*IJST (2013) 37A4: 477-482*
***Iranian Journal of Science & Technology***

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# Numerical study of some nonlinear wave equations via Chebyshev collocation method

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## Abstract

The numerical methods are of great importance for approximating the solutions of nonlinear ordinary or partial differential equations, especially when the nonlinear differential equation under consideration faces difficulties in obtaining its exact solution. In this latter case, we usually resort to one of the efficient numerical methods. In this paper, the Chebyshev collocation method is suggested to deal numerically with some nonlinear partial differential equations in mathematical physics.

***Keywords****:* Chebyshev Collocation Method; nonlinear wave equation; numerical solution

## 1. Introduction

The physical phenomena in nonlinear science are usually modeled by ordinary (ODEs) or partial differential equations (PDEs). One of such phenomena arises in the soliton problems which are modeled by PDEs with different form of nonlinearities. In order to solve such nonlinear PDEs, various analytical and numerical approaches have been proposed. The analytical methods are important in obtaining the exact solutions. In order to achieve this goal, various direct methods have been proposed such as tanh-function method [1], Jacobi elliptic function method [2], F-expansion method [3, 4], Exp-function method [5-19], the generalized Exp-function method [20], and others. The numerical methods are also of great importance when the given ODE or PDE cannot be solved exactly. Various numerical methods have also been proposed in [10, 11, 21] to obtain the numerical solutions of PDEs. The objective of the current research is to explore the applicability of Chebyshev collocation method for the numerical solutions of nonlinear wave equations. This suggested method has been applied for various scientific problems [22-28]. Chebyshev have proven successfully in the numerical solution of various boundary value problems [29, 30] and in computational fluid problems [31, 32]. The spectral method distinguishes itself from the finite-difference and finite-element methods by the fact that global information is incorporated in computing a spatial derivative. The spectral method can yield greater accuracy for a smooth solution with far fewer nodes and therefore less computational time than the finite-difference and finite-element schemes [33]. Chebyshev pseudospectral methods are widely used in the numerical approximation of many types of ordinary and partial differential equations which arise from the engineering problems [34-37]. Therefore, when many decimal places of accuracy are needed, the contest between pseudospectral algorithms and finite difference is not an even battle but a rout: pseudospectral methods win hands-down. Moreover, engineers and mathematicians who need accurate many decimal places have always preferred spectral methods [38]. Elbarbary and El-Sayed [39] has recently introduced a new pseudospectral differentiation matrix to decrease the round off error, especially on increasing $N$ (the number of degrees-of-freedom) or number of equations. Hence, on solving nonlinear equations in the present results the error becomes nearly zero [40, 41]. Here, we aim to extend its application to a class of nonlinear wave equations, the KdV equation, Burgers' equation, the extended KdV, and the Klein-Gordon equations with nonlinear term of any order. It should be noted that a few numerical methods have been suggested to treat the soliton problems. The results reveal that Chebyshev collocation method is very effective in obtaining accurate numerical soiltary wave solutions for the present physical models.

## 2. Chebyshev collocation method

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Received: 23 April 2013 / Accepted: 7 July 2013

collocation approximations seems to be a very good choice in many practical problems (as described in the literature review and, for example, (Canuto et al. [31] and Peyret [35]). Accordingly, Chebyshev collocation method will be applied for the presented model. The derivatives of the function $\boldsymbol{f(x)}$at the Gauss-Lobatto points, $\boldsymbol{x_k = \cos\left(\frac{k\pi}{L}\right)}$, which are the linear combination of the values of the function $\boldsymbol{f(x)}$ [41]

$$\boldsymbol{f}^{(\mathrm{n})} = \boldsymbol{D}^{(\mathrm{n})}\boldsymbol{f},$$

where,

$$\underline{\boldsymbol{f}} = [\boldsymbol{f(x^0), f(x^1), \ldots, f(x_L)}]^{\boldsymbol{T}},$$

and

$$\underline{\boldsymbol{f}}^{(\mathrm{n})} = \left[\boldsymbol{f}^{(\mathrm{n})}(\boldsymbol{x^0}), \boldsymbol{f}^{(\mathrm{n})}(\boldsymbol{x^1}), \ldots, \boldsymbol{f}^{(\mathrm{n})}(\boldsymbol{x_L})\right]^{\boldsymbol{T}}.$$

