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Approximations in (bi-)hyperideals of Semihypergroups

R. Ameri[1]*, S. Azizpour Arabi[2] and H. Hedayati[3]

[1]*School of Mathematics, Statistics and Computer Science,*
College of Sciences, University of Tehran, P.O. Box 14155-6455, Teheran, Iran
[2]*Department of Mathematics, University of Mazandaran, Babolsar, Iran*
[3]*Department of Mathematics, Babol University of Technology, Babol, Iran*
E-mail: rameri@ut.ac.ir

Abstract

We consider the fundamental relation β^* on a semihypergroup H to interpret the lower and upper approximations as subsets of the fundamental semigroup H/β^* and we give some results in this connection. Also, we introduce the notion of a bi-hyperideal to study the relationship between approximations and bi-hyperideals.

Keywords: Hyperoperation; Semihypergroup; (Bi-)hyperideal; Fundamental relation; Rough set; Approximation Space

1. Introduction

Hyperstructures, in particular hypergroups, were introduced in 1934 by a French mathematician, Marty, at the VIIIth Congress of Scandinavian Mathematicians [1]. Nowadays hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics; for example, semihypergroups are the simplest algebraic hyperstructures to possess the properties of closure and associativity. They are very important in the theory of sequential machines, formal language, and in certain applications. A comprehensive review of the theory of hyperstructures appears in [2-4].

In 1982, Pawlak [5] introduced the concept of a rough set. This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis [6-7]. The idea is to approximate a subset of a universal set by a lower approximation and an upper approximation in the following manner. A partition of the universe is given. The lower approximation is the union of those members of the partition contained in the given subset and the upper approximation is the union of those members of the partition which have a non-empty intersection with the given subset. It is well-known that a partition induces an equivalence relation on a set and vice versa. Since then the subject has been investigated in many papers, and subsequently the algebraic approach to rough sets has been studied by some authors, for example, Ali, Davvaz and Shabir [8], Estaji, Hooshmandasl and Davvaz [9], Iwinski [10], Pomykala and Pomykala [11], Biswas and Nanda [12], Kuroki [13], Kuroki and Wang [14], Comer [15], Davvaz [16-18],Y.B. Jan [19], Kazanci and Davvaz [20] and Xiao and Zhang [21]. In [22-28] Anvariyeh, Mirvakili and Davvaz, Zhan and X. Ma, Zhan and Tan, Rasouli and Davvaz, Leoreanu, Kazanci, Yamak and Davvaz applied the concept of approximation spaces in the theory of algebraic hyperstructures. The fundamental relations use corresponding classical theory of algebraic structures. These relations, on the one hand, connect this theory, in some way with the corresponding classical theory and on the other hand, introduce new important classes. In this paper we consider the fundamental relation β^* on a semihypergroup H to find the lower and upper approximations for subsets of H as subsets of the fundamental semigroup H/β^*, and obtain basic properties of this connection.

*Corresponding author
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2. Preliminaries

In this section we recall some definitions and results of semihypergroups and approximation theory from [2] and [3], which is necessary for the development of our paper.

Recall that A $hypergroupoid$ is a non-empty set H together with a map $:H \times H \to P^*(H)$ where $P^*(H)$ denotes the set of all non-empty subsets of H. The image of the pair (x, y) is denoted by $x \cdot y$. If $x \in H$ and A, B are subsets of H, then by $A \cdot B$,

$A \cdot x$ and $x \cdot B$ we mean $A \cdot B = \bigcup_{a\in A, b\in B} a \cdot b$, $A \cdot x = A \cdot \{x\}, x \cdot B = \{x\} \cdot B$.A hypergroupoid $(H,\cdot)$ is called a *semihypergroup* if associativity holds, that is $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in H$.

The motivating example is the following: Let S be a semigroup and K be any subsemigroup of S. Then $S/K = \{xK | x \in S\}$ becomes a semihypergroup where the hyperoperation is defined in a usual manner $\overline{x} \cdot \overline{y} = \{\overline{z} | z \in \overline{x} \cdot \overline{y}\}$ where $\overline{x} = xK$. A non-empty subset X of the semihypergroup H is called a *subsemihypergroup* of H if $X \cdot X \subseteq X$.

