IJST (2013) 37A3: 237-249
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

A study of fuzzy soft interior ideals of ordered semigroups

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Abstract

In this paper, we present the concepts of a fuzzy soft left (right) ideal and fuzzy soft interior ideal over an ordered semigroup S. Some basic results of fuzzy soft left (right) ideals and fuzzy soft interior ideals are investigated and the supported examples are provided. Different classes, regular, intra-regular, and simple ordered semigroups are characterized by means of fuzzy soft left (right) ideals and fuzzy soft interior ideals. It is shown that an ordered semigroup is simple if and only if it is fuzzy soft simple. Furthermore, left (right) regular and intra-regular ordered semigroups are characterized by means of fuzzy soft left (right) ideals and fuzzy soft ideals.

***Keywords**:* Ordered semigroups; regular (left, right and completely regular) ordered semigroups; fuzzy left (right) ideals; fuzzy interior ideals; soft left (right) ideals; soft interior ideals; fuzzy soft left (right) ideals; fuzzy soft interior ideals

1. Introduction

It is well known that semigroups are basic structures in many applied branches like automata and formal languages, coding theory, finite state machines and others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings (see e.g., monograph [1]). In particular (fuzzy) regular ordered semigroups, being a union of groups etc., play an important role in the mentioned applications. A theory of fuzzy soft sets on ordered semigroups can be developed. Using the concepts of fuzzy soft sets, the notions of a fuzzy soft left (right) ideal and fuzzy soft interior ideal over an ordered semigroup are provided.

The classical method cannot be used to solve complicated problems in economics engineering, and environment, because various uncertainties are typical for those problems. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all these theories have their own difficulties which are pointed out in [2]. Molodtsov [2] and Maji et al. [3] suggested that one reason for these difficulties may be due to the inadequacy of the parameterization tool of the theory. To overcome these difficulties, Molodtsove [2] introduced the concept of soft set as a new mathematical tool for dealing with uncertainities that are free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [4] described the applications of soft set theory to the decision making problems. Maji et al. also studied several operations on the theory of soft sets [5]. Chen et al. [6] presented a new definition of soft set parameterization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. Jun and Park applied the notion of soft set to the theory of *BCK/BCI*-algebra [7]. Ali et al. [8] further studied the soft set theory and investigated some new algebraic operations for soft set theory. Aygunoglu and Aygun [9] discussed the applications of fuzzy soft sets to the group theory and investigated fuzzy soft groups. Shabir and Naz studied soft topological spaces in [10] and investigated the notion of soft open sets, soft closed set, soft closure, soft interior points, soft neighborhood of a point and soft separation axioms. Jun et al. [11] applied fuzzy soft set theory to *BCK/BCI*-algebras and discussed some

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Received: 25 January 2012 / Accepted: 10 June 2012

basic properties using the concept of a fuzzy soft set. For further study on soft sets one can refer to [12-18].

This paper is divided into the following sections, in Section 2, some basics of fuzzy sets (fuzzy ideals and fuzzy interior ideals) are provided. The notions of soft sets and fuzzy soft sets are also revised in this section. In Section 3, the concepts of a fuzzy soft left (right) ideal and fuzzy soft interior ideals with the help of examples, are discussed. Moreover, the basic properties of fuzzy soft left (right) ideals and fuzzy soft interior ideals of ordered semigroups are investigated by using t-level cuts. In Section 4, simple ordered semigroups are studied and it is shown that an ordered semigroup is simple if and only if it is fuzzy soft simple. Some discussion of regular and intra-regular ordered semigroups in terms of fuzzy soft ideals is also provided. In the last Section, by using the concept of a fuzzy soft left (right) ideals and fuzzy soft ideals, the characterizations of left (right) regular and intra-regular ordered semigroups are studied.

2. Some Basic Definitions and Lemmas

By an *ordered semigroup* (or *po-semigroup*) we mean a structure $(S,\cdot,\leq)$ in which the following conditions are satisfied:

(*OS1*) $(S,\cdot)$ is a semigroup,

(*OS2*) $(S,\leq)$ is a poset,

(*OS3*) $a\leq b\to ax\leq bx$ and $a\leq b\to xa\leq xb$ for all $a,b,x\in S$.

For subsets A,B of an ordered semigroup S, we denote by $AB=\{ab\in S\mid a\in A, b\in B\}$. If $A\subseteq S$ we denote $(A]=\{t\in S\mid t\leq h$ for some $h\in A\}$. If $A=\{a\}$, then we write $(a]$ instead of $(\{a\}]$. If $A,B\subseteq S$, then $A\subseteq (A]$, $(A](B]\subseteq (AB]$, and $((A]]=(A]$.

Let $(S,\cdot,\leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $A^2\subseteq A$. A non-empty subset A of S is called *left* (resp. *right*) *ideal* of S if

(i) $(\forall a\in S)(\forall b\in A)\ (a\leq b\to a\in A)$,

(ii) $SA\subseteq A$ (resp. $AS\subseteq A$).

A non-empty subset A of S is called an ideal if it is both a left and a right ideal of S.

We denote by $L(a)$ (resp. $R(a)$ and $I(a)$) the left (resp. right and two-sided)-ideal of S respectively, generated by a $(a\in S)$, and we have $L(a)=(a\cup Sa]$ (resp. $R(a)=(a\cup aS]$ and $I(a)=(a\cup Sa\cup aS\cup SaS])$ (see [19]).

A non-empty subset A of an ordered semigroup S is called an *interior ideal* of S if

(i) $(\forall a\in S)(\forall b\in A)\ (a\leq b\to a\in A)$,

(ii) $A^2\subseteq A$,

(iii) $SAS\subseteq A$.

A non-empty subset T of an ordered semigroup S is called *semiprime* [19], if $a^2\in T$ implies $a\in T$ for every $a\in S$, or equivalently, $A^2\subseteq T$ implies $A\subseteq T$ for every $A\subseteq S$.

An ordered semigroup S is *regular* [20] if for every $a\in S$ there exists $x\in S$ such that $a\leq axa$, or equivalently, we have (i) $a\in (aSa]$ $\forall a\in S$ and (ii) $A\subseteq (ASA]$ $\forall A\subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *regular* [21] if for every $a\in S$ there exists $x\in S$, such that $a\leq xa^2$ (resp. $a\leq a^2x$), or equivalently, (i) $a\in (Sa^2]$(resp. $a\in (a^2S])$ $\forall a\in S$ and (ii) $A\subseteq (SA^2]$(resp. $A\subseteq (A^2S])$ $\forall A\subseteq S$. An ordered semigroup S is called *intra-regular* if for every $a\in S$ there exist $x,y\in S$ such that $a\leq xa^2y$, or equivalently, (i) $a\in (Sa^2S]$ $\forall a\in S$ and (ii) $A\subseteq (SA^2S]$ $\forall A\subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *simple* [21] if for every left (resp. right) ideal A of S we have $A=S$ and S is called *simple* [21] if it is both left and right simple. An ordered semigroup S is called *completely regular*, if it is left regular, right regular and regular.