where

$$\boldsymbol{D}^{(\mathrm{n})} = [\boldsymbol{d_{k,j}^{(n)}}], \text{ or } \boldsymbol{f}^{(\mathrm{n})}(\boldsymbol{x_k}) = \textstyle\sum_{\boldsymbol{j=0}}^{\boldsymbol{L}} \boldsymbol{d_{k,j}^{(n)}}\, \boldsymbol{f(x_j)},$$

where,

$$\mathrm{d}_{\mathrm{k,j}}^{(\mathrm{n})} = \frac{2\gamma_j^*}{L}\sum_{l=n}^{L}\ \sum_{\substack{m=0\\(m=l-n)\,even}}^{l-n} \gamma_l^*\, a_{m,l}^n (-1)^{[\frac{lj}{L}]+[\frac{mk}{L}]}\, x_{lj-L[\frac{lj}{L}]}\, x_{mk-L[\frac{mk}{L}]},$$

and

$$a_{m,l}^n = \frac{2^{\mathrm{n}}\, l}{(n-1)!\,c_m}\, \frac{(s-m+n-1)!\,(s+n-1)!}{s!\,(s-m)!},$$

such that $\boldsymbol{2s = l + m - n}$ and $\boldsymbol{c_o = 2, c_i = 1, i \geq 1}$, where $\boldsymbol{k}$, $\boldsymbol{j = 0, 1, 2, \ldots, L}$ and $\boldsymbol{\gamma_o^* = \gamma_l^* = \frac{1}{2}, \gamma_j^* = 1}$ for $\boldsymbol{j = 0, 1, 2, 3, \ldots, L-1}$. The round off errors incurred during computing differentiation matrices $\boldsymbol{D}^{(\mathrm{n})}$are investigated in [41]. In the next section, Chebyshev pseudospectral method will be introduced for the presented models.

## 3 Application of Chebyshev collocation method

In this section, we aim to explore the effectiveness of the Chebyshev collocation method in solving nonlinear wave equations, where four familiar wave equations (the KdV equation, Burgers' equation, the extended KdV equation and the generalized Klein-Gordon equation) will be investigated. We shall compare the present results utilizing Chebyshev collocation method (*ChCM*) (see [22-28]) with those obtained by [9, 10].

### *3.1. KdV equation*

The nonlinear KdV equation was solved exactly by Ebaid [9] in the form:

$$\boldsymbol{u_t + \alpha u\, u_x + \beta\, u_{xxx} = 0}, \tag{1}$$

which under the following transformation,

$$\boldsymbol{u = u(\eta), \quad \eta = k(x - ct)}, \tag{2}$$

becomes

$$\boldsymbol{-c\, u'(\eta) + \alpha\, u(\eta) u'(\eta) + \beta k^2\, u'''(\eta) = 0}. \tag{3}$$

The exact solution obtained by Ebaid [9] in the form:

$$\boldsymbol{u(\eta) = a_1 + \frac{\frac{6k^2\beta b_0}{\alpha}}{e^{\eta} + b_0 + \frac{1}{4}{b_0}^2 e^{-\eta}}, \ \eta = k(x - ct), \ c = k^2\beta + \alpha a_1}. \tag{4}$$

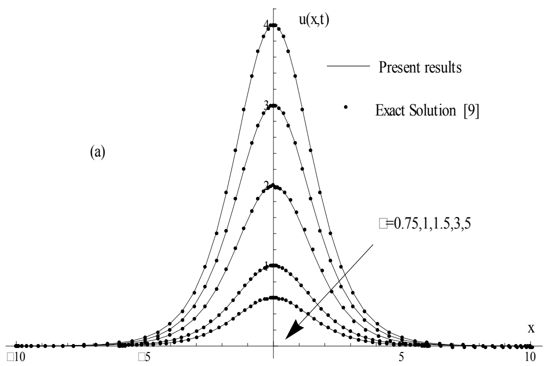
Where $a_1$, $b_0$, $k$, $\alpha$, and $\beta$ are free parameters. If we set $a_1$=0, $b_0$=2, $k$=1, $\alpha$=6, and $\beta$=1,
the traveling wave solution can be readily obtained by Wazwaz [42]:

$$\boldsymbol{u(x,t) = \frac{-2k^2 e^{k(x-k^2t)}}{\left(1+e^{k(x-k^2t)}\right)^2}}. \tag{5}$$

To compare our results with those obtained by Ebaid [9] and by Wazwaz [42], we need three conditions for Eq. (3) at the parameter $\alpha$=6

$$\boldsymbol{u(0) = 1, u(\eta_{max}) = \frac{-2k^2 e^{\eta max}}{(1+e^{\eta max})^2}, \ u'(0) = 0}. \tag{6}$$

The three conditions will be changed with different values of α and for fixed values of the parameters $a_1$=0, $b_0$=2, $k$=1, and $\beta$=1. The nonlinear equation (3) with the conditions (6) are approximated by using Chebyshev collocation method and the solution of the above equation (3) with the conditions (6), are obtained using the Newton-Raphson iteration technique. The present results have been compared with the exact solution in equation (5) as shown as in Fig. 1. The computer program of the numerical method was executed in Mathematica 6$^{TM}$.



