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Graded prime spectrum of a graded module

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Abstract

Let R be a graded ring and M be a graded R-module. We define a topology on graded prime spectrum $G-Spec(M)$ of the graded R-module M which is analogous to that for $G-Spec(R)$, and investigate several properties of the topology.

***Keywords**:* Graded module; graded prime spectrum; graded prime submodule

1. Introduction

Let G be a multiplicative group. A commutative ring R with identity is called a G-*graded ring* if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called *homogeneous elements* of R of degree g. The homogeneous elements of the ring R are denoted by $h(R)$, i.e. $h(R) = \bigcup_{g\in G} R_g$. If $a \in R$, then the element a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is called the g-*component* of a in R_g. Let R be a graded ring and I be an ideal of R. I is called graded prime ideal of R if $I \neq R$ and whenever $ab \in I$, then either $a \in I$ or $b \in I$, where a, $b \in h(R)$. The *graded radical* of I is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that if $r \in h(R)$, then r is an element of graded radical of I if and only if $r^n \in I$ for some $n \in N$. The graded radical of I is denoted by $\sqrt{I}$.

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Let R be a G-graded ring and M an R-module. We recall that M is a G-*graded* R-*module* (or *graded* R-*module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ where $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$ are called *homogeneous*. Let $M = \bigoplus_{g\in G} M_g$ be a graded R-module and N be a submodule of M. Then recall that N is a *graded submodule* of M if $N = \bigoplus_{g\in G} (N \cap M_g)$. In this case, $N_g = N \cap M_g$ is called the g-*component* of N.

Let M be a graded R-module and N be a graded R-submodule of M. Then recall that N is a *graded prime submodule* of M if $N \neq M$ and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a \in (N :_R M)$ where $(N :_R M) = \{r \in R \mid rM \subseteq N\}$. Graded prime submodules of graded modules have been studied by various authors, see, for example, [1-3].

A graded R-module M is called to be a *multiplication graded module* if for every graded submodule N of M has the form IM for some graded ideal I of R. Multiplication graded modules were characterized in [4]. N is a *graded maximal submodule* of M if $N \neq M$ and there is no graded submodule N' of M such that $N \subset N' \subset M$. A graded R-module M is called *graded finitely generated* if there are $x_1, x_2, \cdots, x_k$ in $h(M)$ such that $M = \sum_{i=1}^{k} Rx_i$. An element $m \in M$ is called *nilpotent* if $(Rm)^k = 0$ for some positive integer k [5]. It is clear that, if M is graded multiplication then $Nil(M) = \bigcap P$ where the intersection runs over all graded prime submodules of M. Moreover, a faithful graded R-module M is multiplication if and only if $\bigcap_{\lambda \in \Lambda} (I_\lambda M) = \left(\bigcap_{\lambda \in \Lambda} I_\lambda \right) M$ where I_λ is a graded ideal of R [6, Theorem 8].

The graded prime spectrum $G - Spec(R)$ of a graded ring R consists of all graded prime ideals of R and similarly the graded prime spectrum $G - Spec(M)$ of a graded module M consists of all graded prime submodules of M. For each graded ideal I of R, if we introduce the G-variety $V_G^R(I) = \{ p \in G - Spec(R) \mid p \supseteq I \}$ then the collection $\zeta(R) = \{ V_G^R(I) \mid I \lhd_G R \}$ satisfies the topology axioms for closed sets. This topology is called a *Zariski topology* on $G - Spec(R)$. In this study, we generalize this prime spectrum to graded R-modules. For a graded submodule N of M we define the variety $V_G^*(N) = \{ P \in G - Spec(M) \mid P \supseteq N \}$ where the collection $\zeta^*(M) = \{ V_G^*(N) \mid N$ is a submodule of $M \}$ does not satisfy all of the topology axioms for closed sets. Whenever $\zeta^*(M)$ is closed under finite union, then this topology is called a *quasi-Zariski topology* and the module M is called a G*-top module*. After this, we define another variety $V_G(N) = \{ P \in G - Spec(M) \mid (P:M) \supseteq (N:M) \}$ of the graded module M, the collection $\zeta(M) = \{ V_G(N) \mid N$ is a submodule of $M \}$ satisfies all of the topology axioms for the closed sets. Hence we obatain a topology on $G - Spec(M)$ called a *Zariski topology*. Some properties of these topologies are given and we obtain some relations between properties of the graded prime spectrum $G - Spec(R)$ and $G - Spec(M)$ by using the map $\phi : G - Spec(M) \to G - Spec\left(R / Ann(M) \right)$ defined by $P \mapsto (P:M)$ for $P \in G - Spec(M)$. Finally, we give some results that determine under what conditions the graded prime spectrum $G - Spec(M)$ is T_0, T_1 or T_2-space.

Throughout this paper, we deal with G-graded rings and graded R-modules. If I is a graded ideal of R and N is a graded submodule of M we write respectively, $I \lhd_G R$ and $N <_G M$. Throughout this paper we assume that $G - Spec(M)$ is nonempty.

2. The Zariski topology on $G - Spec(R)$

In this section we will give some properties of the G-variety $V_G^R(S) = \{ p \in G - Spec(R) \mid p \supseteq S \}$ for a homogeneous subset S of R. Note that, if the graded ideal I is generated by S, then it is clear that $V_G^R(S) = V_G^R(I)$. Also, $V_G^R(I) = V_G^R(\sqrt{I})$ for any graded ideal I of R. Therefore, we can easily see that $V_G^R(rR) = V_G^R(r)$ for any $r \in h(R)$. We show that the set $G - Spec(R)$ is a topology for the closed sets $V_G^R(I)$.

