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An optimal control approach for arbitrary order singularly perturbed boundary value problems

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Abstract

The aim of this paper is to introduce a new approach for obtaining the numerical solution of singulary perturbed boundary value problems based on an optimal control technique. In the proposed method, first the mentioned equations are converted to an optimal control problem. Then, control and state variables are approximated by Chebychev series. Therefore, the optimal control problem is reduced to a parametric optimal control problem (POC) subject to algebric constraints. Finally, the obtained POC is solved numerically using an iterative optimization technique. In this method, a new idea is proposed which enables us to apply the new technique for almost all kinds of singularly perturbed boundary value problems. Some numerical examples are solved to highlight the advantages of the proposed technique.

***Keywords**:* Singularly perturbed boundary value problem; parametric optimal control; optimal control problem

1. Introduction

Various slow-fast systems naturally apear in the modelling of real world-processes. Typical examples involve climate systems, celestial mechanics, enzyme kinetics and etc. These systems have to be formulated by means of singularly perturbed boundary value problems.

Here, we consider the optimal solution of the general form of mth order linear and non-linear singularly perturbed differential equations

$$\varepsilon y^{(m)}(t) = f(t, y(t), y'(t), \ldots, y^{(m-1)}(t)), \quad a \le t \le b, \quad 0 < \varepsilon << 1, \tag{1}$$

subject to the separated boundary conditions

$$\sum_{j=0}^{m-1} c_{ij} y^{(j)}(t_i, \varepsilon) = C_i, \quad 1 \le i \le m, \tag{2}$$

where $a \le t_1 \le t_2 \le \cdots \le t_{m-1} \le t_m \le b.$ For different kinds of boundary conditions (2) we refer to [1]. Up to now, a great deal of effort has been spent on the development of numerical techniques for obtaining a suitable approximate solution of (1) and (2). Some of these techniques use basis functions to represent the solution in analytical forms, while some others produce a solution in the form of an array that contains the value of the solution at a selected group of points. Lately, many computational techniques have been introduced for solving optimal control problems [2-6]. By using the mentioned techniques, new numerical methods have been developed for solving different kinds of ordinary differential equations, partial differential equations, integral and integro-differential equations by converting them into optimal control problems [1, 7-10]. The aim of this paper is to apply an optimal control technique for solving (1) and (2). To do so, first an optimal control problem must be defined using (1) and (2). The approximate solution of (1) and (2) is considered as state function and the boundary conditions (2) are used as control. Now, many computational techniques are available for solving the so-called conjugate problem, mostly using Bellman's dynamic programming [2, 6] and Pontryagin's maximum principle method [6, 11]. Here, by extending the work of El-kady et al. [8] an alternative general algorithm is presented to solve the obtained optimal control problem(conjugate problem) by parameterizing both state and control variables. In fact, our approach is based upon the expansion of both state and control functions in terms of Chebychev series with unknown coefficients. Therefore the conjugate problem is converted into a parametric optimization problem (POC) which consists of the minimization of the performance index subject to equality algebric constraints. The obtained POC can then be replaced by an unconstrained minimization problem by applying

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the method of Lagrange [12, 13] or a penalty function technique [14]. Eventually, unknown system parameters which have to be optimized can be determined within this procedure. In this research, by proposing a new idea, the previous methods are generalized. Furthermore, the mentioned idea enables us to apply the proposed method to arbitrary order singularly perturbed equations of the form (1) with different kinds of boundary conditions. The technique has been tested on problems of all kinds, and shows very promissing results. The remainder of this paper is organised as follows. In Section 2, the mathematical description of the method will be presented. Analysis of convergence of the proposed method will be investigated in Section 3. Later, in Section 4, different kinds of test problems will be solved to illustrate the accuracy and efficiency of the method. Finally, in the last section, the paper is concluded by summarizing the main points of the presented method.

2. Matheatical formulation

In this section, first some definitions of optimal control problems are presented briefly. Then, the above-mentioned procedure is formulated mathematically step by step. Let us consider the following optimal control problem [15] with state conditions. Minimize the continuous-time cost functional

