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Symplectic Hodge theory, harmonicity, and Thom duality

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Abstract

We study the notion of harmonicity in the sense of symplectic geometry, and investigate the geometric properties of harmonic Thom forms and distributional Thom currents, dual to different types of submanifolds. We show that the harmonic Thom form associated to a symplectic submanifold is nowhere vanishing. We also construct symplectic smoothing operators which preserve the harmonicity of distributional currents and using these operators, construct harmonic Thom forms for co-isotropic submanifolds, which unlike the harmonic forms associated with symplectic submanifolds, are supported in an arbitrary tubular neighborhood of the manifold.

Keywords: Harmonicity; duality; Thom class; Hodge theory; symplectic; distributional currents; smoothing operators; oriented submanifold

1. Introduction

Let $(\mathcal{M}^{2n}, \omega)$ be a compact symplectic manifold, and $\mathcal{X}$a closed oriented submanifold of $\mathcal{M}$, of codimension k. From $\mathcal{X}$, one gets a Thom class, denoted by $\mathcal{T}_\mathcal{X}$, satisfying the property that for every cohomology class $c \in H^{2n-k}(\mathcal{M})$,

$$\imath^*_\mathcal{X} c[\mathcal{X}] = c\mathcal{T}_\mathcal{X}[\mathcal{M}],$$

ι_X being the inclusion map of $\mathcal{X} \hookrightarrow \mathcal{M}$. In this paper we will study the question of whether or not $\mathcal{T}_\mathcal{X}$ admits a “harmonic" representative, $\mathcal{T}_\mathcal{X} \in \Omega^k(\mathcal{M})$, and if so, what kind of geometric properties this “harmonic" representative satisfies. Here, harmonicity is in the sense of symplectic geometry, i.e., having the property that $d\tau_\mathcal{X} = 0$, and $d * \tau_\mathcal{X} = 0$, where $* = *_s$ is the symplectic star operator

$$*\colon \Lambda^k(\mathcal{T}^*_p) \to \Lambda^{2n-k}(\mathcal{T}^*_p)$$

associated with the symplectic quadratic form ω_p [1-3]). We will review the basic definitions and some of the basic results in symplectic Hodge theory in section 2.

Since harmonicity is a much flabbier property in symplectic Hodge theory than in Riemannian Hodge theory, one does not expect a harmonic (in the symplectic sense) representative $\tau_\mathcal{X}$ of the class $\mathcal{T}_\mathcal{X}$ to exhibit any interesting global features, however, we will show that this is not entirely the case. In particular, in section 4, we show that whenever the submanifold $\mathcal{X}$ is symplectic as well, then the support of any harmonic representative $\tau_\mathcal{X}$ is the whole space $\mathcal{M}$. Note that for Riemannian manifolds this type of statement is true for all non-zero harmonic form (in the Riemannian sense), due to the ellipticity of the Hodge Laplacian. (This result is somewhat unexpected, since one can construct, for every neighborhood U of $\mathcal{X}$, a representative $\tau'_\mathcal{X}$ of $\mathcal{T}_\mathcal{X}$ with the support entirely inside U (see [4]).)

This statement is, in fact, a particular case of a much stronger theorem:

Theorem 1. Let $\mathcal{X}$ be a closed oriented submanifold of even codimension 2m in $\mathcal{M}$. If $\int_\mathcal{X} \imath^*_\mathcal{X}\omega^{n-m} \neq 0$, then $\mathcal{X}$ does not vanish anywhere on $\mathcal{M}$. In particular, a harmonic Thom form for a symplectic submanifold does not vanish anywhere on $\mathcal{M}$.

In section 3, co-isotropic submanifolds are studied and a strikingly different result is proved.

Theorem 2. Let $\mathcal{X}$ be a co-isotropic submanifold of $\mathcal{M}$. Then, for every neighborhood U of, there exists a harmonic representative $\tau_\mathcal{X}$ of $\mathcal{T}_\mathcal{X}$ supported entirely inside U.

We recall that for every oriented submanifold $\mathcal{X}$ of $\mathcal{M}$, one has a canonical distributional Thom form $\tau^0_\mathcal{X}$, i.e. a current in deRham's sense which is entirely supported on $\mathcal{X}$. We will deduce theorem 2 from the following sharper result.