Definition 2.1. Let H be a semihypergroup. A non-empty subset $I$of H is called a *left* (resp. *right*) *hyperideal* if $H \cdot I \subseteq I$ (resp. $I \cdot H \subseteq I$), and a *hyperideal* of H if it is both a left and a right hyperideal of H.

Example 2.2. Let $H = \{1,2,3,4\}$ be a set with the hyperoperation " $\cdot$ " defined as follows:$1 \cdot 1 = 2 \cdot 2 = 3 \cdot 3 = 3 \cdot 4 = 4 \cdot 3 = 4 \cdot 4 = \{1,2,3\}; 1 \cdot 2 = 2 \cdot 1 = \{1,2\}; 1 \cdot 3 = 1 \cdot 4 = 3 \cdot 1 = 4 \cdot 1 = \{1,3\}$, and$2 \cdot 3 = 2 \cdot 4 = 3 \cdot 2 = 4 \cdot 2 = \{2,3\}$. Then $(H,\cdot)$ is a semihypergroup such that the set $\{1,2,3\}$ is a hyperideal of it.

Definition 2.3. Let H and H' be semihypergroups. A function $f: H \to H'$ is called a *homomorphism* if it satisfies the condition $f(x \cdot y) \subseteq f(x) \cdot f(y)$; f is a *stronghomomorphism* if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in H$.

3. Approximations in semihypergroups

Let θ be an equivalence relation defined on the semihypergroup H and $\theta(x)$ be the equivalence class of the relation θ generated by an element $x \in H$. Any finite union of equivalence classes of H is called a *definable* set in H. Let X be any subset of H. In general, X is not a definable set in H. However, the set X may be approximated by two definable sets in H. The first one is called a $\theta - lowerapproximation$ of X in H, denoted by $\underline{\theta(x)}$ and defined as follows: $\underline{\theta(X)} = \{x \in H | \theta(x) \subseteq X\}$. The second set is called a $\theta - upperapproximation$ of X in H, denoted by $\overline{\theta(X)}$ and defined as follows: $\overline{\theta(X)} = \{x \in H | \theta(x) \cap X \neq \emptyset\}$. The θ-lower approximation of X in H is the greatest definable set in H contained in X. The θ-upper approximation of X in H is the least definable set in H containing X. The difference $\widehat{\theta(X)} = \overline{\theta(X)} - \underline{\theta(X)}$ is called the $\theta - boundaryregion$ of X. In the case when $\widehat{\theta(X)} = \emptyset$ the set X is said to be $\theta - exact$, otherwise X is $\theta - rough$.

Let H_1 and H_2 be semihypergroups and T be a strong homomorphism from H_1 into H_2. The relation $T \circ T^{-1}$ is an equivalence relation θ on H_1 ($a\theta b$ if and only if $T(a) = T(b)$) known as the kernel of T.

Theorem 3.1. Let H_1 and H_2 be semihypergroups and T be a strong homomorphism from H_1 into H_2. If X is a non-empty subset of H_1; then$T(\overline{\theta(X)}) = T(X)$.

Proof: Since $X \subseteq \overline{\theta(X)}$, it follows that $T(X) \subseteq T(\overline{\theta(X)})$. To see the converse inclusion holds, let y be any element of $T(\overline{\theta(X)})$. Then there exists an element $x \in \overline{\theta(X)}$ such that $T(x) = y$. Therefore there exists $a \in H_1$ such that $a \in \theta(x) \cap X$, and so $T(a) = T(x)$ and $a \in X$. Then we obtain $y = T(x) = T(a) \in T(X)$, and so $T(\overline{\theta(X)}) \subseteq T(X)$.

Theorem 3.2. Let θ_1, θ_2 be equivalence relations on a semihypergroup H. If X is a non-empty subset of H, then $\overline{(\theta_1 \cap \theta_2)(X)} \subseteq \overline{\theta_1(X)} \cap \overline{\theta_2(X)}$.