2.1. Lemma [20]

Let S be an ordered semigroup. Then the following statements are equivalent:

(1) S is left (resp. right) regular.

(2) Every left ideal of S is semiprime.

(3) $L(a)$ is semiprime left ideal of S for every $a\in S$.

(4) $L(a^2)$ is semiprime left ideal of S for every $a\in S$.

2.2. Lemma [20]

Let S be an ordered semigroup. Then the following statements are equivalent:

(1) S is intra-regular.

(2) Every ideal of S is semiprime.

(3) $I(a)$ is semiprime ideal of S for every $a \in S$.

(4) $I(a^2)$ is semiprime left ideal of S for every $a \in S$.

2.3. Lemma [21]

An ordered semigroup $(S,\cdot,\leq)$ is left (right) simple if and only if for every $a \in S$, we have $S = (Sa](S = (aS])$.

From Lemma 2.3, it is clear that if S is left and right simple then S is regular. Indeed, if $a \in S$, then $a \in S = (aS]$, since $S = (Sa]$, we have $a \in S = (aS] = (a(Sa]] = (aSa]$, and $a \in (aSa]$, implies that S is regular.

A function $\lambda : S \to [0,1]$ is called a *fuzzy subset* of S.

If λ and μ are fuzzy subsets of S then $\lambda \leq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets:

$$\lambda \wedge \mu : S \to [0,1] \mid x \mapsto (\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) = \min\{\lambda(x), \mu(x)\}$$
$$\lambda \vee \mu : S \to [0,1] \mid x \mapsto (\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) = \max\{\lambda(x), \mu(x)\},$$

for all $x \in S$.

A fuzzy subset λ of S is called a *fuzzy subsemigroup* if $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$ for all $x, y \in S$. A fuzzy subset λ of S is called a *fuzzy left (resp. right)-ideal* of S if

(i) $x \leq y \to \lambda(x) \geq \lambda(y)$,

(ii) $\lambda(xy) \geq \lambda(y)$(resp. $\lambda(xy) \geq \lambda(x)$ for all $x, y \in S$.

A fuzzy subset λ of S is called a *fuzzy ideal* if it is both a fuzzy left and a fuzzy right ideal of S. It is easy to see that every fuzzy left (right) ideal of S is a fuzzy subsemigroup of S.

A fuzzy subsemigroup λ is called a *fuzzy interior ideal* of S if

(i) $x \leq y \to \lambda(x) \geq \lambda(y)$,

(ii) $\lambda(xay) \geq \lambda(a)$ for all $x, a, y \in S$.

Obviously every fuzzy ideal is a fuzzy interior ideal of S.

A fuzzy subset λ of S is called *semiprime* if $\lambda(a) \geq \lambda(a^2)$ for every $a \in S$.

3. Soft sets

Throughout this paper, U refers to an initial universe, E is a set of parameters, S is an ordered semigroup, unless otherwise stated.

3.1. Definition [2].

Let U be an initial universe and E be the set of parameters. Let $P(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of sets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

3.2. Definition [5].

For two soft sets (F, A) and (G,B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$ and

(2) for all $e \in A$, $F(e)$ and $G(e)$ are identical approximations.

We write $(F, A) \tilde{\subset} (G, B)$.

(F, A) is said to be soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A) \tilde{\supset} (G, B)$.

3.3. Definition [5].

Two soft sets (F, A) and (G, B) over a common universe U are said to soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

3.4. Definition [5].

Let $E = \{e_1, e_2, ..., e_n\}$ be a set of parameters. The NOT set of E denoted by $\rceil E$ is defined $\rceil E = \{\rceil e_1, \rceil e_2, \ldots, \rceil e_n\}$ where $\rceil e_i =$ not e_i for all i.

The following results are obvious:

3.5. Lemma [5].

(1) $\rceil(\rceil A)=A$;

(2) $\rceil(A \cup B)=\rceil A \cup \rceil B$;

(3) $\rceil(A \cap B)=\rceil A \cap \rceil B$.

3.6. Definition [5].

The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \rceil A)$, here $F^c: \rceil A \to P(U)$ is a mapping given by F^c (e)

$=U\setminus F(\rceil e)$ for all $e \in \rceil A$.

We call F^c the soft complement function of F. Clearly, $(F^c)^c$ is the same as F and $((F,A)^c)^c = (F,A)$.

3.7. Definition [5]

A soft set (F,A) over U is said to be a $NULL$ soft set denoted by Φ if for all $e \in A$, $F(e) = \phi$ (null set).

3.8. Definition [5].

A soft set (F,A) over U is said to be an absolute soft set, denoted by $\widetilde{A}$ if for all $e \in A$, $F(e) = U$. Clearly, $\widetilde{A}^c = \Phi$ and $\Phi^c = \widetilde{A}$.

3.9. Definition [5].

If (F,A) and (G,B) are two soft sets, then (F,A) *AND* (G,B) denoted by $(F,A) \wedge (G,B)$ is defined by $(F,A) \wedge (G,B)$ where $H((\alpha,\beta)) = F(\alpha) \cap G(\beta)$ for all $(\alpha,\beta) \in A \times B$.

3.10. Definition [5].

If (F,A) and (G,B) are two soft sets, then (F,A) *OR* (G,B) denoted by $(F,A) \vee (G,B)$ is defined by (G,B) where $O((\alpha,\beta)) = F(\alpha) \cup G(\beta)$ for all $(\alpha,\beta) \in A \times B$.

3.11. Definition [5].

The union of two soft sets of (F,A) and (G,B) over the common universe U is the soft set (H,C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), \text{ if } e \in A - B, \\ G(e), \text{ if } e \in B - A, \\ F(e) \cup G(e) \text{ if } e \in A \cap B. \end{cases}$$

We write $(F,A) \cup (G,B) = (H,C)$.

3.12. Definition [22].

A soft set (F,A) over an ordered semigroup S is called a *soft ordered semigroup* over S if and only if $F(e)$ is a subsemigroup of S for all $e \in A$.

3.13. Definition [22].

A soft set (F,A) over S is called a *soft left (right) ideal* over S if and only if $F(e)$ is a left (right) ideal of S, for all $e \in A$.

A soft set (F,A) over S is called a *soft ideal* over S if and only if $F(e)$ is both a left and a right ideal of S, for all $e \in A$.

3.14. Definition [22].

A soft set (F,A) over S is called a *soft interior ideal* over S if and only if $F(e)$ is an interior ideal of S, for all $e \in A$.

3.15. Definition

A soft set (F,A) over S is called *semiprime soft ideal* over S if and only if $F(e)$ is semiprime ideal of S for all $e \in A$. That is, for each $e \in A$, we have $a^2 \in F(e)$ implies $a \in F(e)$ for every $a \in S$.

4. Fuzzy soft interior ideals

In this section, we define the fuzzy soft left (right) and fuzzy soft interior ideals over an ordered semigroup S. We provide the basic results of fuzzy soft left (right) and fuzzy soft interior ideals and support them by examples.

