where $q$ is even. Then $OC(P) - \{m_1(P)\} = \{q^2 + 1, q^2 + q + 1\}$, so by (1), $2^{n-1} + 1 \in OC(P) - \{m_1\}$. If $2^{n-1} + 1 = q^2 + q + 1$, then $2^{n-1} = q^2(q - 2) + 1$, which is impossible. If $2^{n-1} + 1 = q^2 + 1$, then $q^2 = 2^{n-1}$. If $n = 5$, then $|P|^2 = 2^4$ which does not divide $|G_2| = 2^20$. Thus $n \geq 6$ and, hence, by Zsigmondy’s Theorem allows us to assume that $r$ is a primitive prime divisor of $2^{3(n-1)} - 1$, but considering $|P|$ and $|G|$, we have

$$2^{3(n-1)} - 1, \quad \text{where } i_0 = \frac{n-1}{2}, i_1 = 3(n - 1) - i_0.$$  

where $i_0 = \frac{n-1}{2}, i_1 = 3(n - 1) - i_0$. Thus:

(i) if $|P|^2 \leq 2(n-1)(n-6)$, then $P = 2$, which is a contradiction;

(ii) if $|P|^2 > 2(n-1)(n-6)$, then $P > 2$, which is a contradiction;

(iii) if $|P|^2 \leq 2(n-1)(2^{n-1} - 1)$, then $P = 2^n - 1$ and so $n \leq 3$, contradicting;

(iv) if $|P|^2 \leq 2(n-1)(2^{n-1} - 1)$, then $P = 2^{n-1} - 1$ and hence, by (2), we get a contradiction in a similar manner.

This shows that $|P| \nmid |G|$, which is a contradiction.

Case 10.

**P** ≠ $E_0(q)$, where $q = 2, 3(mod 5)$, then the order components of $P$ are $q^3 - q^2 - q + 1, q^2 - q + 1, q - 1$ and $q - 2$. If $q^3 - q^2 - q + 1 = 2^n - 1$, then $q^3 - q^2 - q + 1 \equiv 1 (mod 5)$, but $2^{n-1} + 1 \equiv 2 (mod 5)$, which is a contradiction.

Therefore,

$$2^{n-1} + 1 = q^3 - q^2 - q + 1 \quad \text{or} \quad 2^{n-1} + 1 = q^2 - q^2 + q^2 + q^2 + 1,$$

the same reasoning as above leads to a contradiction. If $P = E_0(q)$, where $q = 0, 1, 4 (mod 5)$, then we get a contradiction in a similar manner.

The above contradictions imply that $t(G) = 2$, so $\pi_1(G) = \pi_1(M)$ and $\pi_2(G) = \pi_2(M)$. Thus $OC(G) = OC(M)$, so the main theorem in [14] shows that $G \cong M$, as claimed.

3.2. Proof of Theorem B
1). We obtain that $3^p = 2 \cdot 3^{n-1} + 1 \equiv 1 (\text{mod } 3)$, which is a contradiction. Thus both cases are ruled out.

Case 4.
$P \equiv A_1(q)$ and $2 < q$ is even. Then $OC(P) - (m_1(P)) = \{q - 1, q + 1\}$. Thus (2) shows that $
 \frac{3^{q-1} + 1}{2} \in \{q - 1, q + 1\}. If q - 1 = \frac{3^{q-1} + 1}{2}, then
3^{n-1} + 3 = 2q. So 3\{q\}, which is a contradiction.

Therefore, $q + 1 = \frac{3^{q-1} + 1}{2}$, so $3^{n-1} - 2q = 1.$ Since $q$ is a power of 2, then by Lemma 2.7(i), $n = 3,$ which is not the case.