Proposition 2.1. Let I, J and $\{I_i\}_{i \in \Lambda}$ be graded ideals of the graded ring R. Then the following hold for G-variety of ideals:

(1) $V_G^R(0) = G - Spec(R)$ and $V_G^R(R) = \varnothing$,

(2) $\bigcap_{i \in \Lambda} V_G^R(I_i) = V_G^R\left(\sum_{i \in \Lambda} I_i \right) = V_G^R\left(\bigcup_{i \in \Lambda} I_i \right)$,

(3) $V_G^R(I) \cup V_G^R(J) = V_G^R(I \cap J) = V_G^R(IJ)$.

Proof: (1) For any $p \in G - Spec(R)$, $0 \subseteq p$, so $p \in V_G^R(0)$. Hence $G - Spec(R) = V_G^R(0)$.

Suppose that $V_G{}^R(R)\neq\varnothing$. Then there is $p\in V_G{}^R(R)$. Hence $1\in R\subseteq p$, a contradiction.

(2) Let $p\in\bigcap_{i\in\Lambda}V_G^R(I_i)$. Then $p\in V_G^R(I_i)$ and we obtain that $I_i\subseteq p$ for all $i\in\Lambda$. Hence $\sum_{i\in\Lambda}I_i\subseteq p$, so that $p\in V_G^R\left(\sum_{i\in\Lambda}I_i\right)$. Conversely, let $p\in V_G^R\left(\sum_{i\in\Lambda}I_i\right)$. Then $\sum_{i\in\Lambda}I_i\subseteq p$ and so $I_i\subseteq p$ for all $i\in\Lambda$. This shows that $p\in V_G^R(I_i)$ for all $i\in\Lambda$ and hence $p\in\bigcap_{i\in\Lambda}V_G^R(I_i)$.

(3) Since $V_G^R(I)\subseteq V_G^R(I\cap J)$ and $V_G^R(J)\subseteq V_G^R(I\cap J)$, $V_G^R(I)\cup V_G^R(J)\subseteq V_G^R(I\cap J)$. For the reverse inclusion, let $p\in V_G^R(I\cap J)$. Then $I\cap J\subseteq p$. Since p is a graded prime ideal, then $I\subseteq p$ or $J\subseteq p$. So that $p\in V_G^R(I)$ or $p\in V_G^R(J)$. We obtain $V_G^R(I\cap J)\subseteq V_G^R(I)\cup V_G^R(J)$.

Corollary 2.2. Let R be a graded ring. The collection $\zeta(R)=\left\{V_G^R(I)\,\middle|\, I\lhd_G R\right\}$ of all varieties of graded ideals of R satisfies the axioms of topological space for closed sets. We call this topology the *Zariski topology* on $G-Spec(R)$.

Theorem 2.3. Let R be a graded ring. For any homogeneous elements r and s of R, we have the following properties:

(1) The set $D_r=G-Spec(R)\setminus V_G^R(rR)$ is open in $G-Spec(R)$ and the family $\left\{D_r\,\middle|\, r\in h(R)\right\}$ is the basis for the Zariski topology on $G-Spec(R)$.

(2) For the open sets D_r and D_s, we have $D_r\cap D_s=D_{rs}$.

(3) For the open sets D_r and D_s, we have $D_r=D_s$ if and only if $\sqrt{rR}=\sqrt{sR}$.

(4) The open set D_r is quasi compact for all $r\in h(R)$.

(5) The space $G-Spec(R)$ is a T_0-space for the Zariski topology.

Proof: (1) Assume that U is an open set in $G-Spec(R)$. Thus $U=G-Spec(R)\setminus V_G^R(I)$ for some graded ideal I of R. Notice that $I=\bigcup_{g\in G}I_g=\langle h(I)\rangle$. Then $V_G^R(I)=V_G^R(h(I))=\bigcap_{r\in h(I)}V_G^R(r)$. Hence $U=\bigcup_{r\in h(I)}\left(G-Spec(R)\setminus V_G^R(r)\right)=\bigcup_{r\in h(I)}D_r$. This implies that $\left\{D_r\,\middle|\, r\in h(R)\right\}$ is a basis for the Zariski topology on $G-Spec(R)$.

(2) Let $p\in D_r\cap D_s$ for the open sets D_r and D_s. Then $r\notin p$ and $s\notin p$, so that $rs\notin p$. It follows that $p\in D_{rs}$ and hence $D_r\cap D_s\subseteq D_{rs}$. For reverse inclusion, assume that $p\in D_{rs}$. Then $rs\notin p$, namely $r\notin p$ and $s\notin p$. Hence $p\in D_r$ and $p\in D_s$, so that $D_{rs}\subseteq D_r\cap D_s$.

(3) Suppose that $D_r=D_s$. Then $V_G^R(rR)=V_G^R(sR)$, so that $r\in p$ if and only if $s\in p$. This implies $\sqrt{rR}=\sqrt{sR}$. Conversely, assume that $\sqrt{rR}=\sqrt{sR}$. It follows that $r\in p$ if and only if $s\in p$. Then $V_G^R(rR)=V_G^R(sR)$ and hence $D_r=D_s$.