A pair (λ,Σ) is called a *fuzzy soft set* over S [3], where $\lambda : \Sigma \to \widetilde{P}(S)$ is a mapping, and $\widetilde{P}(S)$ is the set of all fuzzy sets of S.

Let (λ,Σ) be a fuzzy soft set over S. For every $t \in [0,1]$, the set $(\lambda,\Sigma)^t = (\lambda^t,\Sigma)$ is called a t-*level set* [9] of (λ,Σ) where $\lambda^t(\varepsilon) = \{x \in S \mid \lambda(\varepsilon)(x) \geq t\}$ for every $\varepsilon \in \Sigma$. The set $(\lambda,\Sigma)^t$ is obviously a soft set over S.

4.1. Definition [3]

Let (λ,Σ) and (μ,Ω) be two fuzzy soft sets over S, (λ,Σ) is called a *fuzzy soft subset* of (μ,Ω), denoted by $(\lambda,\Sigma) \mathrel{\widetilde{\subseteq}} (\mu,\Omega)$,

if (i) $\Sigma \subseteq \Omega$ and (ii) $\lambda(\varepsilon) \leq \mu(\varepsilon)$ for all $\varepsilon \in \Sigma$.

4.2. Definition [3]

Let (λ,Σ) and (μ,Ω) be two fuzzy soft sets over S with $\Sigma \cap \Omega \neq \phi$. The *intersection* of (λ,Σ) and (μ,Ω), denoted by $(\lambda,\Sigma) \mathbin{\widetilde{\wedge}} (\mu,\Omega) = (\Theta,\Xi)$, is a fuzzy soft (Θ,Ξ) over S, where $\Xi=\Sigma\cap\Omega$ and $\Theta(\varepsilon)=\lambda(\varepsilon)\wedge\mu(\varepsilon)$ for every $\varepsilon\in\Xi$.

4.3. Definition [3]
Let (λ,Σ) and (μ,Ω) be two fuzzy soft sets over S. The *union* of (λ,Σ) and (μ,Ω) over S, denoted by $(\lambda,\Sigma)\tilde{\vee}(\mu,\Omega)=(\Theta,\Xi)$, is a fuzzy soft set (Θ,Ξ) over S, where $\Xi=\Sigma\cup\Omega$ and for every $\varepsilon\in\Xi$,

$$\Theta(\varepsilon)=\begin{cases}\lambda(\varepsilon), & \text{if } \varepsilon\in\Sigma-\Omega,\\ \mu(\varepsilon), & \text{if } \varepsilon\in\Omega-\Sigma,\\ \lambda(\varepsilon)\vee\mu(\varepsilon), & \text{if } \varepsilon\in\Sigma\cap\Omega.\end{cases}$$

4.4. Definition
A fuzzy soft set (λ,Σ) over S is called a *fuzzy soft ordered semigroup* if and only if $\lambda(\varepsilon)$ is a fuzzy subsemigroup of S, for every $\varepsilon\in\Sigma$.

4.5. Definition
A fuzzy soft set (λ,Σ) over S is called a *fuzzy soft left* (*right*) *ideal* over S if and only if $\lambda(\varepsilon)$ is a fuzzy left (right) ideal of S, for every $\varepsilon\in\Sigma$.

A fuzzy soft set (λ,Σ) over S is called a *fuzzy soft ideal* over S if and only if $\lambda(\varepsilon)$ is a fuzzy left and a fuzzy right ideal of S, for every $\varepsilon\in\Sigma$.

4.6. Definition
If (F,A) is a soft set over S, then for each $e\in A$, the set $(\chi_{F(e)},\Sigma)$ is a fuzzy soft set over S, where $\chi_{F(e)}$ is the characteristic function of F and for each $\varepsilon\in\Sigma$, we define $\chi_{F(e)}$ as follows:

$$\chi_{F(e)}(\varepsilon)(a)=\begin{cases}1, \text{ if } a\in F(e),\\ 0, \text{ if } a\notin F(e).\end{cases}$$

4.7. Lemma
A soft set (F,A) over S is a soft left (right) ideal over S if and only if $(\chi_{F(e)},\Sigma)$ is a fuzzy soft left (right) ideal over S.

Proof: We only prove the case for soft left ideals. The proof of soft right ideals is similar.

Suppose that (F,A) is a soft left ideal over S. Let $a,b\in S$ and $a\le b$. If $\chi_{F(e)}(\varepsilon)(b)=0$, for each $\varepsilon\in\Sigma$ then $b\notin F(e)$. Since $\chi_{F(e)}(\varepsilon)(a)\ge 0$, for each $a\in S$ and $\varepsilon\in\Sigma$, we have $\chi_{F(e)}(\varepsilon)(a)\ge\chi_{F(e)}(\varepsilon)(b)$. If $\chi_{F(e)}(\varepsilon)(b)=1$, then $b\in F(e)$. Since $F(e)$ is a left ideal of S for each $e\in A$ and $a\le b\in F(e)$, we get $a\in F(e)$. Thus, $\chi_{F(e)}(\varepsilon)(a)=1\ge\chi_{F(e)}(\varepsilon)(b)$.

Let $a,b\in S$ be such that $\chi_{F(e)}(\varepsilon)(b)=1$ for each $\varepsilon\in\Sigma$, then $b\in F(e)$. Since $F(e)$ is a left ideal of S, we get $ab\in F(e)$. Then $\chi_{F(e)}(\varepsilon)(ab)=1\ge\chi_{F(e)}(\varepsilon)(b)$. If $\chi_{F(e)}(\varepsilon)(b)=0$, then $\chi_{F(e)}(\varepsilon)(ab)\ge 0$ for all $a,b\in S$ and $\varepsilon\in\Sigma$. Thus, $\chi_{F(e)}(\varepsilon)(ab)\ge\chi_{F(e)}(\varepsilon)(b)$. Therefore $\chi_{F(e)}(\varepsilon)$ is a fuzzy left ideal of S and hence $(\chi_{F(e)},\Sigma)$ is a fuzzy soft left ideal over S.

Conversely, assume that $(\chi_{F(e)},\Sigma)$ is a fuzzy soft left ideal over S. Let $a,b\in S$, $a\le b$. If $b\in F(e)$ for each $e\in A$. Then $\chi_{F(e)}(\varepsilon)(b)=1$. Since $\chi_{F(e)}(\varepsilon)$ is a fuzzy left ideal of S and $a\le b$, we have $\chi_{F(e)}(\varepsilon)(a)\ge\chi_{F(e)}(\varepsilon)(b)$. So $\chi_{F(e)}(\varepsilon)(a)=1$, and $a\in F(e)$.

Let $a,b\in S$ such that $b\in F(e)$, then $\chi_{F(e)}(\varepsilon)(b)=1$. By hypothesis, $\chi_{F(e)}(\varepsilon)(ab)\ge\chi_{F(e)}(\varepsilon)(b)$, then $\chi_{F(e)}(\varepsilon)(ab)=1$, and so $ab\in F(e)$. Thus, $F(e)$ is a left ideal of S for each $e\in A$. Therefore (F,A) is a soft left ideal over S. This completes the proof.
Similarly, we can prove the following result:

4.8. Lemma
A soft set (F,A) over S is a soft interior ideal over S if and only if $(\chi_{F(e)},\Sigma)$ is a fuzzy soft interior ideal over S.