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Theorem 3. Let $\mathcal{X}$ be an oriented submanifold in $\mathcal{M}$. The canonocal distributional Thom form (i.e., the current) $\tau^0_{\mathcal{X}}$, is harmonic if and only if $\mathcal{X}$ is co-isotropic.

In section 4 we prove theorem 1, and in section 5, we prove a slightly stronger version of theorem 1, namely, $\omega^m\tau^0_{\mathcal{X}}$ is harmonic for some m if and only if $\mathcal{X}$ is co-isotropic or $\imath^*_{\mathcal{X}}\omega^m = 0$.

2. Symplectic hodge theory and symplectic harmonic forms

Let $(\mathcal{M}^{2n}, \omega)$ be a compact symplectic manifold with a symplectic form $\omega \in \Omega^2(\mathcal{M})$. Associated with this form, the manifold is equipped with a nondegenerate bilinear form $\mathcal{B}_r$ on $\Lambda^r(T^*(\mathcal{M}))$. The bilinear form naturally gives us a symplectic Hodge star operator,*, as well as a symplectic co-boundary operator, δ, whose definitions are as follows;

$$u \wedge * v = \mathcal{B}_r(u, v)\omega^n$$

and

$$\delta u = (-1)^r * d *$$

for $u, v \in \Omega^r$. (See [2] and [1].)

Symplectic Hodge theory and Riemannian Hodge theory are fundamentally different in their nature, due to the fact that the Laplacian operator, the operator $d\delta + \delta d$ which is elliptic in Riemannian geometry, is identically zero in symplectic geometry. In other words, d and δ anti-commute. A form α on $\mathcal{M}$ is called symplectic harmonic or just harmonic, if it is closed and co-closed, i.e. $d\alpha = 0$ and $\delta\alpha = 0$. One of the most remarkable results in this Hodge theory is Mathieu's theorem, which investigates whether or not every cohomology class on $\mathcal{M}$ has a harmonic representative. (see [3] and [5].)

Theorem. (Mathieu) Every cohomology class on $\mathcal{M}$ has a (symplectic) harmonic representative, if and only if $\mathcal{M}$ satisfies the hard Lefshetz property, i.e.,

$$[\omega]^r : H^{n-r}(\mathcal{M}) \to H^{(n+r)}(\mathcal{M})$$

is bijective for all $r \le n$.

In particular, for $\mathcal{M}$ satisfying the hard Lefshetz property the Thom class dual to any arbitrary closed oriented submanifold has a harmonic representative.

Let $\Lambda : \Omega^r(\mathcal{M}) \to \Omega^{r-2}$ be the dual to the operator
$\mathcal{L} : \Omega^k(\mathcal{M}) \to \Omega^{k+2}(\mathcal{M})$

$$\mathcal{L}(\alpha) = \omega \wedge \alpha$$

In the sense that
$\Lambda = - * \mathcal{L} *$.

The following lemma gives a precise description of the action of Λ on $\mathcal{M}$, given a local Darboux coordinate system (see [5]).

Lemma 1. Let $(x_1, y_1, \dots, x_n, y_n)$ be a Darboux coordinate system for an open neighborhood U of $\mathcal{M}$. Then for a form on $\mathcal{M}$,

$$\Lambda\alpha = \sum_i \imath\left(\frac{d}{dx_i}\right)\imath\left(\frac{d}{dy_i}\right)\alpha.$$

In addition, these operators on $\Omega^r(\mathcal{M})$ satisfy the following relations and properties (for proofs see [1], [5]).

- $[d, \Lambda] = -\delta$
- $[\delta, L] = d$
- $[d, L] = 0$
- $[\delta, \Lambda] = 0$
- $*^2 = 1$

3. Smoothing operators and co-isotropic submanifolds

Let $(\mathcal{M}, \omega)$ be a symplectic manifold, and consider a closed submanifold $\mathcal{X}$ of dimension l. One can define the canonical Thom current associated with $\mathcal{X}$, denoted by $\tau^0_{\mathcal{X}}$, as the following linear functional:

$$\tau^0_{\mathcal{X}} : \Omega^l(\mathcal{M}) \to R$$
$$\tau^0_{\mathcal{X}}(\alpha) = \int_{\mathcal{X}} \imath^*\alpha$$

where $\imath : \mathcal{X} \to \mathcal{M}$ is the imbedding map.