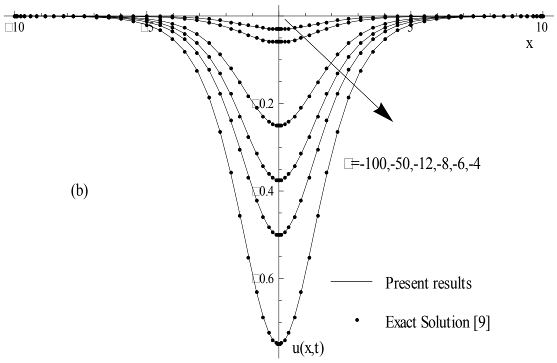



**Fig .1.** Comparison with Kd V equation at different values of α and c=1, k=1, β=1

### 3.2. Burgers' equation

A well-known model is the one-dimentional Burgers'.equation:

$$\boldsymbol{u_t + uu_x - \nu u_{xx} = 0}, \qquad (7)$$

where, ν > 0 is the coefficient of the kinemitics viscosity of the fluid. This equation was formulated by Burgers in an attempt to model turbulent flow in a channel [43]. Using the transformation (2), Eq. (7) becomes:

$$\boldsymbol{-c\,u'(\eta) + u(\eta)u'(\eta) - \nu k\,u''(\eta) = 0}. \qquad (8)$$

The general solution is given by Ebaid [9] as:

$$\boldsymbol{u(\eta) = a_1 + \frac{\nu\,k\left(b_0 \pm \sqrt{{b_0}^2 - 4b_{-1}}\right) + 2\nu\,k\,e^{-\eta}}{e^{\eta} + b_0 + b_{-1}e^{-\eta}},\ \eta = k(x - ct),\ c = \nu\,k + a_1}. \qquad (9)$$

Setting $a_1$=0, $b_0$=2, $b_{-1}$=1 in Eq. (9), we obtain

$$\boldsymbol{u(\eta) = \frac{2\nu\,k\,e^{-\eta/2}}{e^{\eta/2} + b_{-1}e^{-\eta/2}},\ \eta = k(x - ct),\ c = \nu\,k}. \qquad (10)$$

For purpose of comparison with results in Ebaid [9] and Drazin [43], we use here three conditions for Eq. (8) at $a_1$=0.001, -1for different values of ν and $k$= 1,

$$\boldsymbol{u(0) = a_1 + \nu\,k,\ u(\eta_{max}) = a_1 + \frac{2\nu\,k\,e^{-\eta max/2}}{e^{\eta max/2} + b_{-1}e^{-\eta max/2}},\ u'^{(0)} = -\frac{\nu\,k}{2}}. \qquad (11)$$

The results are depicted in Fig. 2, where the numerical solutions obtained via the present numerical approach are found to coincide with the exact solution.