(4) Let $r\in h(R)$ and suppose that $\left\{D_{s_i}\,\middle|\, i\in\Lambda\right\}$ is an open cover of D_r, where for each $i\in\Lambda$, $s_i\in h(R)$. Then,

$$G-Spec(R)\setminus V_G^R(rR)=D_r\subseteq\bigcup_{i\in\Lambda}D_{s_i}=\bigcup_{i\in\Lambda}\left(G-Spec(R)\setminus V_G^R(s_iR)\right)$$

$$=G-Spec(R)\setminus V_G{}^R\left(\sum_{i\in\Lambda}s_iR\right)$$

and hence $V_G^R\left(\sum_{i\in\Lambda}s_iR\right)\subseteq V_G^R(R)$. It follows from (3) that $\sqrt{rR}\subseteq\sqrt{\sum_{i\in\Lambda}s_iR}$, then there exists a positive integer n such that $r^n\in\sum_{i\in\Lambda}s_iR$. Then there exists $i_1,i_2,\cdots,i_m\in\Lambda$, $t_{i_1},t_{i_2},\cdots,t_{i_m}\in h(R)$

such that $r^n = s_{i_1}t_{i_1} + s_{i_2}t_{i_2} + \cdots + s_{i_m}t_{i_m}$. Let $\Delta = \{i_1, i_2, \cdots, i_m\} \subseteq \Lambda$. Notice that $p \in V_G^R(r)$ iff $p \in V_G^R(r^n)$. $(rR)^n \subseteq \sum_{j\in\Delta} s_j R$ implies $V_G^R\left(\sum_{j\in\Delta} s_j R\right) \subseteq V_G^R(r^n) = V_G^R(r)$. Therefore $\bigcap_{j\in\Delta} V_G^R(s_j) \subseteq V_G^R(r)$, so $\bigcup_{i\in\Delta}(G-Spec(R)\setminus V_G^R(s_i)) \supseteq G-Spec(R) \setminus V_G^R(r)$ and hence $D_r \subseteq \bigcup_{i\in\Delta} D_{s_i}$. Since Δ is finite set, D_r is quasi compact.

(5) Let $p,\ q \in G-Spec(R)$ and $p \neq q$. Then $p \setminus q \neq \varnothing$ or $q \setminus p \neq \varnothing$. Suppose that $p \setminus q \neq \varnothing$. Then there exists an element $r \in p \setminus q$ for $r \in h(R)$. Then $p \notin D_r$ and since $rR \not\subset q$, we get $q \notin V_G^R(rR)$. So $q \in D_r$ and since D_r is an open set, $G-Spec(R)$ is a T_0-space for the Zariski topology.

3. The Zariski topology on $G-Spec(M)$

In this section we will give different varieties for any graded submodule of a graded module. Also, we investigate under what conditions these varieties give a topology on $G-Spec(M)$. Now we give some relations between graded ideals of R and graded submodules of graded R-modules M.

Lemma 3.1. Let R be a G-graded ring, M be a graded R-module, and N be a graded R-submodule of M. Then the following hold:

(i) $(N :_R M) = \{r \in R \mid rM \subseteq N\}$ is a graded ideal of R,

(ii) If I is a graded ideal of R, $r \in h(R)$ and $x \in h(M)$, then IN, rN, and Rx are graded submodules of M.

Proof: One can look for the proof of (i) and (ii) to [1, Lemma 2.1], [7, Lemma 2.2], and [6, Lemma 1]. Also, for the proof of (i), see [5, Lemma 1.2 (iii)].

Theorem 3.2. Let M be a graded R-module. If N is a graded prime submodule of M then $(N :_R M)$ is a graded prime ideal of R. The converse part is true when M is a multiplication graded R-module.

Proof: One can look for the proof to [6, Theorem 3].

Proposition 3.3. Let M be a graded R-module. For any graded submodule N of M, we define the *variety* of N to be $V_G^*(N) = \{P \in G-Spec(M) \mid P \supseteq N\}$. Then the following hold:

(1) $V_G^*(0) = G-Spec(M)$ and $V_G^*(M) = \varnothing$.

(2) $\bigcap_{i\in\Lambda} V_G^*(N_i) = V_G^*\left(\sum_{i\in\Lambda} N_i\right)$, for any family $\{N_i\}_{i\in\Lambda}$ of graded submodules.

(3) $V_G^*(N) \cup V_G^*(L) \subseteq V_G^*(N \cap L)$ for any graded submodules N, L of M.

Proof: (1) Trivial.

(2) Let $P \in \bigcap_{i\in\Lambda} V_G^*(N_i)$. Then, $P \in V_G^*(N_i)$ gives us $N_i \subseteq P$ for all $i \in \Lambda$. It follows that $\sum_{i\in\Lambda} N_i \subseteq P$ and hence $P \in V_G^*\left(\sum_{i\in\Lambda} N_i\right)$. Conversely, assume that $P \in V_G^*\left(\sum_{i\in\Lambda} N_i\right)$. Then $\sum_{i\in\Lambda} N_i \subseteq P$ and so, $N_i \subseteq P$ for all $i \in \Lambda$. Thus $P \in \bigcap_{i\in\Lambda} V_G^*(N_i)$ and equality holds.

(3) Since $N \cap L \subseteq N$ and $N \cap L \subseteq L$, then $V_G^*(N) \subseteq V_G^*(N \cap L)$ and $V_G^*(L) \subseteq V_G^*(N \cap L)$. Hence $V_G^*(N) \cup V_G^*(L) \subseteq V_G^*(N \cap L)$.