Proof: Note that $\theta_1 \cap \theta_2$ is also an equivalence relation on H. Let $a \in \overline{(\theta_1 \cap \theta_2)(X)}$. Then $(\theta_1 \cap \theta_2)(a) \cap X \neq \emptyset$, and so there exists $x \in (\theta_1 \cap \theta_2)(a) \cap X$. Since $(x, a) \in \theta_1 \cap \theta_2$, we have $(x, a) \in \theta_1$ and $(x, a) \in \theta_2$. Therefore we have $x \in \theta_1(a)$ and $x \in \theta_2(a)$. Since $x \in X$, then $\theta_1(a) \cap X \neq \emptyset$, and $\theta_2(a) \cap X \neq \emptyset$. Thus $a \in \overline{\theta_1(X)}$ and $a \in \overline{\theta_2(X)}$. Therefore we obtain $\overline{(\theta_1 \cap \theta_2)(X)} \subseteq \overline{\theta_1(X)} \cap \overline{\theta_2(X)}$. This completes the proof.

Theorem 3.3. Let θ_1, θ_2 be equivalence relations on a semihypergroup H. If X is a non-empty subset of H; then $\underline{(\theta_1 \cap \theta_2)(X)} = \underline{\theta_1(X)} \cap \underline{\theta_2(X)}$.

Proof: We have$x \in \underline{(\theta_1 \cap \theta_2)(X)} \Leftrightarrow (\theta_1 \cap \theta_2)(x) \subseteq X \Leftrightarrow \theta_1(x) \subseteq X$ and $\theta_2(x) \subseteq X \Leftrightarrow x \in \underline{\theta_1(X)}andx \in \underline{\theta_2(X)} \Leftrightarrow x \in \underline{\theta_1(X)} \cap \underline{\theta_2(X)}$.

4. On the Fundamental Relation β^*

Throughout this section we let H be a semihypergroup.

The relation β^* is the smallest equivalence relation on H such that the quotient H/β^*, the set of all equivalence classes, is a semigroup. β^* is called the $fundamentalequivalencerelation$ on H. This relation is studied by Corsini [2] concerning hypergroups, see also [4] and [29].

According to [4] if U denotes the set of all the finite products of elements of H, then a relation β can be defined on H whose transitive closure is the fundamental relation β^*. The relation β is as follows: for x and y in H we write $x\beta y$ if and only if $\{x,y\} \in u$, for some $u \in U$. We can rewrite the definition of β^* on H as follows: $a\beta^* b$ iff there exist $z_1,\ldots,z_{n+1} \in H$ with $z_1 = a, z_{n+1} = b$ and $u_1,\ldots,u_n \in U$ such that $\{z_i, z_{i+1}\} \subseteq u_i (i = 1,\ldots,n)$.

Suppose $\beta^*(a)$ is the equivalence class containing $a \in H$. Then the product $\odot$ on H/β^* is defined as follows:$\beta^*(a) \odot \beta^*(b) = \{\beta^*(c) | c \in \beta^*(a)\cdot\beta^*(b)\}$ for all $a, b \in H$.

For a subset $X \subseteq H$ we define the approximations of X relative to the fundamental equivalence relation β^* as follows: $\underline{\beta^*(X)} = \{x \in H | \beta^*(x) \subseteq X\}$ and $\overline{\beta^*(X)} = \{x \in H | \beta^*(x) \cap X \neq \emptyset\}$.

Since the proof of the following Theorem is similar to Theorem 2.1 of [13] and Proposition 2.2 of [6], we omit it here.

Theorem 4.1. Let X and Y be non-empty subsets of H. Then the following hold:

1)$\underline{\beta^*(X)} \subseteq X \subseteq \overline{\beta^*(X)}$;

2)$\underline{\beta^*(\emptyset)} = \emptyset = \overline{\beta^*(\emptyset)}$ and $\underline{\beta^*(H)} = H = \overline{\beta^*(H)}$;

3)$\overline{\beta^*(X \cup Y)} = \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}$;

4)$\underline{\beta^*(X \cap Y)} = \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}$;

5)$X \subseteq Y$ implies $\underline{\beta^*(X)} \subseteq \underline{\beta^*(Y)}$ and $\overline{\beta^*(X)} \subseteq \overline{\beta^*(Y)}$;

6)$\overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(X)} \cap \overline{\beta^*(Y)}$;

7)$\underline{\beta^*(X \cup Y)} \supseteq \underline{\beta^*(X)} \cup \underline{\beta^*(Y)}$;

8)$\overline{\beta^*(X^c)} = (\underline{\beta^*(X)})^c$, (X^c) is the complement of X in H;

9)$\underline{\beta^*(X^c)} = (\overline{\beta^*(X)})^c$;

10) $\overline{\beta^*(\overline{\beta^*(X)})} = \overline{\beta^*(X)} = \underline{\beta^*(\overline{\beta^*(X)})}$;

11)$\underline{\beta^*(\underline{\beta^*(X)})} = \underline{\beta^*(X)} = \overline{\beta^*(\underline{\beta^*(X)})}$.