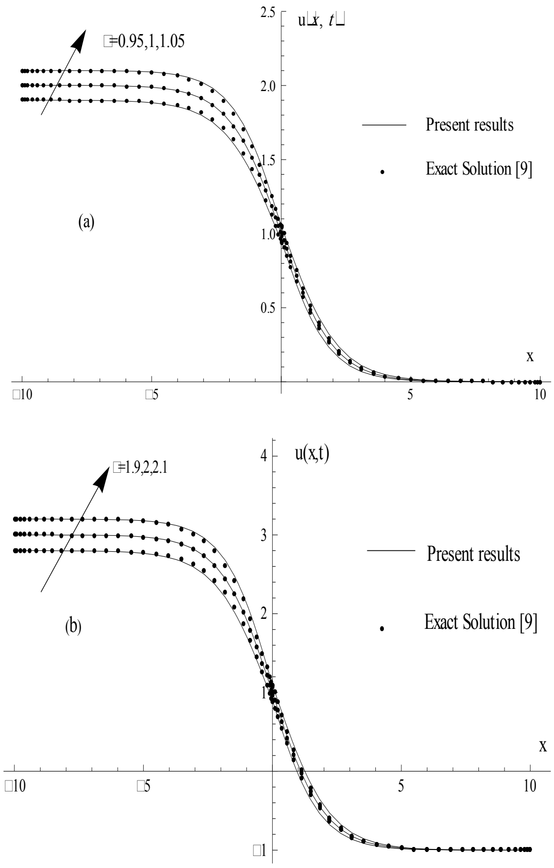



**Fig. 2.** Comparison with Burgerse quation at different values of ν and $\boldsymbol{a_1 = 0.001, a_1 = -1}$

### 3.3. Extended KdV equation

The extended KdV equation (eKdV) in the form:

$$\boldsymbol{u_t + p\,u\,u_x + qu^2u_x + u_{xxx} = 0}, \qquad (12)$$

has recently become a popular model for the description of internal solitary waves in shallow seas. Transformation (2) converts Eq. (12) into the nonlinear ODE:

$$\boldsymbol{-c\,u'(\eta) + p\,u(\eta)u'(\eta) + qu^2(\eta)u'(\eta) + k^2u'''(\eta) = 0}. \qquad (13)$$

The exact solution is given by Ebaid [9] as:

$$\boldsymbol{u(\eta) = -\frac{p}{2q} \pm \frac{\left(\pm\sqrt{\frac{3k^2(4b_{-1} - {b_0}^2)}{2q}} \pm \sqrt{\frac{24k^2{b_{-1}}^2}{q(4b_{-1} - {b_0}^2)}}\right) \pm \sqrt{\frac{3k^2{b_0}^2}{2q(4b_{-1} - {b_0}^2)}}(e^{\eta} + b_{-1}e^{-\eta})}{e^{\eta} + b_0 + b_{-1}e^{-\eta}}}. \qquad (14)$$

At $b_0$=0,$b_{-1}$=1, the exact solution is given as:

$$\boldsymbol{u(0) = -\frac{p}{2q} \pm k\sqrt{\frac{6}{q}}\,\mathrm{sech}(\eta),\ \eta = k(x - ct),\ c = k^2 - \frac{p^2}{4q^2}}. \qquad (15)$$

Consequently, under the three conditions for Eq. (13) at $p$= 0.1 for different values of $q$ and $k$= 1,

$$u(0) = -\frac{p}{2q} - k\sqrt{\frac{6}{q}},\ u(\eta_{max}) = -\frac{p}{2q} - k\sqrt{\frac{6}{q}}\operatorname{sech}(\eta_{max}),\ u'(0) = 0. \qquad (16)$$

a comprehensive numerical is conducted and the results are reported in terms of graphs as in Fig. 3, in which we get good agreement results with those obtained by Ebaid [9] and Xu et. al. [44].

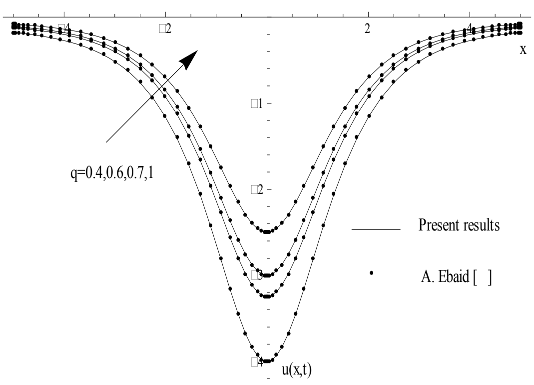



**Fig. 3.** Combined KdV at p=0.1 and different values of q

### *3.4. Klien-Gordon equation*

The Klein-Gordon equation plays an important role in mathematical physics. The equation has attracted much attention in studying solitons [43] in condensed matter physics, in investigating the interaction of solitons in a collisionless plasma, and the recurrence of initial states. The generalized nonlinear Klein-Gordon equation is given by:

$$u_{tt} - c_0{}^2 u_{xx} + \alpha u - \beta u^\gamma = 0,\ \gamma \in \mathbb{R},\ \gamma \neq \pm 1. \quad (17)$$

Assume that Eq. (17); has the travelling wave solution (2) consequently, Eq. (17) is reduced to the ODE:

$$k(c^2 - c_0{}^2)u'' - \alpha u - \beta u^\gamma = 0,\ c = \pm\sqrt{c_0{}^2 - \frac{\alpha(\gamma-1)^2}{4k^2}}. \qquad (18)$$

