On the multiplication operator on analytic function spaces

Kh. Jahedi

Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran
E-mail: mjahedi80@yahoo.com

Abstract

Let $H$ be a Hilbert space of functions analytic on a plane domain $\Omega$ such that for every $\lambda$ in $\Omega$ the functional of evaluation at $\lambda$ is bounded. Assume further that $H$ contains the constants and admits multiplication by the independent variable $z$, $M_z$, as a bounded operator. We give sufficient conditions for $M_z^n$ to be reflexive for all positive integers $n$.

Keywords: Hilbert spaces of analytic functions; multiplication operators; reflexive operator; multipliers; Caratheodory hull; bounded point evaluation; spectral set

1. Introduction

By a domain we understand a connected open subset of the plane. If $B$ is a bounded domain in the plane, then the Caratheodory hull (or $C^*$-hull) of $B$ is the complement of the closure of the unbounded component of the complement of the closure of $B$. The $C^*$-hull of $B$ is denoted by $B^*$. Intuitively, $B^*$ can be described as the interior of the outer boundary of $B$, and in analytic terms it can be defined as the interior of the set of all points $z_0$ in the plane such that $|p(z_0)| \leq \sup \{|p(z)| : z \in B\}$ for all polynomials $p$. The components of $B^*$ are simply connected; in fact, one can easily see that each of these components has a connected complement. The component of $B^*$ that contains $B$ is denoted by $B_1$. Note that for all polynomials $p$, $\|p\|_{B_1} = \|p\|_{B_1}$. Note that $B_1$ is a Caratheodory domain and so by the Farrel-Rubel-Shields Theorem ([1, Theorem 5.1, p.151]), each bounded analytic function on $B_1$ can be approximated by a sequence of polynomials pointwise boundedly.

Now let $H$ be a separable Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on $H$. Recall that if $A \in B(H)$, then $Lat(A)$ is by definition the lattice of all invariant subspaces of $A$, and $AlgLat(A)$ is the algebra of all operators $B$ in $B(H)$ such that $Lat(A) \subset Lat(B)$. An operator $A$ in $B(H)$ is said to be reflexive if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $B(H)$ that contains $A$ and the identity $I$ is closed in the weak operator topology. For some sources on these topics see [2-7].

Consider a Hilbert space $H$ of functions analytic on a plane domain $G$, such that for each $\lambda \in G$ the linear functional, $e_\lambda$, of evaluation at $\lambda$ is bounded on $H$. Assume further that $H$ contains the constant functions and multiplication by the independent variable $z$ defines a bounded linear operator $M_z$ on $H$. The continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in G$ there is a unique function $k_\lambda \in H$ such that $e_\lambda(f) = f(\lambda) = \langle f, k_\lambda \rangle$, $f \in H$. The function $k_\lambda$ is called the reproducing kernel for the point $\lambda$.

A complex valued function $\varphi$ on $G$ for which $\varphi f \in H$ for every $f \in H$ is called a multiplier of $H$ and the collection of all these multipliers is denoted by $M(H)$. Each multiplier $\varphi$ of $H$ determines a multiplication operator $M_\varphi$ on $H$ by $M_\varphi f = \varphi f$, $f \in H$. It is well known that each multiplier is a bounded analytic function on $G$ ([8]). In fact, $\|\varphi\|_G \leq \|M_\varphi\|$. We shall use the
following notation for the norm of the operator $M_{\varphi}$:

$$\|\varphi\|_\infty = \|M_{\varphi}\|.$$  

We also point out that if $\varphi$ is a multiplier and $x \in G$, then

$$M_{\varphi}k_z = \varphi(x)k_z.$$  

Also, we say that $M(H)$ is isometrically rotation invariant if whenever $\varphi \in M(H)$, $\varphi_u \in M(H)$ and $\|\varphi\|_\infty = \|\varphi_u\|_\infty$ where $\varphi_u(x) = \varphi(e^{i\theta}x)$. By $H(G)$ and $H_u(G)$ we mean respectively the set of analytic functions on a plane domain $G$ and the set of bounded analytic functions on $G$.

2. Main results

In this article, we investigate the reflexivity of the powers of the multiplication operator $M_z$ acting on a Hilbert function space.

From now on, let $\Omega$ be a domain in the complex plane such that $\Omega = D$. Also, suppose that the Hilbert space $H$ under consideration satisfy the following axioms:

Axiom 1. $H$ is a subspace of the space of all analytic functions on $\Omega$.

Axiom 2. For each $\lambda \in \Omega$, the linear functional of evaluation at $\lambda$, $e_{\lambda}$, is bounded on $H$.

Axiom 3. The uniform limits of polynomials on $\Omega$ is contained in $M(H)$ and $M(H)$ is isometrically rotation invariant.

Axiom 4. The sequence $\{f_k\}_{k \in \mathbb{Z}}$ is an orthogonal basis for $H$ where $f_k(z) = z^k$ for all integers $k$.