Proposition 4.2. If X, Y are non-empty subsets of H, then $\overline{\beta^*(X)} \cdot \overline{\beta^*(Y)} \subseteq \overline{\beta^*(X \cdot Y)}$.

Proof: Suppose z be any element of $\overline{\beta^*(X)} \cdot \overline{\beta^*(Y)}$. Then $z \in x \cdot y$ with $x \in \overline{\beta^*(X)}$ and $y \in \overline{\beta^*(Y)}$. Thus there exist $a, b \in H$ such that $a \in \beta^*(x) \cap X$ and $b \in \beta^*(y) \cap Y$. Therefore $a \cdot b \subseteq \beta^*(x) \cdot \beta^*(y) \subseteq \beta^*(x \cdot y)$. Since $a \cdot b \subseteq X \cdot Y$, we have $a \cdot b \subseteq \beta^*(x \cdot y) \cap (X \cdot Y)$ and so $\beta^*(x \cdot y) \cap X \cdot Y \neq \emptyset$. Therefore for any $z \in x \cdot y$, we have $\beta^*(z) \cap (X \cdot Y) \neq \emptyset$, which implies $z \in \overline{\beta^*(X \cdot Y)}$. That is $x \cdot y \subseteq \overline{\beta^*(X \cdot Y)}$. Thus we have $\overline{\beta^*(X)} \cdot \overline{\beta^*(Y)} \subseteq \overline{\beta^*(X \cdot Y)}$.

Theorem 4.3. If X is a subsemihypergroup of H, then $\overline{\beta^*(X)}$ is also a subsemihypergroup of H.

Proof: Since X is a subsemihypergroup of $H, X \cdot X \subseteq X$, then it follows from Theorem 4.1(5) and Proposition 4.2 that$\overline{\beta^*(X)} \cdot \overline{\beta^*(X)} \subseteq \overline{\beta^*(X \cdot X)} \subseteq \overline{\beta^*(X)}$,This means that $\overline{\beta^*(X)}$ is a subsemihypergroup of H.

Theorem 4.4. If X is a hyperideal of H, then $\overline{\beta^*(X)}$ is also a hyperideal of H.

Proof: First note that $\overline{\beta^*(H)} = H$. Let X be a left hyperideal of H, that is $H \cdot X \subseteq X$. Then by Theorem 4.1 and Proposition 4.2, we have$H \cdot \overline{\beta^*(X)} = \overline{\beta^*(H)} \cdot \overline{\beta^*(X)} \subseteq \overline{\beta^*(H \cdot X)} \subseteq \overline{\beta^*(X)}$.This means that $\overline{\beta^*(X)}$ is a left hyperideal of H. The case of right hyperideal can be seen in a similar way.

The lower and upper approximations can be presented in an equivalent form as shown below. Let X be a non-empty subset of H. Then we have:$\underline{\underline{\beta^*(X)}} = \{\beta^*(x) \in H/\beta^* | \beta^*(x) \subseteq X\}$and $\overline{\overline{\beta^*(X)}} = \{\beta^*(x) \in H/\beta^* | \beta^*(x) \cap X \neq \emptyset\}$.

Now, we discuss these sets as subsets of the fundamental semigroup H/β^*.

Proposition 4.5. If X and Y are non-empty subsets of H, then$\overline{\overline{\beta^*(X)}} \odot \overline{\overline{\beta^*(Y)}} \subseteq \overline{\overline{\beta^*(X \cdot Y)}}$.

