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On the multiplication operator on analytic function spaces

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Abstract

Let H be a Hilbert space of functions analytic on a plane domain Ω such that for every λ in Ω the functional of evaluation at λ is bounded. Assume further that H contains the constants and admits multiplication by the independent variable z, M_z, as a bounded operator. We give sufficient conditions for M_{z^n} to be reflexive for all positive integers n.

Keywords: Hilbert spaces of analytic functions; multiplication operators; reflexive operator; multipliers; Caratheodory hull; bounded point evaluation; spectral set

1. Introduction

By a domain we understand a connected open subset of the plane. If B is a bounded domain in the plane, then the Caratheodory hull (or $\mathbb{C}$-hull) of B is the complement of the closure of the unbounded component of the complement of the closure of B. The $\mathbb{C}$-hull of B is denoted by B^*. Intuitively, B^* can be described as the interior of the outer boundary of B, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \le \sup\{|p(z)| : z \in B\}$ for all polynomials p. The components of B^* are simply connected; in fact, one can easily see that each of these components has a connected complement. The component of B^* that contains B is denoted by B_1. Note that for all polynomials p, $\|p\|_B = \|p\|_{B_1}$. Note that B_1 is a Caratheodory domain and so by the Farrel-Rubel-Shields Theorem ([1, Theorem 5.1, p.151]), each bounded analytic function on B_1 can be approximated by a sequence of polynomials pointwise boundedly.

Now let H be a separable Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H. Recall that if $A \in B(H)$, then $Lat(A)$ is by definition the lattice of all invariant subspaces of A, and $AlgLat(A)$ is the algebra of all operators B in $B(H)$ such that $Lat(A) \subset Lat(B)$. An operator A in $B(H)$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $B(H)$ that contains A and the identity I is closed in the weak operator topology. For some sources on these topics see [2-7].

Consider a Hilbert space H of functions analytic on a plane domain G, such that for each $\lambda \in G$ the linear functional, e_λ, of evaluation at λ is bounded on H. Assume further that H contains the constant functions and multiplication by the independent variable z defines a bounded linear operator M_z on H. The continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in G$ there is a unique function $k_\lambda \in H$ such that $e_\lambda(f) = f(\lambda) = \langle f, k_\lambda \rangle$, $f \in H$. The function k_λ is called the *reproducing kernel* for the point λ.

A complex valued function φ on G for which $\varphi f \in H$ for every $f \in H$ is called a *multiplier* of H and the collection of all these multipliers is denoted by $M(H)$. Each multiplier φ of H determines a multiplication operator M_φ on H by $M_\varphi f = \varphi f$, $f \in H$. It is well known that each multiplier is a bounded analytic function on G ([8]). In fact, $\|\varphi\|_G \le \|M_\varphi\|$. We shall use the

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following notation for the norm of the operator M_φ:

$$\|\varphi\|_\infty = \|M_\varphi\|.$$

We also point out that if φ is a multiplier and $\lambda \in G$, then

$$M_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda.$$

Also, we say that $M(H)$ is isometrically rotation invariant if whenever $\varphi \in M(H)$, $\varphi_\theta \in M(H)$ and $\|\varphi\|_\infty = \|\varphi_\theta\|_\infty$ where $\varphi_\theta(z) = \varphi(e^{-i\theta} z)$. By $H(G)$ and $H^\infty(G)$ we mean respectively the set of analytic functions on a plane domain G and the set of bounded analytic functions on G.

2. Main results

In this article, we investigate the reflexivity of the powers of the multiplication operator M_z acting on a Hilbert function space.

From now on, let Ω be a domain in the complex plane such that Ω_1 is equal to the open unit disc D. Also, suppose that the Hilbert space H under consideration satisfy the following axioms:

Axiom 1. H is a subspace of the space of all analytic functions on Ω.

Axiom 2. For each $\lambda \in \Omega$, the linear functional of evaluation at λ, e_λ, is bounded on H.

Axiom 3. The uniform limits of polynomials on Ω is contained in $M(H)$ and $M(H)$ is isometrically rotation invariant.

Axiom 4. The sequence $\{f_k\}_{k \in Z}$ is an orthogonal basis for H where $f_k(z) = z^k$ for all integers k.

Note that by axiom 4, each function $f \in H$ can be represented by series expansion $f = \sum_n \hat{f}(n) f_n$. For $h \in M(H)$ and $w \in \partial D$, define h_w by $h_w(z) = h(wz)$. Thus $\hat{h}_w(n) = w^n \hat{h}(n)$ for all n. Also, since $|w| = 1$ we have

$$\|h_w\|^2 = \sum_n |\hat{h}_w(n)|^2 \|f_n\|^2 = \sum_n |\hat{h}(n)|^2 \|f_n\|^2 = \|h\|^2.$$

The following theorem extends the results obtained by Allen Shields [9] that have been proved only for the special case where H is the Hilbert space of formal Laurent series.