4.9. Theorem [23]
Let (λ,Σ) be a soft set over S, (λ,Σ) is a fuzzy soft semigroup if and only if $(\lambda,\Sigma)^t$ is a soft semigroup over S for every $t\in[0,1]$.
Similarly we have the following Theorem:

4.10. Theorem

A fuzzy soft set (λ, Σ) over S is a fuzzy soft ordered semigroup if and only if for every $t \in [0,1]$, the soft set $(\lambda, \Sigma)^t (\neq \phi)$ is a soft ordered semigroup over S.

Proof: The proof follows from Theorem 4.9.

4.11. Theorem

A fuzzy soft set (λ, Σ) over S is a fuzzy soft left (right) ideal over S if and only if for every $t \in [0,1]$, the soft set $(\lambda, \Sigma)^t (\neq \phi)$ is a soft left (right) ideal over S.

Proof: We discuss only the case of a fuzzy soft left ideal. The case for a fuzzy soft right ideal is similar.

Suppose that (λ, Σ) is a fuzzy soft left ideal over S. Let $x, y \in S$, $x \le y$ be such that $y \in \lambda^t(\varepsilon)$, for all $\varepsilon \in \Sigma$ and $t \in [0,1]$, then $\lambda(\varepsilon)(y) \ge t$. Since (λ, Σ) is a fuzzy soft left ideal over S. Definition 4.5, $\lambda(\varepsilon)$ is a fuzzy left ideal of S, and we have $\lambda(\varepsilon)(x) \ge \lambda(\varepsilon)(y) \ge t$. It follows that $\lambda(\varepsilon)(x) \ge t$, and $x \in \lambda^t(\varepsilon)$.

Let $x, y \in S$ be such that $y \in \lambda^t(\varepsilon)$ for every $\varepsilon \in \Sigma$ and $t \in [0,1]$, then $\lambda(\varepsilon)(y) \ge t$. Using Definition 4.5, we have $\lambda(\varepsilon)(xy) \ge \lambda(\varepsilon)(y) \ge t$, and so $xy \in \lambda^t(\varepsilon)$. It follows that $S\lambda^t(\varepsilon) \subseteq \lambda^t(\varepsilon)$, and hence $(\lambda, \Sigma)^t$ is a soft left ideal over S.

Conversely, assume that for all $\varepsilon \in \Sigma$ and $t \in [0,1]$ $(\lambda, \Sigma)^t (\neq \phi)$ is a soft left ideal over S. Let $x, y \in S$, and $x \le y$. If $\lambda(\varepsilon)(y) = 0$, then $\lambda(\varepsilon)(x) \ge 0$ for all $x \in S$ and $\varepsilon \in \Sigma$. Hence $\lambda(\varepsilon)(x) \ge \lambda(\varepsilon)(y)$. If $\lambda(\varepsilon)(y) = t$, then $y \in \lambda^t(\varepsilon)$. Since $(\lambda, \Sigma)^t$ is a soft left ideal over S and $x \le y \in \lambda^t(\varepsilon)$, so by Definition 3.13, we have $x \in \lambda^t(\varepsilon)$. Then $\lambda(\varepsilon)(x) \ge t = \lambda(\varepsilon)(y)$.

Let $x, y \in S$. If $\lambda(\varepsilon)(y) = 0$, then $\lambda(\varepsilon)(xy) \ge 0$ for all $x, y \in S$ and $\varepsilon \in \Sigma$. Hence $\lambda(\varepsilon)(xy) \ge \lambda(\varepsilon)(y)$. If $\lambda(\varepsilon)(y) = t$, then $y \in \lambda^t(\varepsilon)$. By Definition 3.13, $xy \in \lambda^t(\varepsilon)$ and hence $\lambda(\varepsilon)(xy) \ge t = \lambda(\varepsilon)(y)$. Thus $\lambda(\varepsilon)$ is a fuzzy left ideal of S for all $\varepsilon \in \Sigma$. Consequently, (λ, Σ) is a fuzzy soft left ideal over S.

4.12. Example

Let $S = \{a, b, c, d, e\}$ be a semigroup with the following multiplication table

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Define the order relation "$\le$" as: $a \le c \le e$, $a \le d$ and $a \le e$. Then $(S, \cdot, \le)$ is an ordered semigroup. Let $E = \{e_1, e_2, e_3\}$ and define a fuzzy soft set (λ, Σ) as follows:

$$\lambda(e_1)(x) = \begin{cases} 0.8, \text{ if } x = a, \\ 0.7 \text{ if } x = c, \\ 0.6, \text{ if } x = e, \\ 0.5, \text{ if } x = d, \\ 0.3, \text{ if } x = b. \end{cases}$$

Then it is easy to verify that $\lambda(e_1)$ is a fuzzy ideal of S. Thus, (λ, Σ) is a fuzzy soft ideal over S.

4.13. Definition

A fuzzy soft set (λ, Σ) over S is called a *fuzzy soft interior ideal* over S if and only if $\lambda(\varepsilon)$ is a fuzzy interior ideal of S, for every $\varepsilon \in \Sigma$.

4.14. Theorem

A fuzzy soft set (λ, Σ) over S is a fuzzy soft interior ideal over S if and only if for every $t \in [0,1]$, the set $(\lambda, \Sigma)^t (\neq \phi)$ is a soft interior ideal over S.

Proof: The proof follows from Theorem 4.11.

4.15. Example

Consider the ordered semigroup $S = \{a, b, c, d\}$

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b)\}$.

Let $E = \{e_1, e_2, e_3\}$ and define a fuzzy subset $\lambda(e_1)$ of S as follows:

$$\lambda(e_1)(x) = \begin{cases} 0.7, & \text{if } x = a, \\ 0.6, & \text{if } x = d, \\ 0.3, & \text{if } x = c, \\ 0.2, & \text{if } x = b. \end{cases}$$

Then it is easy to verify that $\lambda(e_1)$ is a fuzzy interior ideal of S. Thus, (λ, Σ) is a fuzzy soft interior ideal over S.

4.16. Lemma

Every fuzzy soft ideal over S is a fuzzy soft interior ideal over S.

Proof: The proof is straightforward.

4.17. Remark

The converse of Lemma 4.16, is not true in general, as shown in the following example.

4.18. Example

Consider the ordered semigroup $S = \{0, a, b, c\}$ with the multiplication table and order relation "$\leq$" as below:

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

$\leq = \{(0,0),(a,a),(b,b),(c,c),(0,a)\}$.