$$J = \phi(x(a), a, x(b), b) + \int_a^b F(x(t), u(t), t)dt, \quad (3)$$

subject to the dynamic constraints

$$x'(t) = f(x(t), u(t), t), \quad (4)$$

the algebric constraints

$$g(x(t), u(t), t) \le 0, \quad (5)$$

and the boundary conditions

$$h(x(a), a, x(b), b) = 0. \quad (6)$$

Here the function $f : R^n \times R^k \times R \to R^n$ describes the system dynamics. $g : R^n \times R^k \times R \to R^s$ and $h : R^n \times R^k \times R \to R^q$ describe the inequality mixed constraints and equality boundary conditions respectively. Also, $F : R^n \times R^k \times R \to R$ and $\phi : R^n \times R \times R^n \times R \to R$ are called cost and Lagrangian functions. We assume that all above-mentioned functions are continuously differentiable with respect to all their arguments. Here, we are looking for $x : [a,b] \to R^n$, the state function which is an absolutely continuous function and $u : [a,b] \to R^k$, the measurable control function, such that constraints (4)-(6) are satisfied and the objective functional (3) takes its minimum value. We call $\{x(.), u(.)\}$ a feasible pair. If this feasible pair minimizes (3) globally then it will be called an optimal pair. For the optimal control theory and analytical background one can see [15-18]. Also, it must be noted that, neccessary conditions of optimality for these kinds of optimal control problems have been the focus of attention since the work of Pontryagin and his associates [11] and their applicabality has been extended by a number of authors [2, 4, 15]. First, we start our method by converting (1) and (2) to a conjugate optimal control problem. Therefore, an appropriate performance index should be defined which has to be relevant to the given equation [11]. Now, (1) and (2) play the role of state and control equations. In fact we have the following control problem

$$\text{Minimize } J := \int_a^b F(y(t), u(t), t)dt, \quad (7)$$

$$\text{subject to } \varepsilon y^{(m)}(t) = f(t, y(t), y'(t), \ldots, y^{(m-1)}(t)), \quad a \le t \le b, \quad 0 < \varepsilon << 1, \quad (8)$$

$$\sum_{j=0}^{m-1} c_{ij} u^{(j)}(a_i, \varepsilon) = C_i, \quad 1 \le i \le m. \quad (9)$$

Obviously, an optimal control problem subject to equality constraints is obtained. It must be noted that, in the above control problem $y(t)$ is the exact solution of (1) and (2) and plays the role of state function. Also, $u(t)$ is the control trajectory. We remark that the main difference of our method with existing similiar methods [8] is the way that we treat the boundary conditions (2). In the presented method, by proposing a new idea, all kinds of boundary conditions can be easily handled. We will explain this matter later. Different options for choosing the performance index (7) are available. To see these options we refer to [1, 7-10]. Here we choose

$$J = \int_a^b |y(t) - u(t)|^2 \, dt, \quad (10)$$

as our performance index. Clearly, the objective function (10) plays the role of least square error which has to be minimized by finding the state function and control trajectory $y(t)$ and $u(t)$ [10].

Now the conjugate optimal control problem is reduced to a parametric optimization probelm(POC) using Chebychev polynomials. To

do so, the Chebychev polynomials $\{T_n(t) = cos(ncos^{-1}(t))\}$ defined on the interval $(-1,1)$ is introduced. Also, by letting $t = \frac{b-a}{2}\tau + \frac{b+a}{2}$, the interval $[a,b]$ in (7)-(9) is transformed to $[-1,1]$. Now, if we let

$$y^{(m)}(\tau) = \varphi(\tau), \tag{11}$$

then we are able to approximate $y^{(m)}(\tau)$ in the obtained optimal control problem by using the Chebychev approximation of $\varphi(\tau)$. It should be noted that m is the maximum degree of derivatives appearing in the given equation and $\varphi(\tau)$ is an unknown function. In fact, we have

$$\varphi(\tau) = \sum_{r=0}^{N}{}'' a_r T_r(\tau)$$

in which

$$a_r = \frac{2}{N}\sum_{j=0}^{N}{}'' \varphi(\tau_j)T_r(\tau_j), \quad r = 0,\ldots,N$$

and

$$\tau_j = cos(\frac{\pi j}{N}), \quad j = 0,\ldots,N.$$

A summation symbol with double primes denotes a sum with first and last terms halved. Simply, by successive integration from (11), $y(\tau), y'(\tau),\ldots,y^{(m-1)}(\tau)$ can be approximated in terms of polynomials. Fortunately, by using the famous Khalifa theorem [6], we can see that the successive integration of the Chebychev polynomial can be expressed in terms of Chebychev polynomials.