Lemma 2.1. Let $\varphi \in M(H)$. If g is a continuous complex valued function on ∂D and $d\lambda = |dw|/2\pi$ is the normalized Lebesgue measure on ∂D, then the operator

$$\int_{\partial D} \varphi_w \, g(w) d\lambda$$

defined by

$$\left(\int_{\partial D} \varphi_w \, g(w) d\lambda\right) f = \int_{\partial D} g(w) M_{\varphi_w} f d\lambda$$

is in $M(H)$ and

$$\left\| \int_{\partial D} \varphi_w \, g(w) d\lambda \right\|_\infty \leq \|M_\varphi\| \int_{\partial D} |g| \, d\lambda.$$

Proof: Note that the strong operator continuity of φ_w allows us to define

$$\int_{\partial D} \varphi_w \, g(w) f d\lambda$$

for all $f \in H$. If $f, h \in H$, then

$$< \int_{\partial D} \varphi_w \, g(w) f d\lambda, h > = \int_{\partial D} g(w) < \varphi_w f, h > d\lambda.$$

So we get

$$\left\| \int_{\partial D} \varphi_w \, g(w) f d\lambda \right\| \leq \|M_\varphi\| \|f\| \int_{\partial D} |g| \, d\lambda.$$

Hence

$$\left(\int_{\partial D} \varphi_w \, g(w) d\lambda\right) f = \int_{\partial D} g(w) M_{\varphi_w} f d\lambda \leq \|M_\varphi\| \|f\| \int_{\partial D} |g| \, d\lambda.$$

This completes the proof.

Lemma 2.2. If $\varphi \in H(\Omega_1) \cap M(H)$, then there exists a sequence of polynomials $\{r_n\}$ such that $\hat{r}_n(j) = (1 - \frac{j}{n+1}) \hat{\varphi}(j)$ whenever $j = 0, \ldots, n$

and is 0 else, and $M_{r_n} \to M_\varphi$ in the weak operator topology.

Proof: Let $\varphi \in H(\Omega_1) \cap M(H)$. Since $\Omega_1 = D$, we can represent φ by a power series $\sum_{k=0}^{\infty} \hat{\varphi}(k) z^k$. Put

$$P_n(\varphi) = \sum_{k=0}^{n} (1 - \frac{k}{n+1}) \hat{\varphi}(k) z^k, \qquad n \geq 0$$

and

$$K_n(w) = \sum_{|k| \leq n} (1 - \frac{|k|}{n+1}) w^k, \qquad w \in \partial U, \; n \geq 0.$$

Then

$$\int_{\partial D} \varphi_w K_n(\overline{w}) d\lambda = M_{\varphi_* K_n}, \qquad n \geq 0$$

where

$$(\varphi_* K_n)(z) = \sum_{j=0}^{n} \hat{\varphi}(j) \hat{K}_n(j) z^j = P_n(\varphi).$$

Note that $K_n \geq 0$ and

$$\int_{\partial D} K_n d\lambda = 1.$$

For all $n \geq 0$, $P_n(\varphi) \in M(H)$ and by Lemma 2.1, we get

$$\| M_{P_n(\varphi)} \| = \| M_{\varphi_* K_n} \| \leq \| M_\varphi \| \int_{\partial D} K_n d\lambda = \| M_\varphi \|.$$

Put $r_n = P_n(\varphi)$. Note that M_{r_n} is represented by the matrix whose (i,j)-th entry is

$$< M_{r_n} f_j, f_i > = \hat{r}_n(i-j) \|f_i\|^2 = (1 - \frac{i-j}{n}) \hat{\varphi}(i-j) \|f_i\|^2.$$

Hence

$$\lim_n < M_{r_n} f_j, f_i > = < M_\varphi f_j, f_i >$$

for all base elements f_j and f_i in H. By the boundedness of the sequence $\{M_{r_n}\}$ we have $M_{r_n} \to M_\varphi$ in the weak operator topology. This completes the proof.

Theorem 2.3. If $\{e_\lambda : \lambda \in \Omega\}$ is norm bounded, then M_{z^k} is reflexive for all $k \geq 1$.