Proof of theorem 3. Assume that $\mathcal{X}$ is a co-isotropic submanifold. Consider a local Darboux coordinate system in a neighborhood of $\mathcal{X}$ in $\mathcal{M}$, i.e., the local coordinates $(x_1, y_1, \dots, x_n, y_n)$ for $\mathcal{M}$ in which the symplectic form can be written in the standard form in terms of this system;

$$\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n.$$

Moreover, $\mathcal{X}$ is given locally by the equations $y_{k+1} = .. = y_n = 0$. In this coordinate system, the current $\tau^0_{\mathcal{X}}$ can be written in the following way,

$$\tau^0_{\mathcal{X}} = \delta_0 dy_{k+1} \wedge \dots \wedge dy_n$$

where

$$\delta_0 = \delta_0(y_{k+1}, \dots, y_n)$$

is the distribution on $\mathcal{R}^{n-k}$ given by

$$\delta_0(f) = f(0), \quad f \in \mathcal{C}^\infty(\mathcal{R}^{n-k}).$$

On the other hand, locally we have

$$\delta\tau_{\mathcal{X}}^{0} = -*d*\tau_{\mathcal{X}}^{0} = \pm d(\delta_0\, dx_1 \wedge \ldots \wedge dx_k \wedge dy_1 \wedge \ldots \wedge dy_n) = 0,$$

which together with the closedness of this form shows that the canonical Thom current for any co-isotropic submanifold is harmonic.

For the converse, consider a closed oriented submanifold $\mathcal{X}$ in $\mathcal{M}$, and assume that $\tau_{\mathcal{X}}^{0}$ is harmonic. The rank of the two form $\imath^*\omega$ is locally constant on some open dense subset of, where $\imath$ is the embedding map (see [6]). If $\mathcal{X}$ is not co-isotropic, then there exists a non-empty open subset, namely U, and a local coordinate system on it with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ so that

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

and also $\mathcal{X}$ is locally given by

$$x_r = \cdots = x_n = 0, \quad y_s = \cdots = y_n = 0$$

where $r, s \le n$ (see [6]). Hence, on U,

$$\tau_{\mathcal{X}}^{0} = \delta_0 dx_r \wedge \ldots \wedge dx_n \wedge dy_s \wedge \ldots \wedge dy_n$$

where

$$\delta_0 = \delta_0(x_r, \ldots, x_n, y_s, \ldots, y_n)$$

is the distribution given by

$$\delta_0(f) = f(0).$$

Consequently,

$$d*\tau_{\mathcal{X}}^{0} = \pm d\delta_0 \wedge dx_1 \wedge \ldots \wedge dx_{s-1} \wedge dy_1 \wedge \ldots \wedge dy_{(r-1)} \neq 0,$$

since

$$\frac{d}{dx_n}\delta_0 \neq 0.$$

Hence $\tau_{\mathcal{X}}^{0}$, the canonical Thom current, is not harmonic for a non-coisotropic submanifold.

To deduce theorem 2 from theorem 3, we introduce a symplectic smoothing operator, which applied to a current τ, gives a form cohomologous to the current.

Proof of Theorem 2. Let $\imath : \mathcal{M} \hookrightarrow \mathcal{R}^N$ be an imbedding and let

$$f_i = \imath^* x_i \in C^\infty(M).$$

Let $v_i \in \mathcal{X}(M)$ be the hamiltonian vector field:

$$\imath_{v_i}\omega = df_i, \quad i = 1, \ldots, N.$$

The vector fields $v_1, \ldots, v_n$ span the space $T^*(M)$. Let U be a small neighborhood of $0 \in \mathcal{R}^N$, and consider the family of maps,

$$F : \mathcal{M} \times U \to \mathcal{M}, \quad F(m, s) = f_s(m)$$

defined by

$$f_s(m) = \exp(s_1 v_1 + \cdots + s_N v_N)(m), \quad s = (s_1, \ldots., s_N).$$

The map f_0 is the identity and M is compact, so one can choose a neighborhood U of $0 \in \mathcal{R}^N$, so that the maps

$$g_m : U \to \mathcal{M}, g_m(s) = f_s(m)$$

are submersions (see [7]).