Theorem 2.1. ([5]) The successive integration of Chebychev polynomials is expressed in terms of Chebychev polynomials as follows:

$$\int_{-1}^{x}\int_{-1}^{t_{k-1}}\int_{-1}^{t_{k-2}}\cdots\int_{-1}^{t_2}\int_{-1}^{t_1}T_n(t_0)dt_0dt_1\cdots dt_{k-2}dt_{k-1}$$

$$= \sum_{m=0}^{k}\frac{(-1)^m\binom{k}{m}}{2^k\chi_m}\xi_{k,m,n}(x), \quad n > k,$$

where

$$\xi_{k,m,n}(x) = T_{n+k-2m}(x) - \sum_{i=0}^{k-1}\eta_i T_{n+k-2m}^{(i)}(-1)$$

and

$$\eta_i = \sum_{j=0}^{i}\frac{x^j}{(i-j)!\,j!}.$$

By using the above theorem, $y(\tau)$ can be fomulated in terms of Chebychev polynomials as follows

$$y(\tau) = \sum_{i=0}^{N}a_i^{[m]}T_i(\tau), \tag{12}$$

in which $a_i^{[m]}$ shows the coefficients of Chebychev approximation after m successive integration. We can also determine the following approximation for the control function

$$u(\tau) = \sum_{i=0}^{M}b_iT_i(\tau). \tag{13}$$

We also note that $a_i^{[m]}$'s are expressible in terms of unknowns $\varphi(\tau_j)$. Now by substituting the approximations for the state and control functions from (12) and (13) into (7)-(9), the following POC is obtained

$$\text{minimize } J = \int_{-1}^{1}F(\sum_{i=0}^{M}a_i^{[m]}T_i(\tau), \sum_{i=0}^{M}b_iT_i(\tau), \tau), \tag{14}$$

$$\text{subject to } \varepsilon\sum_{i=0}^{M}a_iT_i(\tau) = f(\tau, \sum_{i=0}^{M}a_i^{[m]}T_i(\tau), \sum_{i=0}^{M}a_i^{[m-1]}T_i(\tau),\ldots,\sum_{i=0}^{M}a_i^{[1]}T_i(\tau)) \quad -1 \le \tau \le 1, \tag{15}$$

$$\sum_{j=0}^{m-1}\sum_{k=0}^{M}c_{i,j}b_kT_k^{(j)}(\tau_i,\varepsilon) = C_i, \ 1 \le i \le m. \tag{16}$$

Clearly, in the above POC, the same degree of expansion is used for the state and control. In fact, the choice of M depends on the required accuracy. If we increase the number of terms, the approximation will improve and will tend to the exact solution. However there is a certain limit beyond which increasing M will not result in any improvement. On the contrary, this will cause degradation of performance due to roundoff errors. Also, by defining $\alpha = (\varphi(t_0),\ldots,\varphi(t_M))$ and $\beta = (b_0,\ldots,b_M)$, we may write (14)-(16) in the

following form

$$\text{Minimize } J := J(\alpha,\beta), \tag{17}$$

$$\text{Subject to } G(\alpha,\beta) = 0. \tag{18}$$

The performance index (14) can be also approximated by Chebychev polynomials. As we mentioned before, our performance index is characterized by equation (10). In fact, we consider the expression

$$J(\alpha,\beta) = \int_{-1}^{1} (\sum_{i=0}^{M} a_i^{[m]} T_i(\tau) - \sum_{i=0}^{M} b_i T_i(\tau))^2 dt.$$

Let

$$B_n(\alpha,\beta) = \frac{2}{\pi}\int_{-1}^{1} \frac{(\sum_{i=0}^{M} a_i^{[m]} T_i(\tau) - \sum_{i=0}^{M} b_i T_i(\tau))^2 T_n(\tau)}{\sqrt{1-\tau^2}}$$

represent the Chebychev coefficients of $(\sum_{i=0}^{M} a_i^{[m]} T_i(\tau) - \sum_{i=0}^{M} b_i T_i(\tau))^2$, then according to a well-known theorem for the integration of Chebychev series[6], one has

$$J(\alpha,\beta) = B_0(\alpha,\beta) - \sum_{n=2}^{\infty} [\frac{1+(-1)^n}{n^2-1}] B_n(\alpha,\beta). \tag{19}$$

The computation of Chebychev coefficients $B_n(\alpha,\beta)$ given in (19) is carried out as follows. Putting $\tau = \cos(\theta)$ and using the property $T_n(\cos\theta) = \cos(n\theta)$, the Chebychev coefficients $B_n(\alpha,\beta)$, $n = 0,1,\ldots,M_1$ can be computed by the following approximation formula [6]

$$B_n(\alpha,\beta) = \frac{2}{N}\sum_{i=1}^{N} [\sum_{j=0}^{M} (a_j^{[m]} \cos j\theta_i - b_j \cos j\theta_i)]^2 \cos n\theta_i,$$

$$n = 0,1,\ldots,M_1,\ N > M_1,\ \theta_i = \frac{(2i-1)\pi}{2N}.$$

Obviously, the optimal control problem is now reduced to a parametric optimization problem subject to equality constraints which may be written in the form (17)-(18). In most cases, J is non-linear in α and β. Clearly, equation (10) is a quadratic performance index. If (1) and (2) are linear, then $G(\alpha,\beta)$ will be linear in α and β, otherwise $G(\alpha,\beta)$ will be non-linear.