Proof: We have $\overline{\overline{\beta^*(X)}} \odot \overline{\overline{\beta^*(Y)}} = \left\{\beta^*(x) \odot \beta^*(y) \,\middle|\, \beta^*(x) \in \overline{\overline{\beta^*(X)}},\ \beta^*(y) \in \overline{\overline{\beta^*(Y)}}\right\} = \{\beta^*(x) \odot \beta^*(y) | \beta^*(x) \cap X \neq \emptyset, \beta^*(y) \cap Y \neq \emptyset\}$. Therefore $(\beta^*(x) \cdot \beta^*(y)) \cap (X \cdot Y) \neq \emptyset$. Since $\beta^*(x) \cdot \beta^*(y) \subseteq \beta^*(x \cdot y)$, we obtain $\beta^*(x \cdot y) \cap (X \cdot Y) \neq \emptyset$. Thus $\beta^*(x \cdot y) = \beta^*(x) \odot \beta^*(y) \in \overline{\overline{\beta^*(X \cdot Y)}}$, and so $\overline{\overline{\beta^*(X)}} \odot \overline{\overline{\beta^*(Y)}} \subseteq \overline{\overline{\beta^*(X \cdot Y)}}$.

Corollary 4.6. If X is a non-empty subset of H and Y is a hyperideal of H, then $\overline{\overline{\beta^*(X)}}e\overline{\overline{\beta^*(Y)}} \subseteq \overline{\overline{\beta^*(Y)}}$.

Proof: Since Y is a hyperideal of H, $X \cdot Y \subseteq H \cdot Y \subseteq Y$. Then by Proposition 4.5, we have $\overline{\overline{\beta^*(X)}} \odot \overline{\overline{\beta^*(Y)}} \subseteq \overline{\overline{\beta^*(X \cdot Y)}} \subseteq \overline{\overline{\beta^*(H \cdot Y)}} \subseteq \overline{\overline{\beta^*(Y)}}$.

Corollary 4.7. If X and Y are two hyperideals of H, then $\overline{\overline{\beta^*(X)}} \odot \overline{\overline{\beta^*(Y)}} \subseteq \overline{\overline{\beta^*(X)}} \cap \overline{\overline{\beta^*(Y)}}$.

Proof: Immediately follows from Corollary 4.6.

Proposition 4.8. If X and Y are a right hyperideal and a left hyperideal of H, respectively, then $\overline{\overline{\beta^*(X \cdot Y)}} \subseteq \overline{\overline{\beta^*(X)}} \cap \overline{\overline{\beta^*(Y)}}$.

Proof: Suppose $\beta^*(a) \in \overline{\overline{\beta^*(X \cdot Y)}}$. Then we have $\beta^*(a) \cap (X \cdot Y) \neq \emptyset$. Since X is a right hyperideal of H, $X \cdot Y \subseteq X \cdot H \subseteq X$. Then $\beta^*(a) \cap X \neq \emptyset$, and so $\beta^*(a) \in \overline{\overline{\beta^*(X)}}$. Since Y is a left hyperideal of H, $X \cdot Y \subseteq H \cdot Y \subseteq Y$. Then $\beta^*(a) \cap Y \neq \emptyset$, and so $\beta^*(a) \in \overline{\overline{\beta^*(Y)}}$. Therefore $\beta^*(a) \in \overline{\overline{\beta^*(X)}} \cap \overline{\overline{\beta^*(Y)}}$. This means that $\overline{\overline{\beta^*(X \cdot Y)}} \subseteq \overline{\overline{\beta^*(X)}} \cap \overline{\overline{\beta^*(Y)}}$.

Proposition 4.9. Let X and Y be a right hyperideal and a left hyperideal of H, respectively, then $\underline{\underline{\beta^*(X \cdot Y)}} \subseteq \underline{\underline{\beta^*(X)}} \cap \underline{\underline{\beta^*(Y)}}$.

Proof: Suppose $\beta^*(a) \in \underline{\underline{\beta^*(X \cdot Y)}}$. Then we have $\beta^*(a) \subseteq X \cdot Y$. As in the proof of the above proposition, we have $X \cdot Y \subseteq X$ and $X \cdot Y \subseteq Y$. Then $\beta^*(a) \subseteq X$ and $\beta^*(a) \subseteq Y$. Thus $\beta^*(a) \in \underline{\underline{\beta^*(X)}}$ and $\beta^*(a) \in \underline{\underline{\beta^*(Y)}}$ and so $\beta^*(a) \in \underline{\underline{\beta^*(X)}} \cap \underline{\underline{\beta^*(Y)}}$. This means that $\underline{\underline{\beta^*(X \cdot Y)}} \subseteq \underline{\underline{\beta^*(X)}} \cap \underline{\underline{\beta^*(Y)}}$.