Remark that, the reverse inclusion in Proposition 3.3 (3) is not true in general. For this, if we take the Z_2-graded Z-module $M = Z \times Z$ and $N = 4Z \times \{0\}$, $L = \{0\} \times 4Z$ as graded submodules of M, then $P = \{0\} \times \{0\} \in V^*(N \cap L)$ but $P \notin V_G^*(N) \cup V_G^*(L)$ since $N \not\subset P$ and $L \not\subset P$, where $P \in G-Spec(M)$.

Definition 3.4. Let M be a graded R-module and $\zeta^*(M)$ be the set of all varieties $V_G^*(N)$ of M, i.e., $\zeta^*(M) = \{V_G^*(N) \mid N <_G M\}$.

M is called a G -*top module* if the set $\zeta^*(M)$ is closed under finite union. Then $\zeta^*(M)$ is a topology on $G-Spec(M)$ and this topology is called a *quasi Zariski topology* on $G-Spec(M)$, denoted by τ^*.

Theorem 3.5. If M is a multiplication graded R -module, then M is a G -top module.

Proof: It is enough to prove that the inclusion $V_G^*(N\cap L)\subseteq V_G^*(N)\cup V_G^*(L)$ is satisfied. Let $P\in V_G^*(N\cap L)$. Then $N\cap L\subseteq P$ and we get $(N\cap L:M)\subseteq(P:M)$. Since $(P:M)$ is a graded prime ideal and $(N\cap L:M)=(N:M)\cap(L:M)$, we get $(N:M)\subseteq(P:M)$ or $(L:M)\subseteq(P:M)$. Then $(N:M)M\subseteq(P:M)M$ or $(L:M)M\subseteq(P:M)M$. Since M is graded multiplication module, then $N\subseteq P$ or $L\subseteq P$. Hence $P\in V_G^*(N)\cup V_G^*(L)$.

Proposition 3.6. Let M be a graded R -module. Then the family $\zeta'(M)=\{V_G^*(IM)\mid I\lhd_G R\}$ is closed under finite union. Further, $\zeta'(M)$ is a topology on $G-Spec(M)$ denoted by τ'.

Proposition 3.7. Let M be a graded R -module. If M is a G -top module then the quasi Zariski topology τ^* on $G-Spec(M)$ is finer than τ'.

Now we define another variety for a graded submodule N of a graded module M. We define the variety of N to be $V_G(N)=\{P\in G-Spec(M)\mid (P:M)\supseteq(N:M)\}$

The following proposition shows that this variety satisfies the topology axioms for closed sets.

Proposition 3.8. Let M be a graded R -module. Then the following hold:

(1) $V_G(0)=G-Spec(M)$ and $V_G(M)=\varnothing$.

(2) $\bigcap_{i\in\Lambda}V_G(N_i)=V_G\left(\sum_{i\in\Lambda}(N_i:M)M\right)$, for any family $\{N_i\}_{i\in\Lambda}$ of graded submodules.

(3) $V_G(N)\cup V_G(L)=V_G(N\cap L)$ for any graded submodules $N,\ L$ of M.

Proof: (1) It is clear.

(2) Let $P\in\bigcap_{i\in\Lambda}V_G(N_i)$. For all $i\in\Lambda$, $P\in V_G(N_i)$ implies $(N_i:M)\subseteq(P:M)$. Then $(N_i:M)M\subseteq(P:M)M$. It follows that $\sum_{i\in\Lambda}(N_i:M)M\subseteq(P:M)M\subseteq P$ for all $i\in\Lambda$. Therefore $P\in V_G\left(\sum_{i\in\Lambda}(N_i:M)M\right)$, so $\bigcap_{i\in\Lambda}V_G(N_i)\subseteq V_G\left(\sum_{i\in\Lambda}(N_i:M)M\right)$.

Conversely, let $P\in V_G\left(\sum_{i\in\Lambda}(N_i:M)M\right)$. Then $\left(\sum_{i\in\Lambda}(N_i:M)M:M\right)\subseteq(P:M)$. Since $(N_i:M)\subseteq\left(\sum_{i\in\Lambda}(N_i:M)M:M\right)$, we get $(N_i:M)\subseteq(P:M)$ for all $i\in\Lambda$. Thus $P\in V_G(N_i)$, for all $i\in\Lambda$. Hence $P\in\bigcap_{i\in\Lambda}V_G(N_i)$.

(3) Let $P\in V_G(N\cap L)$. Then $(N\cap L:M)\subseteq(P:M)$, so that $(N:M)\cap(L:M)\subseteq(P:M)$. Since $(P:M)$ is graded prime ideal, then $(N:M)\subseteq(P:M)$ or $(L:M)\subseteq(P:M)$. It follows that $P\in V_G(N)$ or $P\in V_G(L)$. Hence $P\in V_G(N)\cup V_G(L)$. Reverse inclusion is clear.

Definition 3.9. Let M be a graded R -module. Since $\zeta(M)=\{V_G(N)\mid N<_G M\}$ is closed under finite union, the family $\zeta(M)$ satisfies the axioms of topological space for closed sets. So, there exists a topology on $G-Spec(M)$ called the *Zariski topology* and denoted by τ.

Definition 3.10. Let M be a graded R -module and $p\in G-Spec(R)$. Then the set

$G-Spec_p(M)$ is defined to be $\{P \in G-Spec(M) \mid (P:M)=p\}$.

Now we give some relations between the varieties $V_G^*(N)$ and $V_G(N)$ for any submodule N of the graded R-module M.

Lemma 3.11: Let M be a graded R-module and N, L be graded submodules of M.