3. Anlysis of convergence

Many computational techniques can be used to solve the obtained constrained minimization problem (17) and (18), such as Lagrange multipliers, penalty function, etc. The solution proposed by Lagrange is to form an unconstrained problem by appending the constraints to the performance index by means of Lagrange multipliers. To convert (17) and (18) to an uconstrained optimization problem, first we define $G(\alpha,\beta) = (g_0(\alpha,\beta), g_1(\alpha,\beta),\ldots,g_m(\alpha,\beta))$, where m is the number of constraints and depends on the collocation points used in the POC. Now by introducing Lagrange multipliers $\lambda \equiv (\lambda_0,\lambda_1,\ldots,\lambda_m)$, we can define our unconstrained optimization problem as follows

$$L(\alpha,\beta,\lambda) = J(\alpha,\beta) + \sum_{l=0}^{m} \lambda_l g_l(\alpha,\beta). \tag{20}$$

The necessary conditions for stationarity are given by

$$\frac{\partial L(\alpha,\beta,\lambda)}{\partial \varphi_j} = 0, \quad (j = 0,\ldots,M)$$

$$\frac{\partial L(\alpha,\beta,\lambda)}{\partial b_j} = 0, \quad (j = 0,\ldots,M)$$

$$g_l(\alpha,\beta) = 0, \quad (l = 0,\ldots,m).$$

Hence the determining equations for the unknowns are

$$\frac{\partial J(\alpha,\beta)}{\partial \varphi_j} + \sum_{l=0}^{m} \lambda_i \frac{\partial g_i(\alpha,\beta)}{\partial \varphi_j} = 0, \quad (j = 0,\ldots,M), \tag{21}$$

$$\frac{\partial J(\alpha,\beta)}{\partial b_j} + \sum_{l=0}^{m} \lambda_i \frac{\partial g_i(\alpha,\beta)}{\partial b_j} = 0, \quad (j = 0,\ldots,M), \tag{22}$$

$$g_l(\alpha,\beta) = 0, \quad (l = 0,\ldots,m). \tag{23}$$

Sufficient conditions for a local minimum are the stationarity conditions (21)-(23) and the convexity condition expressing the positive (negative) definiteness of a certain quadratic form [8, 18]. Starting values for α and β can be chosen regarding some physical insight in the problem or by applying the proposed method for very low order. Once these initial values are given, starting values for λ can be obtained by selecting any $m+1$ equations from (21) and (22) and solving the resulting linear system for λ. As we mentioned before, in our proposed method $J(\alpha,\beta)$ turns out

to be a quadratic function. Also it should be noted again that, if a linear equation of (1) is transformed to a POC, then clearly $G(\alpha,\beta)$ will be linear in α and β. Thus, (20) can be considered as a quadratic function

$$F(X)=\frac{1}{2}X^TQX-X^Tb+c, \tag{24}$$

where $X\equiv(\alpha,\beta)$ and Q is symmetric and is called Hessian matrix.

Theorem 3.1. (see [19]) If the eigenvalues of the Hessian matrix are all positive, then (24) is a strictly convex function and will have a single strong minimum.

If the above theorem holds for (24), then it is guaranteed that different iteration methods such as steepest descent method, Newton's method and etc, will converge to this strong minimum. This strong minimum can be directly found by setting $\nabla F(X)=0$. If we call this minimum point X^* then it will satisfy $QX^*=b$. In case of dealing with large scale problems, it is almost impossible to calculate the minimum point directly. Thus, we use the steepest descent iteration method to achieve the minimum point. This method is easy to implement. Also, this method is important from a theoretical view point, because it is simple to analyse and many developed techniques are proposed by modifying this method. We have to note that, the convergence properties of (24) are investigated based on steepest descent method. In order to start investigating the convergence properties of the proposed method, first we define the function

$$E(X)=\frac{1}{2}(X-X^*)^TQ(X-X^*). \tag{25}$$

Also, the following relation holds for every X

$$E(X)=F(X)+\frac{1}{2}X^{*T}QX^*-c. \tag{26}$$

Equations (25) and (26) show, that the difference between E and F is a constant value. Obviously, we are able to investigate the convergence properties by minimizing E instead of F. Steepest descent method is defined by the following iteration algorithm

$$X_{k+1}=X_k-\alpha_k g(X_k), \tag{27}$$

in which α_k is a non-negative scalar.