By a *subsemigroup* of a semigroup S we mean a non-empty subset A of S such that $AA \subseteq A$, and by a *left*(resp. *right*)*ideal* of S we mean a non-empty subset A of S such that $SA \subseteq A$(resp. $AS \subseteq A$). By an *ideal*, we mean a non-empty subset A of a semigroup S which is both a left and a right ideal of S (see [30]).

Theorem 4.10. If X is a subsemihypergroup of $(H,\cdot)$, Then $\overline{\overline{\beta^*(X)}}$ is a subsemigroup of $(H/\beta^*, \odot)$.

Proof: Suppose that $\beta^*(a)$, $\beta^*(b) \in \overline{\overline{\beta^*(X)}}$, we show that $\beta^*(a) \odot \beta^*(b) \in H/\beta^*$. In fact, we have $\beta^*(a) \cap X \neq \emptyset$ and $\beta^*(b) \cap X \neq \emptyset$, then there exist $x, y \in H$ such that $x \in \beta^*(a) \cap X$ and $y \in \beta^*(b) \cap X$. It follows that $x \in \beta^*(a), x \in X, y \in \beta^*(b)$ and $y \in X$.So$x \cdot y \subseteq \beta^*(a) \cdot \beta^*(b) \subseteq \beta^*(a \cdot b) = \beta^*(a)e\beta^*(b)$.For every $z \in a \cdot b$ we have $\beta^*(z) = \beta^*(a) \odot \beta^*(b)$. Hence we get $x \cdot y \subseteq \beta^*(z)$ and $x \cdot y \subseteq X$. Thus, $\beta^*(z) \cap X \neq \emptyset$, which yields $\beta^*(z) \in \overline{\overline{\beta^*(X)}}$. Therefore $\beta^*(a) \odot \beta^*(b) \in \overline{\overline{\beta^*(X)}}$.

Theorem 4.11. If X is a hyperideal of H, then $\overline{\overline{\beta^*(X)}}$ is an ideal of$(H/\beta^*, \odot)$.

Proof: Assume that X is a left hyperideal of H. We show that $H/\beta^* \odot \overline{\overline{\beta^*(X)}} \subseteq \overline{\overline{\beta^*(X)}}$. Let $\beta^*(a)$ and $\beta^*(b)$ be any elements of H/β^* and $\overline{\overline{\beta^*(X)}}$, respectively. Then $\beta^*(b) \cap X \neq \emptyset$, and so there exists $x \in H$ such that $x \in \beta^*(b) \cap X$. Thus $x \in \beta^*(b)$ and $x \in X$. Let y be any element of $\beta^*(a)$. Since X is a left hyperideal of $H, y \cdot x \subseteq \beta^*(a) \cdot X \subseteq H \cdot X \subseteq X$. Since $y \cdot x \subseteq \beta^*(a) \cdot \beta^*(b) \subseteq \beta^*(a \cdot b) = \beta^*(a) \odot \beta^*(b)$, we have $y \cdot x \subseteq (\beta^*(a) \odot \beta^*(b)) \cap X$. This implies that $\beta^*(a) \odot \beta^*(b) \in \overline{\overline{\beta^*(X)}}$. Therefore $H/\beta^* \odot \overline{\overline{\beta^*(X)}} \subseteq \overline{\overline{\beta^*(X)}}$. This means that $\overline{\overline{\beta^*(X)}}$ is a left ideal of H/β^*. The other case can be seen in a similar way.

Theorem 4.12. Let X and Y be two hyperideals of semihypergroup H, and let $f: X \to Y$ be a strong homomorphism, then f induces a homomorphism$F: \overline{\overline{\beta^*(X)}} \to \overline{\overline{\beta^*(Y)}}$by setting$F(\beta^*(x)) = \beta^*(f(x)), \forall x \in X$.

Proof: The proof is similar to the proof of Proposition 5.5 of [27].