Case 5.
$P \equiv A_1(q)$, where $q \equiv -1 (\text{mod } 4)$. Then $OC(P) - (m_1(P)) = \{q - 1, q + 1\}$. Thus by (2), $\n \frac{3^{q-1} + 1}{2} \in \{q - 1, q + 1\}. If q = \frac{3^{q-1} + 1}{2}, then 2(q + 1) = \n 3^{n-1} + 3. But 2(q + 1) \equiv 0 (\text{mod } 8) and
3^{n-1} + 3 \equiv 4 (\text{mod } 8), which is a contradiction. If
\n$\frac{q - 1}{2} = \frac{3^{q-1} + 1}{2}, then q = 3^{n-1} + 2.$ This shows that
\n$q - 1 = 2 \cdot \frac{3^{q-1} + 1}{2}. But by our assumption
\n$\pi(\frac{3^{q-1} + 1}{2}) = \lfloor r \rfloor$ and hence $q - 1 = 2r^t.$ Assume
\n$q = p^t$, where $p \geq 1$ and obviously $p > 3$, because $q \equiv 1 (\text{mod } 2)$ and $q \equiv 2 (\text{mod } 3).$ So we have the following subcases:

(i) if $p > 1,$ then $p - 1 | q - 1,$ so $p - 1 | 2r^t.$ Hence $p - 1 | 2,$ which is a contradiction;

(ii) if $\alpha = 1,$ we have $q = p = 3^{n-1} + 2.$ Now we set $\n G = t, \text{ so } |G| = t|H||P| \text{. By Lemma 2.6(ii) and Lemma 2.9(i), } t|2.$ Thus
\n$3^n + 1 + t|H| = \frac{|G|}{|P|}$

so for every $r \in R_{2^n(3)}$, $r|H|$. If $S \in SyL_r(H)$, then the order of $S$ is a divisor of $3^n + 1$ and by Lemma 2.8, $m_2 m_3 (|S| - 1),$ which is a contradiction.

Case 6.
$P \equiv A_1(q)$, where $q \equiv 1 (\text{mod } 4)$. Then $OC(P) - (m_1(P)) = \{q - \frac{q + 1}{2}\}$, so by (2), $\n \frac{3^{q-1} + 1}{2} \in \{q - \frac{q + 1}{2}\}. First assume that
\n$\frac{q + 1}{2} = \frac{3^{q-1} + 1}{2},$ so $q = 3^{n-1}.$ Now we set $\n G = t, \text{ so } |G| = t|H||P|$, of course by Lemma 2.6(ii) and Lemma 2.9(i), $t|2^{m+1}.$ Thus
\n$t|H| = \frac{|G|}{|P|} \leq \frac{1}{2} \cdot 3^{(n-1)/2} (3^{n-1} + 1)$

This implies that for every $r \in R_{2^n(3)}$, $r|H|$. If $S \in SyL_r(H)$, then the order of $S$ is a divisor of $3^n + 1$ and by Lemma 2.8, $m_2 m_3 (|S| - 1),$ which is a contradiction. This leads to $q = \frac{3^{n-1} + 1}{2}$ and $q = p^t$, so by Lemma 2.7(ii), $\alpha = 1$ and the same reasoning as above leads to get a contradiction.

Case 7.
$P \equiv G_2(q)$, where $q \equiv 0 (\text{mod } 3)$ or $P \equiv G_2(q)$, where $q = 3 \cdot 2^{m+1} > 3.$ If $P \equiv G_2(q)$, then the same reasoning as above shows that $q^2 + q + 1 = \n \frac{3^{n-1} + 1}{2}$ or $q^2 - q + 1 = \frac{3^{n-1} + 1}{2}$. But $2q^2 + 2q + \n 1, 2q^2 - 2q + 1 \equiv 1 (\text{mod } 3)$ and $3^{n-1} \equiv \n 0 (\text{mod } 3)$ and hence, both cases are ruled out. If $P \equiv G_2(q)$, then the same reasoning as above leads to get a contradiction.

Case 8.
$P \equiv F_4(q)$, where $q = 2 \cdot 2^{m+1} > 2.$ Then $OC(P) - (m_1(P)) = \{q + \sqrt{2q^2 + q + 2\sqrt{2q + 1} \equiv 0 (\text{mod } 8) and \n 1, 2q^2 - 2q + 1 \equiv 1 (\text{mod } 3)$ and $3^{n-1} \equiv \n 0 (\text{mod } 3)$ and hence, both cases are ruled out. If $P \equiv F_4(q)$, then the same reasoning as above leads to get a contradiction.