Let $E = \{e_1, e_2, e_3\}$ and choose the fuzzy soft set (λ, Σ) over S as follows:

$$\lambda(e_1)(x) = \begin{cases} 0.7, & \text{if } x = 0, \\ 0.4, & \text{if } x = a, \\ 0.6, & \text{if } x = b, \\ 0, & \text{if } x = c. \end{cases}$$

Then $\lambda(e_1)(xay) = \lambda(e_1)(0) = 0.7 \geq \lambda(e_1)(a)$ for all $x, a, y \in S$. Also, if $xy = 0$, then $\lambda(e_1)(xy) = \lambda(e_1)(0) = 0.7 > \lambda(e_1)(x) \wedge \lambda(e_1)(y)$, if $xy = a$, then $\lambda(e_1)(xy) = \lambda(e_1)(a) = 0.4 > 0 = \lambda(e_1)(x) \wedge \lambda(e_1)(y)$, and if $xy = b$, then $\lambda(e_1)(xy) = \lambda(e_1)(b) = 0.6 > 0 = \lambda(e_1)(x) \wedge \lambda(e_1)(y)$. Since $0 \leq a$, we have $\lambda(e_1)(0) > \lambda(e_1)(a)$. Thus $\lambda(e_1)$ is a fuzzy interior ideal of S and hence (λ, Σ) is a fuzzy soft interior ideal over S. But $\lambda(e_1)(b.c) = \lambda(e_1)(a) = 0.4 < 0.6 = \lambda(e_1)(b)$.

Hence $\lambda(e_1)$ is not a fuzzy right ideal. Thus, it is not a fuzzy ideal of S. Consequently, (λ, Σ) is not a fuzzy soft ideal over S.

4.19. Proposition

If S is a regular ordered semigroup, then every fuzzy soft interior ideal over S is a fuzzy soft ideal over S.

Proof: Let (λ, Σ) be a fuzzy soft interior ideal over S and let $a, b \in S$. Then there exists $x \in S$ such that $a \leq axa$. Using Definition 4.13, for all $\varepsilon \in \Sigma$, we have

$$\lambda(\varepsilon)(ab) \geq \lambda(\varepsilon)((axa)b) = \lambda(\varepsilon)((ax)ab) \geq \lambda(\varepsilon)(a).$$

In a similar way, we can prove that $\lambda(\varepsilon)(ab) \geq \lambda(\varepsilon)(b)$ for all $a, b \in S$ and $\varepsilon \in \Sigma$. Hence $\lambda(\varepsilon)$ is a fuzzy ideal of S. Thus (λ, Σ) is a fuzzy soft ideal over S.

From Lemma 4.16 and Proposition 4.19, we have the following corollary:

4.20. Corollary

In regular ordered semigroups the concepts of a fuzzy soft ideal and a fuzzy soft interior ideal over S coincide.

4.21. Theorem

Let $(S, \cdot, \leq)$ be an ordered semigroup and

$(\lambda,\Sigma),(\mu,\Omega)$ be two fuzzy soft ordered semigroups over S. If $\Sigma\cap\Omega\neq\phi$, then $(\lambda,\Sigma)\,\tilde{\wedge}\,(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft ordered semigroup over S

Proof: Using Definition 4.2, for each $\varepsilon\in\Xi=\Sigma\cap\Omega$ we have $\Theta(\varepsilon)=\lambda(\varepsilon)\wedge\mu(\varepsilon)$, since $\lambda(\varepsilon)$ and $\mu(\varepsilon)$ are fuzzy subsemigroups of S and the intersection of two fuzzy subsemigroups is again a fuzzy subsemigroup of S. Thus, $\Theta(\varepsilon)$ is a fuzzy subsemigroup of S. Consequently, $(\lambda,\Sigma)\,\tilde{\wedge}\,(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft ordered semigroup over S.

The following example shows that the union of two fuzzy soft ordered semigroups need not be a fuzzy soft ordered semigroup over S.

4.22. Example
Let $S=\{a,b,c,d,e\}$ be a semigroup with the following multiplication table

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

and define the order relation "$\leq$" as: $a\leq c\leq e$, $a\leq d$, $b\leq d\leq e, b\leq e$ and $a\leq e$. Then $(S,\cdot,\leq)$ is an ordered semigroup. Let $E=\{e_1,e_2,e_3\}$ and choose two fuzzy soft sets (λ,Σ) and (μ,Ω) over S, respectively as follows:

$$\lambda(e_1)(x)=\begin{cases}0.40, \text{ if } x=a,c,d\\ 0.35, \text{ if } x=b,\\ 0.30, \text{ if } x=e.\end{cases}\quad \text{and}$$

$$\mu(e_1)(x)=\begin{cases}0.50, \text{ if } x=a,\\ 0.30, \text{ if } x=b,c,d,e.\end{cases}$$

Then it is easy to verify that both $\lambda(e_1)$ and $\mu(e_1)$ are fuzzy subsemigroups of S, respectively. Thus, both (λ,Σ) and (μ,Ω) are fuzzy soft ordered semigroups over S.

Let $(\lambda,\Sigma)\,\tilde{\vee}\,(\mu,\Omega)=(\Theta,\Xi)$, then by Definition 4.3, $\Xi=\Sigma\cup\Omega=\{e_1\}$ and

$$\Theta(e_1)(x)=(\lambda\vee\mu)(e_1)(x)=\begin{cases}0.50, \text{ if } x=a,\\ 0.35, \text{ if } x=b,\\ 0.30, \text{ if } x=c,d,e.\end{cases}$$

Since $\Theta(e_1)(a.b)=\Theta(e_1)(d)=0.30<0.35=\Theta(e_1)(a)\wedge\Theta(e_1)(b).$ It follows that $\Theta(e_1)$ is not a fuzzy subsemigroup of S. Thus the union of two fuzzy soft ordered semigroups is not a fuzzy soft ordered semigroup.

4.23. Theorem
Let (λ,Σ) and (μ,Ω) be two fuzzy soft ordered semigroups over S. If $\Sigma\cap\Omega=\phi$, then $(\lambda,\Sigma)\,\tilde{\vee}\,(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft ordered semigroup over S.

Proof: Since $\Sigma\cap\Omega=\phi$, by Definition 4.3, for each $\varepsilon\in\Xi$ we have $\Theta(\varepsilon)=\lambda(\varepsilon)$ or $\mu(\varepsilon)$. Now both $\lambda(\varepsilon)$ and $\mu(\varepsilon)$ are respectively, fuzzy subsemigroup, hence $\Theta(\varepsilon)$ is a fuzzy subsemigroup of S. Therefore $(\lambda,\Sigma)\,\tilde{\vee}\,(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft ordered semigroup over S.

4.24. Theorem
Let $(S,\cdot,\leq)$ be an ordered semigroup and $(\lambda,\Sigma),(\mu,\Omega)$ be two fuzzy soft left (right) ideals over S. If $\Sigma\cap\Omega=\phi$, then $(\lambda,\Sigma)\,\tilde{\wedge}\,(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft fuzzy left (right) ideal over S.

Proof: Suppose that both (λ,Σ) and (μ,Ω) are two fuzzy soft left ideals over S. Using Definition 4.2, for each $\varepsilon\in\Xi=\Sigma\cap\Omega$ we have $\Theta(\varepsilon)=\lambda(\varepsilon)\wedge\mu(\varepsilon)$, since $\lambda(\varepsilon)$ and $\mu(\varepsilon)$ are fuzzy left ideals of S and the intersection of two fuzzy left ideals is again a fuzzy left ideal of S. Thus, $\Theta(\varepsilon)$ is a fuzzy left ideal of S. Consequently, $(\lambda,\Sigma)\,\tilde{\wedge}\,(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft left ideal over S.