Ebaid [10] solved this nonlinear equation in the form:

$$u(\eta) = \left(\frac{a_0}{2}\operatorname{sech}(\eta)\right)^{\frac{2}{\gamma-1}},\ \eta = \pm\sqrt{\frac{\alpha(\gamma-1)^2}{4(c^2-c_0{}^2)}}(x - ct),\ a_0 = \pm\sqrt{\frac{2\alpha(\gamma+1)}{\beta}}, \qquad (19)$$

where $a_0$ is a free parameter, with the boundary conditions

$$u(0) = 1,\ u(\eta_{max}) = \left(\frac{a_0}{2}\operatorname{sech}(\eta_{max})\right)^{\frac{2}{\gamma-1}}. \qquad (20)$$

The present results have been compared with the exact solution obtained in [10] for Eq. (18) at $c_0=\sqrt{2}$, $c=1$ for different values of $\gamma$ and $k$ as illustrated in Fig. 4. The obtained results using the present method indicate that it is an adequate scheme for the solution of the present problem.

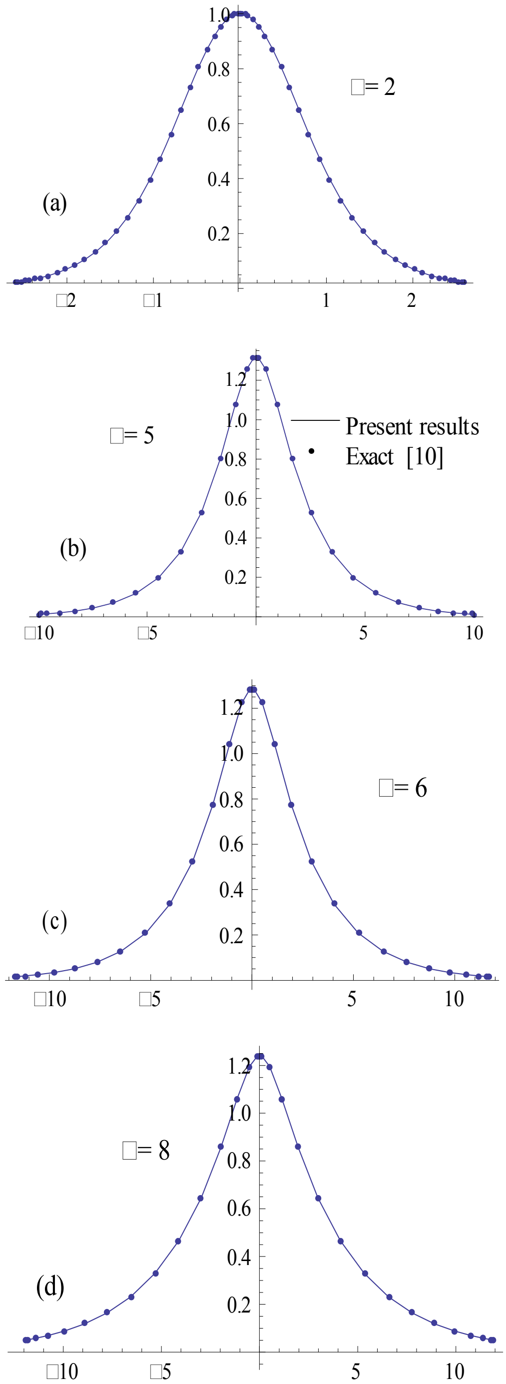



**Fig. 4.** Comparison with the generalized nonlinear Klein-Gordon equation for different values of $\gamma$:
(a)α=4,β=6,k=1,c_0=√2,c=1.
(b)α=1,β=1,k=2,c_0=√2,c=1.
(c)α=1,β=1,k=5/2,c_0=√2,c=1.
(d)α=2,β=2,k=7/2 √2,c_0=√2,c=1

## 4. Conclusions

A comprehensive numerical parametric study for the numerical solutions of a class of nonlinear wave equations is conducted and the results are reported in terms of graphs. This is done in order to illustrate special features of the solutions. So, the numerical solutions by using the Chebyshev collocation method were obtained for the function *u*. In addition, the obtained results indicate that it is an adequate scheme for the solution of the present problems. Although the Chebyshev collocation method previously dealt with a variety of applied problems, it may be the first implementation of it in the current paper for a class of (PDEs) of practical interest in soliton theory.

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