Let H be a semihypergroup with scalar identity. The kernel of the canonical map $\varphi: H \to H/\beta^*$ is called the *core* of H and is denoted by ω_H. Here we also denote by ω_H the unit element of H/β^* (see [2]).

Definition 4.13. Let X, Y and Z be hyperideals of H(a semihypergroup with scalar identity). The sequence of hyperideals and strong homomorphisms$X \xrightarrow{f} Y \xrightarrow{g} Z$is said to be *exact*if for every$x \in X$,

$$(g \circ f)(x) \in \omega_H.$$

Theorem 4.14. Let$X \xrightarrow{f} Y \xrightarrow{g} Z$be an exact sequence of hyperideals of H and strong homomorphisms. Then the sequence$\overline{\overline{\beta^*(X)}} \xrightarrow{F} \overline{\overline{\beta^*(Y)}} \xrightarrow{G} \overline{\overline{\beta^*(Z)}}$is an exact sequence of ideals of $H/\beta^*$where$F(\beta^*(x)) = \beta^*(f(x)), \forall x \in X$
$G(\beta^*(y)) = \beta^*(g(y)), \forall y \in Y$.

Proof: By Theorem 4.12, F and G are well-defined and homomorphisms. Finally, it is enough to show that $ImF = KerG$. We know $\beta^*(\omega_H) = \{\beta^*(e)\}$, so we have $\beta^*(b) \in ImF \Rightarrow \exists a \in H,\ F(\beta^*(a)) = \beta^*(b) \Rightarrow \beta^*(f(a)) = \beta^*(b) \Rightarrow G(\beta^*(f(a)) = G(\beta^*(b)) \Rightarrow \beta^*(g(f(a))) = G(\beta^*(b)) \Rightarrow G(\beta^*(b)) \subseteq \beta^*(\omega_H) \Rightarrow \beta^*(g(b)) \subseteq \beta^*(\omega_H) \Rightarrow G(\beta^*(b)) = \beta^*(\omega_H) = \beta^*(e) \Rightarrow \beta^*(b) \in KerG$ and so $ImF \subseteq KerG$. Conversely, we can show $KerG \subseteq ImF$. Therefore $ImF = KerG$.

5. Approximations in Bi-hyperideals of Semihypergroups

Let S be a semigroup. A subsemigroup B of S is called a $bi-ideal$ of S if $BSB \subseteq B$. The notion of a bi-ideal was first introduced by Good and Hughes [31] as early as 1952, and it has been widely studied. Here we define and study the bi-hyperideals of semihypergroups and its connections with approximations.

Definition 5.1. A subsemihypergroup X of a semihypergroup $(H,\cdot)$ is called a $bi-hyperideal$ of H if $X \cdot H \cdot X \subseteq X$.

Example 5.2. (1)Let $H_1 = \{a,b,c,d\}$ be a set. We define a hyperoperation " $\cdot_1$" on H_1 by $a \cdot_1 a = \{b,c\}$ and $x \cdot_1 y = \{b,d\}$ for all $(x,y) \in H \times H$ with $(x,y) \neq (a,a)$, then it is easy to see that $\{b,d\}$ is a bi-hyperideal of H_1.
(2)Let $H_2 = (\{a,b,c\},\cdot_2)$ be a semihypergroup, where $a \cdot_2 a = a \cdot_2 b = a \cdot_2 c = b \cdot_2 a = c \cdot_2 a = a$ and $b \cdot_2 b = b \cdot_2 c = c \cdot_2 b = c \cdot_2 c = b$. Then it is not difficult to see that the sets $\{a\}$ and $\{a,b\}$ are bi-hyperideals of H_2.

Theorem 5.3. Every left (resp. right) hyperideal X of a semihypergroup H is a bi-hyperideal of it.

Proof: First we show that X is a subsemihypergroup of H. Since X is a left (resp. right) hyperideal of H, we have $X \cdot X \subseteq H \cdot X \subseteq X$ (resp. $X \cdot X \subseteq X \cdot H \subseteq X$). Then X is a subsemihypergroup of H. Now $X \cdot H \cdot X = (X \cdot H) \cdot X \subseteq X \cdot X \subseteq X$ (resp. $X \cdot H \cdot X = X \cdot (H \cdot X) \subseteq X \cdot X \subseteq X$);This completes the proof.