Similarly, one can prove the result for fuzzy soft right ideals over S.

4.25. Theorem
Let $(S,\cdot,\leq)$ be an ordered semigroup and $(\lambda,\Sigma),(\mu,\Omega)$ be two fuzzy soft left (right) ideals

over S. Then $(\lambda,\Sigma)\tilde{\vee}(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft fuzzy left (right) ideal over S.

Proof: Suppose that both (λ,Σ) and (μ,Ω) are two fuzzy soft left ideals over S. By Definition 4.3, for each $\varepsilon\in\Xi=\Sigma\cup\Omega,$

if $\varepsilon\in\Sigma-\Omega$ or $\Omega-\Sigma,$ then $\Theta(\varepsilon)=\lambda(\varepsilon)$ or $\mu(\varepsilon)$. Since $\lambda(\varepsilon)$ and $\mu(\varepsilon)$ both are fuzzy left ideals of S, $\Theta(\varepsilon)$ is a fuzzy left ideal of S.

If $\varepsilon\in\Sigma\cap\Omega,$ then $\Theta(\varepsilon)=\lambda(\varepsilon)\vee\mu(\varepsilon).$ Again, since $\lambda(\varepsilon)$ and $\mu(\varepsilon)$ both are fuzzy left ideals of S, $\Theta(\varepsilon)$ is a fuzzy left ideal of S. Thus, $(\lambda,\Sigma)\tilde{\vee}(\mu,\Omega)=(\Theta,\Xi)$ is a fuzzy soft left ideal over S. This completes the proof.

In a similar way one can prove the result for fuzzy soft right ideals.

5. Fuzzy soft simple ordered semigroups

If $(S,\cdot,\le)$ is an ordered semigroup and (λ,Σ) a fuzzy soft set over S. Then for every $a\in S$ and $\varepsilon\in\Sigma,$ we denote by (I_a,A) the soft set over S, defined as follows:

$(\forall e\in A)(I_a(e):=\{b\in S\mid\lambda(\varepsilon)(b)\ge\lambda(\varepsilon)(a)\}).$

5.1. Proposition

Let $(S,\cdot,\le)$ be an ordered semigroup and (λ,Σ) a fuzzy soft right ideal over S. Then (I_a,A) is a soft right ideal over S for every $a\in S$.

Proof: Let (λ,Σ) be a fuzzy soft right ideal over $S,$ and let $a\in S$. Then for every $e\in A$, $I_a(e)\ne\phi,$ because for every $a\in S$ and $\varepsilon\in\Sigma$ we have $\lambda(\varepsilon)(a)\ge\lambda(\varepsilon)(a),$ so $a\in I_a(e)$ and $I_a(e)\ne\varphi.$ Let $b\in I_a(e)$ and $S\ni t\le b$. Then $t\in I_a(e)$ for every $e\in A$. Indeed, since (λ,Σ) is a fuzzy soft right ideal over S and $t\le b,$ for every $\varepsilon\in\Sigma$ we have $\lambda(\varepsilon)(t)\ge\lambda(\varepsilon)(b).$ Since $b\in I_a(e),$ $\lambda(\varepsilon)(b)\ge\lambda(\varepsilon)(a)$ and so $\lambda(\varepsilon)(t)\ge\lambda(\varepsilon)(a),$ hence $t\in I_a(e).$ Let $b\in I_a(e)$ for every $e\in A$ and $t\in S$. We prove that $bt\in I_a(e).$ Since (λ,Σ) is a fuzzy soft right ideal over $S,$ for every $\varepsilon\in\Sigma$ we have $\lambda(\varepsilon)(bt)\ge\lambda(\varepsilon)(b).$ Since $b\in I_a(e),$ we have $\lambda(\varepsilon)(b)\ge\lambda(\varepsilon)(a)$ and so $\lambda(\varepsilon)(bt)\ge\lambda(\varepsilon)(a),$ hence $bt\in I_a(e).$ Thus, $I_a(e)S\mathrel{\tilde{\subseteq}}I_a(e)$ and $I_a(e)$ is a right ideal of S for every $e\in A$. Therefore, (I_a,A) is a soft right ideal over S.

In a similar way, we can prove the following result:

5.2. Proposition

Let $(S,\cdot,\le)$ be an ordered semigroup and (λ,Σ)a fuzzy soft left ideal over S. Then (I_a,A) is a soft left ideal over S for every $a\in S$.

Combining Propositions 5.1 and 5.2, we have the following corollary:

5.3. Corollary

Let $(S,\cdot,\le)$ be an ordered semigroup and (λ,Σ) a fuzzy soft ideal over S. Then (I_a,A) is a soft ideal over S for every $a\in S$.

5.4. Definition

An ordered semigroup S is called *fuzzy soft left (right) simple* if and only if it is fuzzy left (right) simple. That is, for every fuzzy soft left (right) ideal (λ,Σ) over S, we have

$$\lambda(\varepsilon)(a)=\lambda(\varepsilon)(b),\text{ for every }a,b\in S\text{ and }\varepsilon\in\Sigma.$$

An ordered semigroup S is called *fuzzy soft simple* if and only if it is both a fuzzy left and right simple.

5.5. Example

Consider the set $S=\{a,b,c\}$ with the following multiplication table and order relation as given below:

$\cdot$	a	b	c
a	c	b	a
b	b	b	c
c	c	c	c

$\le:=\{(a,a),(b,b),(c,c),(a,c),(b,c)\}$

Then $(S,\cdot,\le)$ is an ordered semigroup and S has no left or right ideal except S. All non-empty subsets of S are $\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}$ and S. If $A=\{a\}$, then $c=a.a\in SA$ but

$c \notin A$. If $A = \{b\}$, then $c = c.b \in SA$ but $c \notin A$. If $A = \{c\}$ then $a \le c$ but $a \in A$. If $A = \{a,b\}$, then $c = a.a \in SA$ but $c \notin A$. If $A = \{a,c\}$, then $b = b.a \in SA$ but $b \notin A$. Finally, if $A = \{b,c\}$, then $a \le c$ but $a \notin A$. Thus none of the above non-empty subsets of S is a left ideal of S and hence S is left simple. Similarly, we can show that S is right simple. Let $E = \{e_1, e_2, e_3\}$ and choose the fuzzy soft set (λ, Σ) over S as follows:

$$\lambda(e_1)(x) = \begin{cases} t \text{ if } x \in S \\ 0 \text{ if } x \notin S \end{cases}$$

Then $(\lambda, \Sigma)^t$ is a soft left simple ordered semigroup and so (λ, Σ) is a fuzzy soft left simple ordered semigroup.

5.6. Theorem

An ordered semigroup $(S, \cdot, \le)$ is soft simple if and only if it is fuzzy soft simple.