(1) If $(N:M)=(L:M)$, then $V_G(N)=V_G(L)$. The converse is true if N and L are graded prime submodules.

(2) $V_G(N)=V_G\left((N:M)M\right)=V_G^*\left((N:M)M\right)$.

Theorem 3.12. For any graded R-module M, the Zariski topology τ on $G-Spec(M)$ is identical with τ' and the quasi Zariski topology τ^* on $G-Spec(M)$ is finer than the Zariski topology τ.

Proof: It is clear.

Let M be a graded R-module. Now we give the relation between $G-Spec(M)$ and $G-Spec\left(R/Ann(M)\right)$. For this we set X^M and $X^{\bar{R}}$ to represent $G-Spec(M)$ and $G-Spec(\bar{R})$ respectively, where $\bar{R}=R/Ann(M)$.

The map $\varphi: X^M \to X^{\bar{R}}$, defined by $P \mapsto \overline{(P:M)}$ for $P \in X^M$ is called the *natural map* of X^M.

Proposition 3.13. Let M be a graded R-module. The natural map φ of X^M is continuous for the Zariski topologies defined on M and $\bar{R}$. More precisely, $\varphi^{-1}\left(V_G^{\bar{R}}(\bar{I})\right)=V_G(IM)$ for every graded ideal I of R containing $Ann(M)$.

Proof: Let $\bar{I}$ be a graded ideal of $\bar{R}$, $V_G^{\bar{R}}(\bar{I}) \in \zeta(\bar{R})$ and $P \in \varphi^{-1}\left(V_G^{\bar{R}}(\bar{I})\right)$. Then $\overline{(P:M)}=\varphi(P) \in V_G^{\bar{R}}(\bar{I})$, thus $\overline{(P:M)} \supseteq \bar{I}$. It follows that $(P:M) \supseteq I$, that $P \in V_G(IM)$. Therefore $\varphi^{-1}\left(V_G^{\bar{R}}(\bar{I})\right) \subseteq V_G(IM)$. For the converse inclusion, let $P \in V_G(IM)$. Then, $IM \subseteq P \in G-Spec(M)$ and hence $I \subseteq (P:M) \subseteq G-Spec(R)$. And so we get $\varphi(P)=\overline{(P:M)} \in V_G^{\bar{R}}(\bar{I})$. This implies $P \in \varphi^{-1}\left(V_G^{\bar{R}}(\bar{I})\right)$. Hence the proof is completed.

Proposition 3.14. The following statements are equivalent for any graded R-module M and any $P,\ Q \in X^M$:

(1) The natural map φ is injective.

(2) If $V_G(P)=V_G(Q)$, then $P=Q$.

(3) $\left|G-Spec_p(M)\right| \le 1$ for every $p \in G-Spec(R)$.

Proof: (1)⇒(2): Suppose that $V_G(P)=V_G(Q)$. By Lemma 3.11, we get $\overline{(P:M)}=\overline{(Q:M)}$. Thus $\varphi(P)=\varphi(Q)$. Since φ is injective, we obtain $P=Q$.

(2)⇒(3): Let $\left|G-Spec_p(M)\right|>1$ and let P, $Q \in G-Spec_p(M)$ such that $P \ne Q$. So, $(P:M)=(Q:M)=p$. Hence we get $V_G(P)=V_G(Q)$ and by hypothesis we obtain $P=Q$, which is a contradiction.

(3)⇒(1): Let $\varphi(P)=\varphi(Q)$. It follows that $\overline{(P:M)}=\overline{(Q:M)}$. So, we can write $(P:M)=(Q:M)=p$ and since $\left|G-Spec_p(M)\right| \le 1$, we get $P=Q$.

Proposition 3.15. Let M be a graded R-module and let φ be the natural map of X^M. If φ is surjective, then φ is both open and closed, more precisely for every $N <_G M$, $\varphi\left(V_G(N)\right)=V_G^{\bar{R}}\left(\overline{(N:M)}\right)$ and $\varphi(X^M \setminus V_G(N)) = X^{\bar{R}} \setminus V_G^{\bar{R}}\left(\overline{(N:M)}\right)$.

Proof: Since φ is a continuous map such that $\varphi^{-1}\left(V_G^{\bar{R}}(\bar{I})\right)=V_G(IM)$, we get for every $N <_G M$,

$\varphi^{-1}\left(V_G^{\bar{R}}\left(\overline{(N:M)}\right)\right)=V_G\left((N:M)M\right)=V_G(N)$.

As φ is surjective,

$\varphi \circ \varphi^{-1}\left(V_G^{\bar{R}}\left(\overline{(N:M)}\right)\right) = V_G^{\bar{R}}\left(\overline{(N:M)}\right)$. Thus $\varphi(V_G(N)) = V_G^{\bar{R}}\left(\overline{(N:M)}\right)$. Similarly $\varphi\left(X^M \setminus V_G(N)\right) = \varphi\left(\varphi^{-1}(X^{\bar{R}}) \setminus \varphi^{-1}\left(V_G^{\bar{R}}\left(\overline{(N:M)}\right)\right)\right)$ $= \varphi \circ \varphi^{-1}\left(X^{\bar{R}} \setminus V_G^{\bar{R}}\left(\overline{(N:M)}\right)\right)$ and so $\varphi\left(X^M \setminus V_G(N)\right) = X^{\bar{R}} \setminus V_G^{\bar{R}}\left(\overline{(N:M)}\right)$.

Corollary 3.16. Let φ be surjective and M be a graded R -module. Then φ is bijective if and only if φ is homeomorphic.