The following example shows that the converse of Theorem 5.3 is not true.

Example 5.4. (1) Let $H = \{a,b\}$ be a semihypergroup with hyperoperation defined as follows: $a \cdot a = b \cdot a = a$ and $a \cdot b = b \cdot b = \{a,b\}$. Then $(H,\cdot)$ is a semihypergroup such that the set $\{a\}$ is a bi-hyperideal of H, but it is not a right hyperideal of H.
(2) Let $H' = (\{x,y\},\circ)$ be a semihypergroup, where $x \circ x = x, x \circ y = \{x,y\}$ and $y \circ x = y \circ y = y$. Then it is not difficult to see that the set $\{y\}$ is a bi-hyperideal of H, but it is not a left hyperideal of H.

Theorem 5.5. Let $\{X_\alpha\}_{\alpha\in I}$ be a family of bi-hyperideals of H. Then $\bigcap_{\alpha\in I} X_\alpha$ is a bi-hyperideal of H if $\bigcap_{\alpha\in I} X_\alpha \neq \emptyset$.

Proof: It is straightforward.

Theorem 5.6. Let X be a bi-hyperideal of H, then $\overline{\beta^*(X)}$ is also a bi-hyperideal of H.

Proof: Let X be a bi-hyperideal of H, that is $X \cdot H \cdot X \subseteq X$. Note that by Theorem 4.1(2), $\overline{\beta^*(H)} = H$. Then by Theorem 4.1 and Proposition 4.2, we have $\overline{\beta^*(X)} \cdot H \cdot \overline{\beta^*(X)} = \overline{\beta^*(X)} \cdot \overline{\beta^*(H)} \cdot \overline{\beta^*(X)} \subseteq \overline{\beta^*(X \cdot H \cdot X)} \subseteq \overline{\beta^*(X)}$. From this and Theorem 4.3, we obtain that $\overline{\beta^*(X)}$ is a bi-hyperideal of H.

Theorem 5.7. Let X be a bi-hyperideal of H, then $\overline{\overline{\beta^*(X)}}$ is a bi-ideal of $(H/\beta^*,\odot)$.

Proof: Let $\beta^*(x)$ and $\beta^*(z)$ be any elements of $\overline{\overline{\beta^*(X)}}$, and $\beta^*(y)$ be any element of H/β^*. Then $\beta^*(x) \cap X \neq \emptyset$, and $\beta^*(z) \cap X \neq \emptyset$, and so there exist $a,c \in H$ such that $a \in \beta^*(x) \cap X$ and $c \in \beta^*(z) \cap X$. Thus $a \in \beta^*(x), a \in X, c \in \beta^*(z)$ and $c \in X$. Let b be any element of $\beta^*(y)$. Then, since X is a bi-hyperideal of H, $a \cdot b \cdot c \subseteq X \cdot \beta^*(y) \cdot X \subseteq X \cdot H \cdot X \subseteq X$, and since $a \cdot b \cdot c \subseteq \beta^*(x) \cdot \beta^*(y) \cdot \beta^*(z) \subseteq \beta^*(x \cdot y \cdot z) = \beta^*(x) \odot \beta^*(y) \odot \beta^*(z)$, we have $\beta^*(x) \odot \beta^*(y) \odot \beta^*(z) \cap X \neq \emptyset$. This implies that $\beta^*(x) \odot \beta^*(y) \odot \beta^*(z) \in \overline{\overline{\beta^*(X)}}$. From this and Theorem 4.10, we have $\overline{\overline{\beta^*(X)}}$ is bi-ideal of H/β^*.

6. Conclusion

The theory of rough sets is regarded as a generalization of the classical sets theory. In the present paper, we substituted a universe set by a semihypergroup H and consider the fundamental relation β^* on H to find the lower and upper approximations for subsets of H as subsets of the fundamental semigroup H/β^* and obtain basic properties of this connection. We also introduce and study the bi-hyperideals of semihypergroups and its connections with approximations. Our future work on this topic will be focused on the properties of fuzzy rough sets and rough fuzzy sets with respect to semihypergroups and then with the study of fuzzy rough (bi-)hyperideals and rough fuzzy (bi-) hyperideals of a semihypergroup.

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