Given a current γ, let

$$\mathcal{S}(\gamma) = \int_{\mathcal{R}^N} f_s^*\gamma\rho(s)ds,$$

where $\rho \in \mathcal{C}^\infty(U)$ is compactly supported and satisfies

$$\int_{\mathcal{R}^N} \rho(s)ds = 1.$$

The operator $\mathcal{S}$ is a smoothing operator since ρ is compactly supported in $\mathcal{R}^n$ (one can extract a proof of this fact from [8].) Moreover, if W is a neighborhood of Suppγ, then by letting U be sufficiently small, one can arrange for the support of the form

$$\gamma \mapsto \int_{\mathcal{R}^N} f_s^*\gamma\rho(s)ds$$

to be contained in W.

If γ is closed (or co-closed), then so is $f_s^*\gamma$. In addition,

$$[f_s^*\gamma] = [\gamma].$$

Consequently, $\int_{\mathcal{R}^N} f_s^*\gamma\rho(s)ds$ is a smooth form cohomologous to γ, and is harmonic if γ is.

Now let Σ be a co-isotropic submanifold, $\Sigma \hookrightarrow \mathcal{M}$, and U an arbitrary neighborhood of Σ. Applying the operator $\mathcal{S}$ to the canonical Thom current associated to Σ, we end up with a smooth harmonic Thom form supported in U, and this completes the proof of the theorem.

4. Non-vanishing Harmonic Thom Forms

In this section, we prove theorem 1. We first observe:

Lemma 2. Harmonicity of forms is preserved by the operators $\mathcal{L}$ and Λ.

Proof of Lemma 2. Let α be a harmonic form on $\mathcal{M}$. Then,

$$d\Lambda\alpha = \Lambda d\alpha + \delta\alpha = 0,$$
$$\delta\Lambda\alpha = \Lambda\delta\alpha = 0,$$

and hence, $\Lambda\alpha$ is harmonic, and so is $\mathcal{L}\alpha$, since

$$d\mathcal{L}\alpha = \mathcal{L}d\alpha = 0,$$
$$\delta\mathcal{L}\alpha = \mathcal{L}\delta + d\alpha = 0.$$

Proof of Theorem 1. Let $\mathcal{X}$ be an oriented submanifold of $\mathcal{M}$, of even codimension 2m, satisfying

$$\int_{\mathcal{X}} i_{\mathcal{X}}^{*}\omega^{n-m} \neq 0.$$

Let $\tau_{\mathcal{X}}$ be a harmonic Thom form associated to this submanifold and consider the following zero form (i.e. smooth function) on $\mathcal{M}$,

$$f = \Lambda^{m}\tau_{\mathcal{X}} .$$

According to lemma 5, this function is harmonic and thus is closed, so $f \equiv c$ for some constant c. Therefore,

$$\begin{aligned} c.\mathrm{Vol}(\mathcal{M}) &= \int_{\mathcal{M}} * f = \int_{\mathcal{M}} * \Lambda^{m}\tau_{\mathcal{X}} \\ &= \int_{\mathcal{M}} * \tau \wedge \omega^{m} = \int_{\mathcal{M}} \tau \wedge \omega^{m} \\ &= \frac{(n-m)!}{m!}\int_{\mathcal{M}} \tau \wedge \omega^{n-m} \\ &= \frac{(n-m)!}{m!}\int_{\mathcal{X}} i_{\mathcal{X}}^{*}\omega^{n-m} \neq 0. \end{aligned}$$

Hence, $c \neq 0$, which means that

$$\Lambda^{m}\tau(p) \neq 0.$$

Consequently, $\tau(p) \neq 0$ for every $p \in \mathcal{M}$.

5. Criteria for Harmonicity of Canonical Thom Currents

In section 3, we observed that the canonical Thom current associated to a submanifold $\mathcal{X}$ of the manifold $\mathcal{M}$ is harmonic, if and only if $\mathcal{X}$ is co-isotropic. In this section we prove a generalized version of this fact.

Theorem 4. Let $\mathcal{X}$ be a closed oriented submanifold of $\mathcal{M}$, and m be a non-negative integer. Then, $\tau_{\mathcal{X}}^{0} \wedge \omega^{m}$ is harmonic if and only if either $\imath_{\mathcal{X}}^{*}\omega^{m}$ is identically zero, or $\mathcal{X}$ is co-isotropic.