Lemma 3.2. (see [10]) If $F(X)$ is minimized along a line with respect to α_k at each iteration, then steepest decsent formula (27) can be rewritten as

$$X_{k+1}=X_k-\left(\frac{g_k^Tg_k}{g_k^TQ_kg_k}\right)g_k. \tag{28}$$

Lemma 3.3. (see [10]) The iteration formula (28) will result in the following relation:

$$E(X_{k+1})=\{1-\frac{(g_k^Tg_k)^2}{(g_k^TQg_k)(g_k^TQ^{-1}g_k)}\}E(X_k). \tag{29}$$

Now using the Kantrovich inequality, a lower bound for (29) will be achieved.

Lemma 3.4. ([19]) Let Q be $n\times n$ symmetric and positive definite matrix. For all vector X the following inequality holds*:*

$$\frac{(X^TX)^2}{(X^TQX)(X^TQ^{-1}X)}\geq\frac{4aA}{(a+A)^2}, \tag{30}$$

in which a and A are the smallest and largest eigenvalues of Q respectively.

By combining Lemmas 3.3 and 3.4, the following fundamental theorem for the convergency of the steepest descent method will result.

Theorem 3.5. ([19]) For every $x_0\in R^n$ the steepest descent method (27) will converge to x^* which is the unique minimum of L. Furthermore, for $E(x)=\frac{1}{2}(x-x^*)^TQ(x-x^*)$, at stage k we will have:

$$E(x_{k+1})\leq\left(\frac{A-a}{A+a}\right)^2E(x_k).$$

In all numerical examples, the Hessian matrix of the obtained POC is positive definite, which guarantees the convergence. Since the convexity conditions are satisfied in all test problems, this Chebychev approximation offers at least a local minimum. If we deal with nonlinear equations then $L(\alpha,\beta,\lambda)$ will be nonlinear, and can be simply approximated by a quadratic function using the Taylor series expansion and the proposed method becomes applicable for such equations. Here we summarize our proposed method as follows:

Step 1. Convert (1) and (2) to an optimal control

problem which is known as conjugate problem.
Step 2. Find and substitute the Chebychev expansions of $y(t)$ and $u(t)$. Boundary conditions are used as controls and are transformed to new constraints. In this step the conjugate problem will be reduced to an optimization problem.
Step 3. Find the approximate solution of the reduced problem with an arbitrary method. If a tolerance ε is given, then one of the following formulas can be used as the stop condition [5, 6, 10].

$$| J(\alpha_{k+1}, \beta_{k+1}) - J(\alpha_k, \beta_k) | < \varepsilon,$$

$$| X_{K+1} - X_k | < \varepsilon,$$

or

$$| y_M(t) - y_{2M}(t) | < \varepsilon.$$

4. Test problems

In this section different kinds of singularly perturbed equations are solved. In all treated cases, the obtained Hessian matrix (Q) is positive definite. The proposed algorithm is programmed using Maple 13. To measure the accuracy of the obtained numerical solutions, we use the following least square error formula(LSE)

$$\int_a^b (y_M(t) - y(t))^2 dt, \quad t \in [a,b],$$

for different values of M and ε. $y_M(t)$ is the numerical solution of our proposed method and $y(t)$ is the exact solution of (1) and (2). In order to show how this method works and how we deal with boundary conditions, the first test problem is solved in more details.

Test Problem 4.1. [1] Consider the linear boundary value problem

$$\varepsilon^2 y''(t) - y(t) = 0, \quad 0 \le t \le 1$$

$$y(0) = y(1) = 1$$

with the exact solution

$$y(t) = \frac{[(1 - e^{\frac{-1}{\varepsilon}})(e^{\frac{-t}{\varepsilon}} + e^{\frac{-(1-t)}{\varepsilon}})]}{1 - e^{\frac{-2}{\varepsilon}}}.$$