Proposition 3.17: Let M and M' be graded R -modules, $X^M = G-Spec(M)$ and $X^{M'} = G-Spec(M')$. If $f: M \to M'$ is an epimorphism, then the function $\phi: X^{M'} \to X^M$ defined by $P' \to f^{-1}(P')$ is continuous.

Proof: For any $N <_G M$ and $P' \in X^{M'}$ and any closed set $V_G(N)$ of X^M, we have $P' \in \phi^{-1}(V_G(N)) = \phi^{-1}\left(V_G^*\left((N:M)M\right)\right)$ iff $\phi(P') = f^{-1}(P') \supseteq (N:M)M$ iff $P' \supseteq f\left((N:M)M\right) = (N:M)M'$ iff $P' \in V_G^*\left((N:M)M'\right) = V_G\left((N:M)M'\right)$. Thus $\phi^{-1}(V_G(N)) = V_G\left((N:M)M'\right)$. Hence ϕ is continuous.

4. A base for the Zariski topolgy on $G-Spec(M)$

In this section we write $X_r = X^M \setminus V_G(rM)$ of X^M for $r \in h(R)$ and show that $B = \left\{X_r \,\middle|\, r \in h(R)\right\}$ forms a base for X^M. Further, we compare this base with the base of $X^{\bar{R}}$. For each element r of $h(R)$, we write $X_r = X^M \setminus V_G(rM)$. Clearly, every X_r is an open set of X^M and we have $X_0 = \varnothing$ and $X_1 = X^M$ for $0_R, 1_R \in h(R)$.

Proposition 4.1: Let M be a graded R -module with natural map φ on X^M and $r \in h(R)$. Then,

(1) $\varphi^{-1}(D_{\bar{r}}) = X_r$

(2) $\varphi(X_r) \subseteq D_{\bar{r}}$. If φ is surjective, then the equality holds.

(3) The set $B = \left\{X_r \,\middle|\, r \in h(R)\right\}$ is a base for the Zariski topology on X^M.

(4) $X_{rs} = X_r \cap X_s$, for any $r,\ s \in h(R)$.

Proof: (1) $\varphi^{-1}\left(D_{\bar{r}}\right) = \varphi^{-1}\left(X^{\bar{R}} \setminus V_G^{\bar{R}}(\bar{r}\bar{R})\right)$ $= X^M \setminus \varphi^{-1}\left(V_G^{\bar{R}}(\bar{r}\bar{R})\right) = X^M \setminus V_G(rM) = X_r$.

(2) Trivial.

(3) Let U be any open set in X^M. Since $\zeta(M) = \zeta'(M) = \left\{V_G^*(IM) = V_G(IM) \,\middle|\, I \lhd_G R\right\}$ by Lemma 3.11, $U = X^M \setminus V_G(IM)$ for some graded ideal I of R. Notice that $I = \langle h(I) \rangle$. Then, $IM = \langle h(I) \rangle M = \langle h(I)M \rangle$. So, $V_G(IM) = V_G\left(h(I)M\right) = \bigcap_{r \in h(I)} V_G(rM)$. It follows that $U = X^M \setminus V_G(IM) = X^M \setminus \bigcap_{r \in h(I)} V_G(rM) = \bigcup_{r \in h(I)} X_r$ ⋅ Therefore B is a base for the Zariski topology on X^M.

(4) $X_{rs} = \varphi^{-1}\left(D_{\overline{rs}}\right) = \varphi^{-1}\left(D_{\bar{r}} \cap D_{\bar{s}}\right) = \varphi^{-1}\left(D_{\bar{r}}\right) \cap \varphi^{-1}\left(D_{\bar{s}}\right) = X_r \cap X_s$ by (1).

Theorem 4.2. Let M be a graded R -module. If the natural map φ is surjective, then the open set X_r is quasi compact for each $r \in h(R)$. Specifically, X^M is quasi compact.

Proof: As the set $B = \left\{X_r \,\middle|\, r \in h(R)\right\}$ is a base for the Zariski topology by Proposition 4.1(3), for every open cover of X_r, there is a set $\left\{r_\alpha \in h(R) \,\middle|\, \alpha \in \Lambda\right\}$ such that $X_r \subseteq \bigcup_{\alpha \in \Lambda} X_{r_\alpha}$. Then $D_{\bar{r}} = \varphi\left(X_r\right) \subseteq \bigcup_{\alpha \in \Lambda} \varphi\left(X_{r_\alpha}\right) = \bigcup_{\alpha \in \Lambda} D_{\bar{r}_\alpha}$ by Proposition 10(2). Since $D_{\bar{r}}$ is quasi compact, there exists a finite subset $\Lambda' \subset \Lambda$ such that

$D_{\bar{r}} \subseteq \bigcup_{\alpha\in\Lambda'} D_{\bar{r}_\alpha}$. Hence we obtain $X_r = \varphi^{-1}(D_{\bar{r}}) \subseteq \bigcup_{\alpha\in\Lambda'} X_{r_\alpha}$.

Let M be a graded R-module and Y be any subset of X^M. We will denote the intersection of all elements in Y by $\xi(Y)$ and the closure of Y in X^M for the Zariski topology by $Cl(Y)$.

Proposition 4.3. Let M be a graded R-module and $Y \subseteq X^M$. Then $V_G(\xi(Y)) = Cl(Y)$. In particular, Y is closed if and only if $V_G(\xi(Y)) = Y$.