Case 9.
$P \equiv F_4(q)$, where $q$ is even. Then $OC(P) - (m_1(P)) = \{q^4 + 1, q^4 - q^2 + 1\}$, so by (2), $\n \frac{3^{n-1} + 1}{2} \in \text{OC}(P) - (m_1(P)).$ If $\n \frac{3^{n-1} + 1}{2} = q^4 + 1$, then $3^{n-1} - 2q^4 = 1.$ Hence by Lemma 2.7(ii), $n = 3,$ contraction with our assumption. Therefore $\n \frac{3^{n-1} + 1}{2} = q^4 - q^2 + 1,$ so $3^{n-1} = 2q^4 - 2q^2 + 1.$ Since $q$ is a power of 2, an easy computation shows that $\n 2q^4 - 2q^2 + 1 \equiv 1 (\text{mod } 3)$, which is a contradiction.

Case 10.
$P \equiv B_2(q)$, where $q = 2 \cdot 2^{m+1} > 2.$ Then $OC(P) - (m_1(P)) = \{q + \sqrt{2q + 1, q - \sqrt{2q + 1}}, q - 1\}$. By (2), $\n \frac{3^{n-1} + 1}{2} \in \text{OC}(P) - (m_1(P))$. If $\n \frac{3^{n-1} + 1}{2} = q - 1$, then $3^{n-1} + 3 = 2q$, therefore $\n 3|q$, which is a contradiction. If $\n \frac{3^{n-1} + 1}{2} = q + \sqrt{2q + 1}$, then $3^{2m} = 2(m+1) + 2.2^{m+1} + 1$ and hence, $(3^{2m-1})^2 = (2^{m+1} + 1)^2.$ Thus $\n 3^{2m-1} = 2^{m+1} + 1$, so by Lemma 2.7(i), $m = 2$, which is
impossible. If \( \frac{3^{n-1} + 1}{2} = q - \sqrt{2q} + 1 \), then 
\( 3^{n-1} = \frac{2^{2m+2} - 2^{m+2} + 1}{2} \). Thus:

(i) if \( m' \) is odd, then \( 1 \equiv 3^{n-1} = \frac{2^{2m+2} - 2^{m+2} + 1}{2} \) \( \equiv 1 \pmod{5} \), which is a contradiction;

(ii) if \( m' \) is even, then \( 0 \equiv 3^{n-1} = \frac{2^{2m+2} - 2^{m+2} + 1}{2} \equiv 1 \pmod{5} \), which is a contradiction.

**Case 11.**

\[ P \cong E_6(q) \]  
If \( P \cong E_6(q) \) with \( q \equiv 2, 3 \pmod{5} \), then the odd order components of \( P \) are \( q^8 + q^7 - q^5 - q^4 + q + 1 \), \( q^8 - q^7 + q^5 - q^4 + q^3 - q + 1 \) and \( q^8 - q^6 + q^4 - q^2 + 1 \). If \( \frac{3^{n-1} + 1}{2} = q^8 + q^7 - q^5 - q^4 + q^3 + q + 1 \), then:

(i) if \( q \equiv 0 \pmod{3} \), then \( 2(q^8 + q^7 - q^5 - q^4 + q^3 + q + 1) \equiv 2 \pmod{3} \) and hence, \( 3^{n-1} + 1 \equiv 2 \pmod{3} \), which is a contradiction;

(ii) if \( q \equiv 1, 2 \pmod{3} \), then we get a contradiction in a similar manner.

Therefore \( \frac{3^{n-1} + 1}{2} = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1 \) or \( \frac{3^{n-1} + 1}{2} = q^8 - q^6 + q^4 - q^2 + 1 \), then the same reasoning as above leads to get a contradiction. If \( P \cong E_6(q) \), where \( q \equiv 0, 1, 4 \pmod{5} \), then we get a contradiction in a similar manner.

The above contradictions imply that \( t(G) = 2 \), so \( \pi_1(G) = \pi_1(M) \) and \( \pi_2(G) = \pi_2(M) \). Thus \( OC(G) = OC(M) \) and hence, the main theorem in [15] shows that \( G \cong M \), as claimed.

**References**