First, the above problem is formulated as an optimal control problem as follows. The aim is to find the control trajectory $u(t)$ that minimizes the functional J for some positive $t \in [0,1]$, defined by

$$\text{Minimize } J = \int_0^1 (y(t) - u(t))^2, \tag{31}$$

$$\varepsilon^2 y''(t) - y(t) = 0, \tag{32}$$

$$u(0) = 1, \tag{33}$$

$$u(1) = 1. \tag{34}$$

Now using Chebychev approximation for $y(t)$ and $u(t)$, the above control problem will be reduced to a POC. Thus, we need to introduce the following transformation

$$\tau = 2t - 1 \Rightarrow t = \frac{1}{2}(\tau + 1). \tag{35}$$

By using (35), the interval $[0,1]$ will be transformed to $[-1,1]$. Also, we let

$$u(t) = \sum_{i=0}^{M} b_i T_i(t), \tag{36}$$

$$y''(t) = \varphi(t). \tag{37}$$

Successive integration from (37) results in $y'(t)$ and $y(t)$ in terms of Chebychev polynomials. We do the procedure step by step to illustrate how our new idea works on (1) with boundary conditions (2). By the first integration of (37) we have

$$y'(t) = \int_0^t \varphi(t) dt + y'(0). \tag{38}$$

Since the value of $y'(0)$ is unknown, we define a new parameter A and we let $y'(0) = A$. This parameter will be added to the optimal control problem as a new unknown. By another integration from (38), $y(t)$ appears to be

$$y(t) = \int_0^t \int_0^t \varphi(t) dt dt + At + y(0). \tag{39}$$

From the boundary conditions of the given test problem one can easily verify that $y(0) = 1$. Now let

$$\varphi(t) = \sum_{r=0}^{N} a_r T_r(t),$$

then by (37) we have

$$y''(t) = \sum_{r=0}^{N} a_r T_r(t).$$

Clearly using (38) yields

$$y'(t) = \sum_{r=0}^{N} a_r \int_0^t T_r(t)dt + y'(0) = \sum_{r=0}^{N+1} C_r T_r(t) + A, \quad (40)$$

in which, by applying theorem 2.1, the coefficients C_r can be simply obtained as follows

$$C_0 = \sum_{\substack{j=0 \\ j\neq 1}}^{N} \frac{(-1)^{j+1} a_j}{j^2 - 1} - \frac{1}{4} a_1,$$

$$C_k = \frac{a_{k-1} - a_{k+1}}{2k}, k = 1,2,\ldots,N-2,$$

$$C_{N-1} = \frac{a_{N-2} - \frac{1}{2} a_N}{2(N-1)},$$

$$C_N = \frac{a_{N-1}}{2N},$$

$$C_N + 1 = \frac{\frac{1}{2} a_N}{2(N+1)}.$$

By inserting the obtained expressions for C_r in (40) and doing certain arrangements, the elements of matrix L defined in the relation

$$[\int_0^t \varphi(t)dt] = L[\varphi],$$

where L is a square matrix of order $N+1$ are obtained. The elements of the column matrix $[\varphi]$ are given by $\varphi_j = \varphi(t_j), j = 0,1,\ldots,N.$ By continuing this procedure, the following system of equations results:

$$y(t_i) = \sum_{j=0}^{N} l_{ij}^{(2)} \varphi_j + A t_i + 1, \quad (41)$$

where

$$l_{ij}^{(r)} = \frac{(t_i - t_j)^{r-1}}{(r-1)!} l_{ij}, r = 2,3,\cdots \text{ and } i, j = 0,1,\ldots,N.$$

Note that l_{ij}'s are the elements of L matrix. By substituting the Chebychev epxpansions of $u(t)$, $y'(t)$ and $y(t)$ from (36),(40) and (41) into (31)-(34), the optimal control problem is then reduced to the following POC

$$\text{minimize } J = \int_0^1 (\sum_{i=0}^{N+2} a_i^{[2]} T_i(t) - \sum_{i=0}^{M} b_i T_i(t))^2 dt, \quad (42)$$

$$\text{subject to } \varepsilon^2 \varphi(t_i) - \sum_{j=0}^{N} l_{ij}^{(2)} \varphi_j + A t_i + 1 = 0, \; i = 1,\ldots,N, \quad (43)$$