Proof of Theorem 4. Set $\tau_{\mathcal{X}}^{m} = \tau_{\mathcal{X}}^{0} \wedge \omega^{m}$. For the only if part, consider a closed oriented submanifold $\mathcal{X}$ in $\mathcal{M}$, and assume that $\tau_{\mathcal{X}}^{m}$ is harmonic. The rank of the (2m)-form $\imath^{*}\omega^{m}$ is locally constant on some open dense subset of, where $\imath$ is the embedding map (see [6]). If neither $\mathcal{X}$ is co-isotropic nor $\imath_{\mathcal{X}}^{*}\omega^{m}$ is identically zero, then as we pointed out above, there exists an open subset, U, with $U \cap \mathcal{X}$, and a local coordinate system on it with coordinates $(x_1, y_1, \dots, x_n, y_n)$ so that

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

and so that $\mathcal{X}$ is locally given by

$$x_r = \cdots = x_n = 0,\ \ y_s = \cdots = y_n = 0$$

where $r, s \leq n$ and $m < \min(r, s)$, (note that if $m \geq \min(r, s)$ then ω^{m} would be identically zero on $\mathcal{X}$). Hence, on U,

$$\tau_{\mathcal{X}}^{0} = \delta_0 dx_r \wedge \dots \wedge dx_n \wedge dy_n \wedge \dots \wedge dy_n$$

where

$$\delta_0 = \delta_0(x_r, \dots, x_n , y_s , \dots, y_n)$$

is the distribution given by

$$\delta_0(f) = f(0),$$

and therefore,

$$\begin{aligned} \tau_{\mathcal{X}}^{m} = \delta_0 dx_r \wedge \dots \wedge dx_n \wedge dy_s \wedge \dots \wedge dy_n \wedge \omega^{m} \\ = \mathcal{L}^{m}(\delta_0 dx_r \wedge \dots \wedge dx_n \wedge dy_s \\ \wedge \dots \wedge dy_n) \end{aligned}$$

Moreover, $\tau_{\mathcal{X}}^{m}$ is closed, and therefore,

$$\delta\tau_{\mathcal{X}}^{m} = (-1)^{m}\mathcal{L}^{m}\delta\tau_{\mathcal{X}}^{0}$$

On the other hand,

$$\begin{aligned} \delta\tau_{\mathcal{X}}^{0} = - * d * \tau_{\mathcal{X}}^{0} \\ = \pm \\ * (d\delta_0 \wedge dx_1 \wedge \dots \wedge dx_{s-1} \wedge dy_1 \\ \wedge \dots \wedge dy_{r-1}), \end{aligned}$$

and therefore,

$$\begin{aligned} \delta\tau_{\mathcal{X}}^{m} = \pm \mathcal{L}^{m} * (d\delta_0 \wedge dx_1 \wedge \dots \wedge dx_{s-1} \wedge dy_1 \wedge \dots \\ \wedge dy_{r-1}) \\ = \pm \\ * \Lambda^{m}(d\delta_0 \wedge dx_1 \wedge \dots \wedge dx_{s-1} \\ \wedge dy_1 \wedge \dots dy_{r-1}) \neq 0 \end{aligned}$$

Since $m < \min(r, s)$ and also

$$\frac{d}{dx_n}\delta_0 \neq 0.$$

Therefore, the canonical Thom current is not harmonic in this case. Conversely, assume that $\imath_{\mathcal{X}}^{*}\omega^{m}$ is identically zero. Then,

$$\tau_{\mathcal{X}}^{m}(\alpha) = \int_{\mathcal{X}} \imath^{*}(\alpha \wedge \omega^{m}) = \int_{\mathcal{X}} \imath^{*}(\alpha) \wedge \imath^{*}(\omega^{m}) = 0,$$

since $\imath^{*}(\omega^{m}) = 0$. Finally, if the submanifold $\mathcal{X}$ is co-isotropic, then by theorem 3, $\tau_{\mathcal{X}}^{0}$ is harmonic, and so is $\tau_{\mathcal{X}}^{m}$, since

$$\tau_{\mathcal{X}}^{m} = \tau_{\mathcal{X}}^{0} \wedge \omega^{m}.$$

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