Proof: Assume that (λ, Σ) is a fuzzy soft ideal over S, and let $a, b \in S$. By Corollary 5.3, (I_a, A) is a soft ideal over S. Since S is soft simple, it follows that $I_a(e) = S$ for every $e \in A$, and so $b \in I_a(e)$. Thus, we have $\lambda(\varepsilon)(a) \ge \lambda(\varepsilon)(b)$ for every $\varepsilon \in \Sigma$. In a similar way, we can show that $\lambda(\varepsilon)(b) \ge \lambda(\varepsilon)(a)$. Hence, $\lambda(\varepsilon)(a) = \lambda(\varepsilon)(b)$ and so S is fuzzy simple. Consequently, S is fuzzy soft simple.

Conversely, suppose that S contains proper soft ideals and let (I, A) be a soft ideal over S such that $I(e) \ne S$ for every $e \in A$. Since (I, A) is a soft ideal over S, by Definition 3.13, $I(e)$ is an ideal of S. By Lemma 4.7, $\chi_{I(e)}(\varepsilon)$ is a fuzzy ideal of S for each $\varepsilon \in \Sigma$. Let $a \in S$. Since S is fuzzy simple, by hypothesis, $\chi_{I(e)}(\varepsilon)$ is a constant function, that is, $\chi_{I(e)}(\varepsilon)(a) = \chi_{I(e)}(\varepsilon)(b)$ for every $b \in S$. Thus for any $y \in I(e)$, we have $\chi_{I(e)}(\varepsilon)(a) = \chi_{I(e)}(\varepsilon)(y) = 1$ and so $a \in I(e)$. Therefore $S = I(e)$, a contradiction. Consequently, S is soft simple.

5.7. Theorem

An ordered semigroup $(S, \cdot, \le)$ is soft simple if and only if for every fuzzy soft interior ideal (λ, Σ) over S, we have $\lambda(\varepsilon)(a) = \lambda(\varepsilon)(b)$, for every $a, b \in S$ and $\varepsilon \in \Sigma$.

Proof: $\Rightarrow$. Suppose that (λ, Σ) is a fuzzy soft interior ideal over S and let $a, b \in S$. Since S is simple and $b \in S$, we have $S = (SbS]$. Since $a \in S$, we have $a \in (SbS]$, then $a \le xby$ for some $x, y \in S$. Since (λ, Σ) is a fuzzy soft interior ideal over S, for each $\varepsilon \in \Sigma$, we have

$$\begin{aligned}\lambda(\varepsilon)(a) &\ge \lambda(\varepsilon)(xby) = \lambda(\varepsilon)(x(by)) \\ &\ge \lambda(\varepsilon)(b).\end{aligned}$$

In a similar way, we can prove that $\lambda(\varepsilon)(b) \ge \lambda(\varepsilon)(a)$. Thus, $\lambda(\varepsilon)(a) = \lambda(\varepsilon)(b)$.

$\Leftarrow$. Assume that (λ, Σ) is a fuzzy soft interior ideal over S, then by Proposition 4.19, (λ, Σ) is a fuzzy soft ideal over S. By hypothesis, $\lambda(\varepsilon)(a) = \lambda(\varepsilon)(b)$ for every $a, b \in S$ and $\varepsilon \in \Sigma$. Thus, S is fuzzy soft simple and by Theorem 5.6, S is soft simple.

5.8. Theorem

An ordered semigroup $(S, \cdot, \le)$ is intra-regular if and only if for every fuzzy soft ideal (λ, Σ) over S, we have $\lambda(\varepsilon)(a) = \lambda(\varepsilon)(a^2)$, for every $a \in S$ and $\varepsilon \in \Sigma$.

Proof: $\Rightarrow$. Suppose that (λ, Σ) is a fuzzy soft ideal over S, and let $a \in S$. Then there exist $x, y \in S$ such that $a \le xa^2y$. Since (λ, Σ) is a fuzzy soft ideal over S, for every $\varepsilon \in \Sigma$ we have

$$\begin{aligned}\lambda(\varepsilon)(a) &\ge \lambda(\varepsilon)(xa^2y) = \lambda(\varepsilon)(x(a^2y)) \\ &\ge \lambda(\varepsilon)(a^2y), \text{ because } \lambda(\varepsilon) \text{ is a fuzzy left ideal} \\ &\ge \lambda(\varepsilon)(a^2), \text{ because } \lambda(\varepsilon) \text{ is a fuzzy right ideal} \\ &= \lambda(\varepsilon)(a.a) \ge \lambda(\varepsilon)(a).\end{aligned}$$

Thus, $\lambda(\varepsilon)(a) \ge \lambda(\varepsilon)(a^2) \ge \lambda(\varepsilon)(a)$ and hence $\lambda(\varepsilon)(a) = \lambda(\varepsilon)(a^2)$.

$\Leftarrow$. Assume that for every fuzzy soft ideal (λ, Σ) over S, we have $\lambda(\varepsilon)(a) = \lambda(\varepsilon)(a^2)$, for every $a \in S$ and $\varepsilon \in \Sigma$. Then S is intra-regular, in fact, we consider the ideal $I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$ of S, generated by $a^2 (a \in S)$. Then by Lemma 4.7, $(\chi_{I(e)(a^2)}, \Sigma)$ is a fuzzy soft ideal over S and by hypothesis, we have $\chi_{I(e)(a^2)}(a) = \chi_{I(e)(a^2)}(a^2)$. Since $a^2 \in I(a^2)$, we have $\chi_{I(e)(a^2)}(a^2) = 1$, then $\chi_{I(e)(a^2)}(a) = 1$ and hence $a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$. Then $a \le a^2$ or $a \le xa^2$ or $a \le a^2x$ or $a \le xa^2y$ for some $x, y \in S$. If $a \le a^2 = aa \le a^2a^2 = aa^2a \in Sa^2S$ and $a \in (Sa^2S]$. Similarly, for other cases we have $a \le ua^2v$ for some $u, v \in S$. Thus S is intra-regular.

6. Semiprime fuzzy soft ideals

In this section, the concept of a semiprime fuzzy soft ideal in ordered semigroups is provided. It is shown that an ordered semigroup S is left regular if and only if every fuzzy soft left ideal over S is semiprime. It is also shown that S is intra-regular if and only if every fuzzy soft ideal over S is semiprime.

6.1. Definition

A fuzzy soft ideal (λ, Σ) is called *semiprime fuzzy soft ideal* over S if and only if it is semiprime fuzzy ideal of S. That is, for each $\varepsilon \in \Sigma$, we have,

$$\lambda(\varepsilon)(a) \ge \lambda(\varepsilon)(a^2) \text{ for every } a \in S.$$

6.2. Lemma

A soft set (F, A) over S is a semiprime soft ideal over S if and only if $(\chi_{F(e)}, \Sigma)$ is a semiprime fuzzy soft ideal over S.