$$\sum_{j=0}^{N} l_{Nj} \varphi_j + A = 0, \quad (44)$$

$$\sum_{i=0}^{M} b_i T_i(0) - 1 = 0, \quad (45)$$

$$\sum_{i=0}^{M} b_i T_i(1) - 1 = 0. \quad (46)$$

As is obvious, the boundary conditions of any given differential equation are transformed into new constraints in three different ways. A group of boundary conditions appear during the successive integration process. These boundary conditions can be handled in two different ways. The values of some of these boundaries can be substituted directly from the initial form of the given equation. For instance, in the above procedure $y(0)$ appear in (39) and then its value is substituted from the boundary condition $y(0) = 1$. On the other hand, the value of some other boundary conditions appearing in the integration process is unknown. Therefore, we add these boundaries as new parameters to the control problem. In the above example, $y'(0)$ appears in (38) and is substituted by parameter A. The last group of boundary conditions which do not appear in the successive integration process will be approximated and replaced by their Chebychev expansions. Here, $y(1) = 1$ does not appear in the integration process and it is converted to equation (44) using its Chebychev expansion. This new idea results in the ability of solving different kinds of differential equations such as (1) with arbitrary boundary conditions. Finally, by compeleting the above procedure a new quadratic performance index (42) with linear constraints (43)-(46) is encountered. Also, as mentioned before (42) can be rewritten in

terms of Chebychev polynomials using (19). Now, the obtained problem can be converted to an unconstrained optimization problem and is solved by any optimization technique such as steepest descent method. The LSE achieved in [10] with $\varepsilon = \frac{1}{2^{10}}$ for this test problem is 9.58×10^{-13}. Numerical results of this test problem are listed in Table 1.

Test Problem 4.2. [1] Consider the linear boundary value problem

$$-\varepsilon^2 y''(t) + (1+t(1-t))y(t) = f(t),\ 0\le t\le 1,$$

$$y(0) = y(1) = 0,$$

where

$$f(t) = 1+t(1-t)+(2\varepsilon - t(1-t)^2)e^{\frac{-t}{\varepsilon}} + (2\varepsilon - t^2(1-t))e^{\frac{-(1-t)}{\varepsilon}}$$

with the exact solution

$$y(t) = 1+(t-1)e^{\frac{-t}{\varepsilon}} - te^{\frac{-(1-t)}{\varepsilon}}.$$

The LSE achieved in [1] with $\varepsilon = \frac{1}{2^{10}}$ for this test problem is 1.10×10^{-12}. Numerical results are listed in Table 1.

Test Problem 4.3. [1] Consider the linear boundary value problem

$$-\varepsilon y^{(4)} + y''(t) - y(t) = -t - e^{\frac{-t}{\sqrt{\varepsilon}}},\ 0\le t\le 1$$

$$y(0) = 1,\ y(1) = 1+e^{\frac{-1}{\sqrt{\varepsilon}}},$$

$$y''(0) = \frac{1}{\varepsilon},\ y''(1) = \frac{e^{\frac{-1}{\sqrt{\varepsilon}}}}{\varepsilon},$$

with the exact solution

$$y(t) = t + e^{\frac{-t}{\sqrt{\varepsilon}}}.$$

The LSE achieved in [1] with $\varepsilon = \frac{1}{2^{10}}$ for this test problem is 5.78×10^{-10}. Numerical results are listed in Table 2.

Test Problem 4.4. [1] Consider the linear boundary value problem

$$-\varepsilon y^{(8)}(t) + y^{(6)}(t) - y(t) = -t - e^{\frac{-t}{\sqrt{\varepsilon}}},\ 0\le t\le 1,$$

$$y(0) = 1,\ y(1) = 1+e^{\frac{-1}{\sqrt{\varepsilon}}},$$

$$y''(0) = \frac{1}{\varepsilon},\ y''(1) = \frac{e^{\frac{-1}{\sqrt{\varepsilon}}}}{\varepsilon},$$

$$y^{(4)} = \frac{1}{\varepsilon^2},\ y^{(4)} = \frac{e^{\frac{-1}{\sqrt{\varepsilon}}}}{\varepsilon^2},$$

$$y^{(6)} = \frac{1}{\varepsilon^3},\ y^{(6)} = \frac{e^{\frac{-1}{\sqrt{\varepsilon}}}}{\varepsilon^3},$$

with the exact solution

$$y(t) = t + e^{\frac{-t}{\sqrt{\varepsilon}}}.$$

The LSE achieved in [1] with $\varepsilon = \frac{1}{2^{10}}$ for this test problem is 3.52×10^{-2}. Numerical results are listed in Table 2.

Test Problem 4.5. [1] Consider the nonlinear boundary value problem

$$\varepsilon y''(t) - (y'(t))^2 y(t) = \varepsilon^3 e^{\varepsilon t} - \varepsilon^2 (e^{\varepsilon t})^3,\ 0\le t\le 1,$$

$$y(0) = 1,\ y(1) = e^{\varepsilon},$$

with the exact solution

$$y(t) = e^{\varepsilon t}.$$

The LSE reported in [1] with $\varepsilon = \frac{1}{2^{15}}$ for this test problem is 1.06×10^{-4}. Numerical results are listed in Table 3.