Note that by axiom 4, each function $f \in H$ can be represented by series expansion $f = \sum_n f(n)f_n$. For $h \in M(H)$ and $w \in \partial D$, define $h_w$ by $h_w(z) = h(wz)$. Thus $\hat{h}_w(n) = w^n\hat{h}(n)$ for all $n$. Also, since $|w|=1$ we have

$$\|h_w\| = \sum_n |\hat{h}_w(n)|^2 = \sum_n |\hat{h}(n)|^2 = \|h\|^2.$$  

The following theorem extends the results obtained by Allen Shields [9] that have been proved only for the special case where $H$ is the Hilbert space of formal Laurent series.

Lemma 2.1. Let $\varphi \in M(H)$. If $g$ is a continuous complex valued function on $\partial D$ and $d\lambda = |dw|/2\pi$ is the normalized Lebesgue measure on $\partial D$, then the operator

$$\int_{\partial D} \varphi(w) g(w) d\lambda$$

defined by

$$(\int_{\partial D} \varphi(w) g(w) d\lambda)f = \int_{\partial D} g(w) M_{\varphi} f d\lambda$$

is in $M(H)$ and

$$\|\int_{\partial D} \varphi(w) g(w) d\lambda\|_\infty \leq \|M_{\varphi}\| \|\int_{\partial D} g d\lambda\|.$$  

Proof: Note that the strong operator continuity of $\varphi$ allows us to define

$$\int_{\partial D} \varphi(w) g(w) f d\lambda$$

for all $f \in H$. If $f, h \in H$, then

$$<\int_{\partial D} \varphi(w) g(w) f d\lambda, h> = \int_{\partial D} g(w) \varphi(w) f d\lambda, h > d\lambda.$$  

So we get

$$\|\int_{\partial D} \varphi(w) g(w) f d\lambda\| \leq \|M_{\varphi}\| \|\int_{\partial D} g d\lambda\|.$$  

Hence

$$(\int_{\partial D} \varphi(w) g(w) f d\lambda)f = \int_{\partial D} g(w) M_{\varphi} f d\lambda \leq \|M_{\varphi}\| \|\int_{\partial D} g d\lambda\|.$$  

This completes the proof.

Lemma 2.2. If $\varphi \in H(\Omega) \cap M(H)$, then there exists a sequence of polynomials $\{r_n\}$ such that

$r_n(j) = (1 - \frac{j}{n+1})\hat{\varphi}(j)$ whenever $j = 0, \ldots, n$. 

$$\|r_n\| = \sum_n |\hat{r}_n(n)|^2 = \sum_n |\hat{\varphi}(n)|^2 = \|\varphi\|^2.$$  

$$\|M_{r_n}\| = \|M_{\varphi}\|.$$
and is 0 else, and $M_{r_n} \to M_{\varphi}$ in the weak operator topology.

**Proof:** Let $\varphi \in H(\Omega_k) \cap M(H)$. Since $\Omega_k = \mathbb{D}$, we can represent $\varphi$ by a power series

$$\sum_{k=0}^{\infty} \hat{\varphi}(k)z^k.$$ 

Put

$$P_n(\varphi) = \sum_{k=0}^{n} \left(1 - \frac{k}{n + 1}\right) \hat{\varphi}(k)z^k, \quad n \geq 0$$

and

$$K_n(w) = \sum_{|k| \leq n} \left(1 - \frac{|k|}{n + 1}\right) w^k, \quad w \in \partial U, \quad n \geq 0.$$ 

Then

$$\int_{\mathbb{D}} \phi \cdot K_n(\overline{w})d\lambda = M_{\varphi,K_n}, \quad n \geq 0$$

where

$$(\varphi,K_n)(z) = \sum_{j=0}^{n} \hat{\varphi}(j)K_n^j(j)z^j = P_n(\varphi).$$

Note that $K_n \geq 0$ and

$$\int_{\partial D} K_n d\lambda = 1.$$ 

For all $n \geq 0$, $P_n(\varphi) \in M(H)$ and by Lemma 2.1, we get

$$\|M_{P_n(\varphi)}\| = \|M_{\varphi,K_n}\| \leq \|M_{\varphi}\| \int_{\partial D} K_n d\lambda = \|M_{\varphi}\|.$$ 

Put $r_n = P_n(\varphi)$. Note that $M_{r_n}$ is represented by the matrix whose $(i,j)$-th entry is

$$<M_{r_n}f_j,f_i> = \delta_{i,j}(i-j)\|f_i\| = (1-i-j)\|f_i\|.$$ 

Hence

$$\lim_n <M_{r_n}f_j,f_i> = \|M_{\varphi}f_j,f_i>$$

for all base elements $f_j$ and $f_i$ in $H$. By the boundedness of the sequence $\{M_{r_n}\}$ we have $M_{r_n} \to M_{\varphi}$ in the weak operator topology. This completes the proof.

**Theorem 2.3.** If $\{e_{\lambda} : \lambda \in \Omega_k\}$ is norm bounded, then $M_{\varphi}$ is reflexive for all $k \geq 1$.