Proof: Suppose that (F, A) is a semiprime soft ideal over S. By Definition 3.15, for each $e \in A$, $F(e)$ is a semiprime ideal of S. Let a be an element of S such that $a^2 \in F(e)$, then $\chi_{F(e)}(\varepsilon)(a^2) = 1$ for each $\varepsilon \in \Sigma$. Since $F(e)$ is semiprime ideal of S, $a^2 \in F(e)$ implies $a \in F(e)$ and so $\chi_{F(e)}(\varepsilon)(a) = 1 = \chi_{F(e)}(\varepsilon)(a^2)$. If $a^2 \notin F(e)$, then we have $\chi_{F(e)}(\varepsilon)(a) \ge 0 = \chi_{F(e)}(\varepsilon)(a^2)$. Hence $\chi_{F(e)}(\varepsilon)$ is semiprime fuzzy ideal of S. Thus $(\chi_{F(e)}, \Sigma)$ is a semiprime fuzzy soft ideal over S.

Conversely, assume that $(\chi_{F(e)}, \Sigma)$ is a semiprime fuzzy soft ideal over S. Let $a^2 \in F(e)$ for each $e \in A$. Then $\chi_{F(e)}(\varepsilon)(a^2) = 1$ and by hypothesis, we have, $\chi_{F(e)}(\varepsilon)(a) \ge \chi_{F(e)}(\varepsilon)(a^2) = 1$. It follows that $\chi_{F(e)}(\varepsilon)(a) = 1$ and so $a \in F(e)$. Hence $F(e)$ is a semiprime ideal of S for each $e \in A$. Consequently, (F, A) is a semiprime ideal over S.

6.3. Theorem

A fuzzy soft set (λ, Σ) over S is a semiprime fuzzy soft ideal over S if and only if for every $t \in [0,1]$ the set $(\lambda, \Sigma)^t (\ne \phi)$ is a semiprime soft ideal over S.

Proof: Suppose that (λ, Σ) is a semiprime fuzzy soft ideal over S. Then for each $\varepsilon \in \Sigma$, $\lambda(\varepsilon)$ is a semiprime fuzzy ideal of S. Let $a \in S$ be such that $a^2 \in \lambda^t(\varepsilon)$ for each $t \in [0,1]$ and $\varepsilon \in \Sigma$. Then $\lambda(\varepsilon)(a^2) \ge t$. Since $\lambda(\varepsilon)$ is semiprime fuzzy ideal of S, we have $\lambda(\varepsilon)(a) \ge \lambda(\varepsilon)(a^2)$. It follows that $\lambda(\varepsilon)(a) \ge t$ and so $a \in \lambda^t(\varepsilon)$. Hence $\lambda^t(\varepsilon)$ is semiprime ideal of S. Thus, $(\lambda, \Sigma)^t$ is a semiprime soft ideal over S.

Conversely, assume that for all $t \in [0,1]$, $(\lambda, \Sigma)^t (\ne \phi)$ is a semiprime soft ideal over S. Let $a \in S$ be such that $\lambda(\varepsilon)(a^2) = 0$, then we have $\lambda(\varepsilon)(a) \ge 0 = \lambda(\varepsilon)(a^2)$. If $\lambda(\varepsilon)(a^2) = t$, then $a^2 \in \lambda^t(\varepsilon)$. Since $\lambda^t(\varepsilon)$ is semiprime ideal of S, $a^2 \in \lambda^t(\varepsilon)$ implies $a \in \lambda^t(\varepsilon)$. Then $\lambda(\varepsilon)(a) \ge t = \lambda(\varepsilon)(a^2)$ and hence $\lambda(\varepsilon)$ is semiprime fuzzy ideal of S. Therefore, (λ, Σ) is a semiprime fuzzy ideal over S.

6.4. Example

Consider the ordered semigroup S defined by the following multiplication table and order relation "$\leq$" given below:

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	a	d	e	f
c	c	c	c	c	c	c
d	a	d	a	b	e	f
e	a	e	a	e	e	e
f	a	f	a	f	e	f

$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(f,e)\}$.

This is a left regular ordered semigroup and $\{a,c\}$, $\{a,c,e,f\}$ and S are left ideals of S. By Lemma 2.1, $\{a,c\}$, $\{a,c,e,f\}$ and S are semiprime. Let $E=\{e_1,e_2,e_3\}$ and define a fuzzy subset λ as follows:

$$\lambda(e_1)(x)=\begin{cases}0.8 \text{ if } x=a\\ 0.6 \text{ if } x=c\\ 0.4 \text{ if } x=f\\ 0.2 \text{ if } x=e\end{cases}$$

Then

$$(\lambda,\Sigma)^t=\begin{cases}\{a,c,e,f\} \text{ if } 0.2\leq t<0.4\\ \phi \quad\quad \text{ if } 0.8\leq t<1\end{cases}$$

By theorem 6.3, $(\lambda,\Sigma)^t$ is semiprime and hence (λ,Σ) is semiprime fuzzy soft left ideal over S.

6.5. Theorem

An ordered semigroup S is left regular if and only if every fuzzy soft left ideal over S is semiprime.

Proof: $\Rightarrow$. Suppose that S is left regular and $a\in S$. Let (λ,Σ) be a fuzzy soft left ideal over S. Since S is left regular and $a\in S$, there exists $x\in S$ such that $a\leq xa^2$. Then for each $\varepsilon\in\Sigma$, we have $\lambda(\varepsilon)(a)\geq\lambda(\varepsilon)(xa^2)\geq\lambda(\varepsilon)(a^2)$, because $\lambda(\varepsilon)$ is a fuzzy left ideal of S. Thus, (λ,Σ) is semiprime.

$\Leftarrow$. Assume that every fuzzy soft left ideal over S is semiprime. We consider the left ideal $L(a^2)$ of S generated by a^2 $(a\in S)$. By Lemma 4.7, $(\chi_{L(e)(a^2)},\Sigma)$ is a fuzzy soft left ideal over S and by hypothesis, $(\chi_{L(e)(a^2)},\Sigma)$ is semiprime. Then $\chi_{L(e)(a^2)}(\varepsilon)(a)\geq\chi_{L(e)(a^2)}(\varepsilon)(a^2)$, since $a^2\in L(a)=(a^2\cup Sa^2]$, we have $\chi_{L(a^2)}(\varepsilon)(a^2)=1$, it follows that $\chi_{L(a^2)}(\varepsilon)(a)=1$ and so $a\in L(a)=(a^2\cup Sa^2]$. Then $a\leq a^2$ or $a\leq xa^2$ for some $x\in S$. If $a\leq a^2$ then $a\leq a^2=aa\leq aa^2\in Sa^2$ and so $a\in(Sa^2]$. For $a\leq xa^2$, we get $a\in(Sa^2]$. Thus, S is left regular.

By left-right dual we can prove the following:

6.6. Theorem

An ordered semigroup S is right regular if and only if every fuzzy soft right ideal over S is semiprime.

In the following we characterize intra-regular ordered semigroups in terms of semiprime fuzzy soft ideals over S.

6.7. Theorem

An ordered semigroup S is intra-regular if and only if every fuzzy soft ideal over S is semiprime.

Proof: The proof follows from Theorem 6.5.

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