Test Problem 4.6. [1] Consider the nonlinear boundary value problem

$$\varepsilon y'''(t) - (y''(t))^2 y'(t) = \frac{2\varepsilon}{(1+t)^3} - \frac{1}{(1+t)^5},\ 0\le t\le 1,$$

$$y(0) = 0,\ y'(0) = 1,\ y(1) = ln(2),$$

with the exact solution

$$y(t) = ln(1+t).$$

The LSE reported in [1] with $\varepsilon = \frac{1}{2^{15}}$ for this test problem is 7.58×10^{-3}. Numerical results are listed in Table 3.

Table 1. Least square errors for test problems 1 and 2

	TP1.			TP2.		
M	$\varepsilon = \frac{1}{2}$	$\varepsilon = \frac{1}{2^5}$	$\varepsilon = \frac{1}{2^{10}}$	$\varepsilon = \frac{1}{2}$	$\varepsilon = \frac{1}{2^5}$	$\varepsilon = \frac{1}{2^{10}}$
4	9.95e-11	2.32e-4	3.41e-1	5.45e-10	2.66e-4	4.41e-1
8	2.79e-23	6.92e-7	3.81e-4	1.76e-21	8.64e-7	7.31e-4
16	4.81e-48	2.37e-14	5.48e-12	6.83e-46	3.92e-13	8.91e-11
32	1.07e-102	4.95e-36	7.61e-24	1.20e-101	1.47e-31	1.51e-23

Table 2. Least square errors for test problems 3 and 4

	TP3.			TP4.		
M	$\varepsilon = \frac{1}{2}$	$\varepsilon = \frac{1}{2^5}$	$\varepsilon = \frac{1}{2^{10}}$	$\varepsilon = \frac{1}{2}$	$\varepsilon = \frac{1}{2^5}$	$\varepsilon = \frac{1}{2^{10}}$
4	3.61e-6	4.51e-1	1.45e02	5.41e-5	2.13e-1	9.91e02
8	5.81e-12	6.65e-5	8.84e-1	7.94e-11	1.18e-4	6.23e00
16	1.08e-26	4.31e-12	9.85e-7	3.61e-25	3.36e-11	4.71e-6
32	4.41e-42	1.67e-29	2.34e-21	7.31e-40	2.28e-28	1.15e-19

Table 3. Least square errors for nonlinear test problems 5 and 6

	TP5.			TP6.		
M	$\varepsilon = \frac{1}{2}$	$\varepsilon = \frac{1}{2^5}$	$\varepsilon = \frac{1}{2^{10}}$	$\varepsilon = \frac{1}{2}$	$\varepsilon = \frac{1}{2^5}$	$\varepsilon = \frac{1}{2^{10}}$
4	2.41e-17	6.18e-3	3.912e01	1.02e-9	1.31e-10	1.03e-10
8	3.54e-33	9.47e-7	2.27e-2	1.08e-17	2.34e-17	1.37e-17
16	1.16e-49	3.72e-10	3.41e-5	1.61e-24	1.61e-24	6.43e-24
32	7.31e-76	6.37e-13	4.22e-8	4.69e-31	4.96e-31	8.95e-31

5. Conclusion

In this work, a new approach is proposed based on optimal control techniques for solving arbitirary order singularly perturbed differential equations. First, the given differential equation is transformed into a conjugate optimal control problem. Then a Chebychev expansion method is used to solve the obtained optimal control problem. To do so, state and control functions are approximated in terms of Chebychev polynomials with unknown coeffecients. The boundary conditions of the problem are used as controls. After substituting the Chebychev expansions of control and state functions, the conjugate problem is then reduced to a parametric optimization problem. Finally, by using a suitable optimization technique, the optimal solution is achieved with the desired accuracy. In spite of many techniques, the proposed method has no restriction for different kinds of boundary conditions. In fact, by proposing a new idea, any boundary condition can be easily handled and converted to a new constraint in the obtained optimization problem. The other advantage of this method is that, different kinds of linear test problems are converted to quadratic functions whose Hessian matrix is positive definite. This guarantees that after a certain number of iterations, the obtained approximate solution converges to optimal solution. We have to note that, the volume of computations for developing a general computer program, appears to be quite large compared to other methods. However, our solutions for standard examples are much more promising and the errors on the boundaries are insignificant. We also applied this method to nonlinear test problems. The accuracy of results is remarkable. Investigating a convergence theorem for this case can be the subject of future works.

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