Proof: We can see easily that $Y \subseteq V_G(\xi(Y))$. Let $V_G(L)$ be any closed subset of X^M which contains Y. Thus for all $Q \in Y$, we have $(Q:M) \supseteq (L:M)$. This implies that $(L:M) \subseteq \bigcap_{Q\in Y}(Q:M) \subseteq (\xi(Y):M)$. So, $(P:M) \supseteq (\xi(Y):M) \supseteq (L:M)$ for every $P \in V_G(\xi(Y))$, that is, $V_G(\xi(Y)) \subseteq V_G(L)$. Hence $V_G(\xi(Y))$ is the smallest closed subset of X^M including Y, which means $V_G(\xi(Y)) = Cl(Y)$.

Proposition 4.4. Let M be a graded R-module, $P \in X^M$, and $\delta = \{(Q:M) \mid Q \in X^M\} \subseteq X^R$. Then,

(1) $Cl(\{P\}) = V_G(P)$.

(2) For any $Q \in X^M$, $Q \in Cl(\{P\})$, if and only if $(Q:M) \supseteq (P:M)$ if and only if $V_G(P) \supseteq V_G(Q)$.

(3) Let M be a finitely generated graded R-module. The set $\{P\}$ is closed in X^M if and only if

a) $p = (P:M)$ is a maximal element of the set δ, and

b) $G-Spec_p(M) = \{P\}$, that is, $|G-Spec_p(M)| = 1$.

Proof: (1) We can easily see that (1) holds by taking $Y = \{P\}$ from Proposition 4.3.

(2) This follows from (1).

(3) Assume that $\{P\}$ is closed in X^M. Hence $\{P\} = Cl(\{P\}) = V_G(P)$ by (1). Let $q \in \delta$ such that $p \subseteq q$. Then there exists $Q \in X^M$ such that $q = (Q:M)$. So, $(P:M) = p \subseteq (Q:M)$. We have $Q \in V_G(P) = \{P\}$, namely $Q = P$. So, $p = q$ and p is a maximal element of the set δ. Let $P^* \in G-Spec_p(M)$. Then $(P^*:M) = p = (P:M)$ and so $P^* \in V_G(P) = \{P\}$. Hence $G-Spec_p(M) = \{P\}$. Conversely, we suppose that (a) and (b) hold. Since P is graded prime we have $\{P\} \subseteq V_G(P)$. If $Q \in V_G(P)$, then $q = (Q:M) \supseteq (P:M) = p$. Therefore $q = p$ by (a) and $Q = P$ by (b). Thus $V_G(P) \subseteq \{P\}$, so that $V_G(P) = \{P\}$. By (1), $Cl(\{P\}) = \{P\}$. Hence the set $\{P\}$ is closed in X^M.

The following corollary is a result of Proposition 4.4(1).

Corollary 4.5. For every graded prime submodule P of a graded R-module M, $V_G(P)$ is an irreducible closed subset of X^M.

Proposition 4.6. Let M be a graded R-module and Y be a subset of X^M. If $\xi(Y)$ is a graded prime submodule of M, then Y is irreducible.

Proof: Assume that $\xi(Y)$ is a graded prime submodule of M. Then, $V_G(\xi(Y)) = Cl(Y)$ is irreducible by Corollary 4.5 and Proposition 4.3. So Y is irreducible.

Corollary 4.7. Let M be a graded R-module. If $Y = \{P_i \mid i \in \Lambda\}$ is a non-empty family of graded prime submodules P_i of M, which is linearly ordered by inclusion, then Y is irreducible in X^M.

Proof: Let $\xi(Y)=\bigcap_{i\in\Lambda}P_i=P$. P is a proper submodule of M. Suppose that $rm\in P$ but $m\notin P$ where $r\in h(R)$ and $m\in h(M)$. Then $m\notin P_i$ for some $i\in\Lambda$. Since P_i is a graded prime submodule, we get $r\in(P_i:M)$. Let j be any element of Λ such that $j\neq i$. Since Y is linearly ordered by inclusion, we have either $P_i\subseteq P_j$ or $P_j\subseteq P_i$. If $P_i\subseteq P_j$, then we obtain $r\in(P_i:M)\subseteq(P_j:M)$. If $P_j\subseteq P_i$, then since $m\notin P_i$ and P_j is a graded prime submodule, we have $r\in(P_j:M)$. Hence $r\in(P:M)$ and $\xi(Y)$ is a graded prime submodule, so Y is irreducible on X^M by Proposition 4.6.

Proposition 4.8. Let M be a graded multiplication R-module. If $Nil(M)$ is graded prime submodule of M, then X^M is irreducible.

Proof: Let U and V be open subsets of X^M and P_U and P_V be elements of U and V, respectively. Then there exist submodules N and K of M such that $U=X^M\setminus V_G(N)$ and $V=X^M\setminus V_G(K)$. So $P_U\notin V_G(N)$ and $P_V\notin V_G(K)$, that is, $N\not\subset P_U$ and $K\not\subset P_V$. Since $Nil(M)\subseteq P_U$, $N\not\subset Nil(M)$. Hence, we get $Nil(M)\in U$. Similarly $Nil(M)\in V$. Consequently, $Nil(M)\in U\cap V\neq\varnothing$ and we obtain X^M, irreducible.

Proposition 4.9. Let M be a graded R-module. Assume that $G-Spec_p(M)\neq\varnothing$ for some $p\in G-Spec(R)$. Then the following hold:

(a) $G-Spec_p(M)$ is irreducible.

(b) If p is a graded maximal ideal of R, then $G-Spec_p(M)$ is an irreducible closed subset of X^M.