**Proof:** The boundedness of point evaluations and the Closed Graph Theorem ensure that in multiplication by $z$, $M_z$ is a bounded operator on $H$. Let $k \in \mathbb{N}$ and note that $W(M_z) \subset \text{AlgLat}(M_z)$. On the other hand, let $X \in \text{AlgLat}(M_z)$. Since $Lat(M_z) \subset Lat(M_z)$, we have $Lat(M_z) \subset Lat(X)$. This implies that $X \in \text{AlgLat}(M_z)$. Note that since $M_\varphi e_{\lambda} = \lambda e_{\lambda}$ for all $\lambda \in \Omega$, the one dimensional span of $e_{\lambda}$ is invariant under $M_\varphi$. Therefore, it is invariant under $X^*$ and we can write $X^* e_{\lambda} = \varphi(\lambda)e_{\lambda}$, $\lambda \in \Omega$. So

$$<XF,e_{\lambda}> = <F,X^* e_{\lambda}> = \varphi(\lambda) f(\lambda)$$

for all $f \in H$ and $\lambda \in \Omega$. This implies that $X = M_{\varphi}$ and $\varphi \in M(H)$, hence $\varphi \in H^*(\Omega)$.

Now put $N = H^*(\Omega_k)$. Then $N = \emptyset$, since $I \in N$. Note that by axiom 3, $N \subset M(H)$. To see this let $f \in H^*(\Omega_k)$. Since $\Omega_k$ is a Caratheodory domain, by the Farrel-Rubel-Shields Theorem [1, Theorem 5.1, p. 151], there is a sequence $\{p_n\}$ of polynomials converging to $f$ such that for all $n$, $N$ for some $c > 0$. So $\{p_n\}$ is a normal family in $H^*(\Omega)$ and by passing to a subsequence if necessary, we may suppose that for some function $g$, $p_n \to g$ uniformly on compact subsets of $\Omega$, this implies that indeed $g = f$. Hence by axiom 3, $f \in M(H)$ and so $N \subset M(H) \subset H$. Also, it is a closed subspace of $H$, since if $\{h_n\}$ is a normal family in $H$, and $h_n \to f$ in $H$, so for all $n$, $\|h_n - f\| \leq c_1$ for some $c_1 > 0$. Because point evaluations are bounded, for all $\lambda \in \Omega$ we have

$$h_n(\lambda) = <h_n,e_{\lambda}> \to <f,e_{\lambda}> = f(\lambda).$$
Also, we note that for all $\lambda$ in $\Omega$,  
$$ |h_n(\lambda)| |e_\lambda| \leq \|h_n\| \|e_\lambda\| \leq c_2 \|h_n\|_H$$  
where $c_2 = \sup \{ \|e_\lambda\| : \lambda \in \Omega \}$. Thus  
$$ \|h_n\|_2 \leq c_2 \|h_n\|_{\Omega} \leq c_2 c_2$$  
for all $n$. Since $h_n \in H^\infty(\Omega_1)$, $\|h_n\|_3 = \|h_n\|_{\Omega}$ and so $\|h_n\|_2 \leq c_2 c_2$ for all $n$. This implies that  
$$\{h_n\}_n$$  
is a normal family in $H^\infty(\Omega_1)$ and we may assume that for some function $g$, $h_n \to g$ uniformly on compact subsets of $\Omega_1$. Thus $g \in H^\infty(\Omega_1)$. But by pointwise convergence $f = g$ on $\Omega$ and so $f$ can be extended to a bounded analytic function on $\Omega_1$, i.e., $f \in H^\infty(\Omega_1)$ and so $N$ is indeed a closed subspace of $H$. Now clearly $N \in Lat(M_z)$, thus $XN \subset N$. Since $1 \in N$ we get $X1 = \varphi \in N \subset H^\infty(\Omega_1)$. Now by Lemma 2.2, there exists a sequence of polynomials $\{r_n\}$ (indeed $r_n = P_n(\varphi)$) such that $M_{r_n} \to M_\varphi$ in the weak operator topology. Now let $M_k$ be the closed linear span of the set $\{f_{nk} : n \geq 0\}$ (recall that $f_{i}(z) = z^{i}$ for all $i$). We have  
$$M_k f_{nk} = f_{(n+1)k} \in M_k$$  
for all $n \geq 0$. Thus $M_k \in Lat(M_z)$ and so $M_k \in Lat(M_z)$. Let $\varphi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n$. Since $1 \in M_k$, thus $M_1 = \varphi \in M_k$. Hence $\hat{\varphi}(i) = 0$ for all $i \neq nk, \ n \geq 0$. Now, by a consequence of the particular construction of $r_n$ used in Lemma 2.2, each $r_n$ should be a polynomial in $z^k$, i.e., $r_n(z) = q_n(z^k)$ for some polynomial $q_n$. Thus  
$$M_{r_n} = r_n(M_z) = q_n(M_z^k) \to X$$  
in the weak operator topology. Hence $X \in W(M_z^k)$. Thus $M_z^k$ is reflexive and this completes the proof.

Acknowledgment

This investigation has been supported by project 90548 of the Islamic Azad University, Shiraz Branch, Iran.

References