Proof: The boundedness of point evaluations and the Closed Graph Theorem ensure that in multiplication by z, M_z is a bounded operator on H. Let $k \in N$ and note that $W(M_{z^k}) \subset AlgLat(M_{z^k})$. On the other hand, let $X \in AlgLat(M_{z^k})$. Since $Lat(M_z) \subset Lat(M_{z^k})$, we have $Lat(M_z) \subset Lat(X)$. This implies that $X \in AlgLat(M_z)$. Note that since $M_z^* e_\lambda = \overline{\lambda} e_\lambda$ for all λ in Ω, the one dimensional span of e_λ is invariant under M_z^*. Therefore, it is invariant under X^* and we can write $X^* e_\lambda = \overline{\varphi(\lambda)} e_\lambda, \lambda \in \Omega$. So

$$< Xf, e_\lambda > = < f, X^* e_\lambda > = \varphi(\lambda) f(\lambda)$$

for all $f \in H$ and $\lambda \in \Omega$. This implies that $X = M_\varphi$ and $\varphi \in M(H)$, hence $\varphi \in H^\infty(\Omega)$.

Now put $N = H^\infty(\Omega_1)$. Then $N \neq \varnothing$, since $1 \in N$. Note that by axiom 3, $N \subset M(H)$. To see this let $f \in H^\infty(\Omega_1)$. Since Ω_1 is a Caratheodory domain, by the Farrel-Rubel-Shields Theorem [1, Theorem 5.1, p. 151], there is a sequence $\{p_n\}$ of polynomials converging to f such that for all n, N for some $c > 0$. So $\{p_n\}_n$ is a normal family in $H^\infty(\Omega)$ and by passing to a subsequence if necessary, we may suppose that for some function g, $p_n \to g$ uniformly on compact subsets of Ω, this implies that indeed $g = f$. Hence by axiom 3, $f \in M(H)$ and so $N \subset M(H) \subset H$. Also, it is a closed subspace of H, since if $\{h_n\}_n \subset N$ and $h_n \to f$ in H, so for all n, $\|h_n\|_H \leq c_1$ for some $c_1 > 0$. Because point evaluations are bounded, for all λ in Ω we have

$$h_n(\lambda) = < h_n, e_\lambda > \to < f, e_\lambda > = f(\lambda).$$

Also, we note that for all λ in Ω,

$$|h_n(\lambda)|=|<h_n, e_\lambda>|\leq \|h_n\|_H \|e_\lambda\| \leq c_2 \|h_n\|_H$$

where $c_2 = \sup\{\|e_\lambda\| : \lambda \in \Omega\}$. Thus

$$\|h_n\|_\Omega \leq c_2 \|h_n\|_H \leq c_1 c_2$$

for all n. Since $h_n \in H^\infty(\Omega_1)$, $\|h_n\|_{\Omega_1} = \|h_n\|_\Omega$ and so $\|h_n\|_{\Omega_1} \leq c_1 c_2$ for all n. This implies that $\{h_n\}_n$ is a normal family in $H^\infty(\Omega_1)$ and we may assume that for some function g, $h_n \to g$ uniformly on compact subsets of Ω_1. Thus $g \in H^\infty(\Omega_1)$. But by pointwise convergence $f = g$ on Ω and so f can be extended to a bounded analytic function on Ω_1, i.e., $f \in H^\infty(\Omega_1)$ and so N is indeed a closed subspace of H. Now clearly $N \in Lat(M_z)$, thus $XN \subset N$. Since $1 \in N$ we get $X1 = \varphi \in N \subset H^\infty(\Omega_1)$. Now by Lemma 2.2, there exists a sequence of polynomials $\{r_n\}$ (indeed $r_n = P_n(\varphi)$) such that $M_{r_n} \to M_\varphi$ in the weak operator topology. Now let $\mathbf{M}_k$ be the closed linear span of the set $\{f_{nk} : n \geq 0\}$ (recall that $f_i(z) = z^i$ for all i). We have

$$M_{z^k} f_{nk} = f_{(n+1)k} \in \mathbf{M}_k$$

for all $n \geq 0$. Thus $\mathbf{M}_k \in Lat(M_{z^k})$ and so $\mathbf{M}_k \in Lat(M_\varphi)$. Let $\varphi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n$. Since $1 \in \mathbf{M}_k$, thus $M_\varphi 1 = \varphi \in \mathbf{M}_k$. Hence $\hat{\varphi}(i) = 0$ for all $i \neq nk$, $n \geq 0$. Now, by a consequence of the particular construction of r_n used in Lemma 2.2, each r_n should be a polynomial in z^k, i.e., $r_n(z) = q_n(z^k)$ for some polynomial q_n. Thus

$$M_{r_n} = r_n(M_z) = q_n(M_{z^k}) \to X$$

in the weak operator topology. Hence $X \in W(M_{z^k})$. Thus M_{z^k} is reflexive and this completes the proof.

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