Proof: *(a)* Let $G-Spec_p(M)=\{P_i\in G-Spec(M)\,|\,(P_i:M)=p,\ i\in\Lambda\}$. Then $\xi\left(G-Spec_p(M)\right)=\bigcap_{i\in\Lambda}P_i$ is a graded prime submodule. Indeed, we assume $rm\in\bigcap_{i\in\Lambda}P_i$ and $r\notin\left(\bigcap_{i\in\Lambda}P_i:M\right)=\bigcap_{i\in\Lambda}(P_i:M)$, where $r\in h(R)$ and $m\in h(M)$. Notice that $(P_i:M)=p$. Then $r\notin p=(P_i:M)$ for all $i\in\Lambda$. Since $rm\in P_i$ and P_i is graded prime, we get $m\in P_i$ for all $i\in\Lambda$. Hence $m\in\bigcap_{i\in\Lambda}P_i$ and $G-Spec_p(M)$ is irreducible by Proposition 4.6.

(b) To prove this, it suffices to show that $G-Spec_p(M)=V_G(pM)$ for the graded maximal ideal p. Let $N\in V_G(pM)$, that is, $(N:M)\supseteq(pM:M)\supseteq p$. Since p is maximal, $(N:M)=p$. So, $N\in G-Spec_p(M)$. Conversely, let $P\in G-Spec_p(M)$. Then $(P:M)=p\subseteq(pM:M)$ and because of maximality of p, we obtain $p=(pM:M)$ and so $P\in V_G(pM)$.

Proposition 4.10. Let M be a graded R-module and Y be a subset of X^M such that $(\xi(Y):M)=p$ is a graded prime ideal of R. If $G-Spec_p(M)\neq\varnothing$, then Y is irreducible.

Proof: Take $P\in G-Spec_p(M)$. Since $(P:M)=p=(\xi(Y):M)$ we have $V_G(P)=V_G(\xi(Y))=Cl(Y)$ by Lemma 3.11 and Proposition 4.3. Therefore, $Cl(Y)$ is irreducible and so is Y.

Theorem 4.11. Let M be a graded R-module. Then the following statements are equivalent for any $P,\ Q\in X^M$:

(1) X^M is T_0-space.

(2) The natural map φ is injective.

(3) If $V_G(P)=V_G(Q)$, then $P=Q$.

(4) $\left|G-Spec_p(M)\right| \leq 1$ for every $p \in G-Spec(R)$.

Proof: (1)⇔(3) follows from Proposition 4.4 and the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct. The equivalences of (2), (3), and (4) are proved in Proposition 3.14.

Corollary 4.12. Let M be a G-top module, in particular, let M be a graded multiplication module. Then $G-Spec(M)$ is a T_0-space for the Zariski topology.

Proposition 4.13. Let M be a graded R-module and $\delta=\left\{(P:M) \,\middle|\, P \in X^M\right\} \subseteq X^R$. Then $G-Spec(M)$ is a T_1-space if and only if

(1) $(P:M)=p$ is a maximal element of δ for all $P \in X^M$,

(2) $\left|G-Spec_p(M)\right|=1$ for all $p \in G-Spec(R)$.

Proof: If $G-Spec(M)$ is a T_1-space then the singleton sets are closed in X^M. So we obtain (1) and (2) by Proposition 4.4(3). Conversely, (1) and (2) are equivalent so that the singleton set $\{P\}$ is closed in X^M for every $P \in X^M$, that is, X^M is a T_1-space.

Theorem 4.14. Let M be a graded R-module. Then X^M is a T_1-space if and only if every graded prime submodule of M is maximal.

Proof: Assume that X^M is a T_1-space. Let P be any graded prime submodule of M. By Proposition 4.4(1), $Cl(\{P\})=V_G(P)$ and since X^M is a T_1-space, every singleton subset of X^M is closed, that is, $Cl(\{P\})=V_G(P)=\{P\}$. Now, assume that $P \subseteq Q$. It follows that $(P:M) \subseteq (Q:M)$. So $Q \in V_G(P)=\{P\}$ and we obtain $P=Q$. For the converse, suppose that every graded prime submodule of M is maximal. Then for all $P \in X^M$ we have $\{P\}=V_G(P)$, and every singleton subset of X^M is closed. Hence X^M is a T_1-space.

Theorem 4.15. Let M be a graded multiplication R-module. Then X^M is a T_1-space if and only if it is a T_2-space.

Proof: Assume that X^M is a T_2-space. Then it is a T_1-space. Conversely, assume that X^M is a T_1-space. If $\left|X^M\right|=1$ or $\left|X^M\right|=2$, then X^M is a T_2-space. Now assume that $\left|X^M\right|>2$. Then we can take three distinct elements in X^M, say P_1, P_2, and P_3. Since M is graded multiplication, $V_G\left(P_1P_3\right)=\left\{P_1,P_3\right\}=X^M \setminus V_G\left(P_2\right)$, $V_G\left(P_2P_3\right)=\left\{P_2,P_3\right\}=X^M \setminus V_G\left(P_1\right)$ and $V_G\left(P_2\right)=\left\{P_2\right\}=X^M \setminus V_G\left(P_1P_3\right)$ are open sets in X^M. This implies that $P_1 \in V_G\left(P_1P_3\right)$ and $P_2 \in V_G\left(P_2\right)$. Moreover, $V_G\left(P_1P_3\right) \cap V_G\left(P_2\right)=\varnothing$.

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