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\frac{\lambda}{2} - \text{Legendre curves in 3-dimensional Heisenberg group } IN^3
\]

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Abstract

In this study, we focused on \( \frac{\lambda}{2} \)-Legendre curves and non-\( \frac{\lambda}{2} \)-Legendre curves in 3-dimensional Heisenberg group \( IN^3 \). Also, we gave some characterizations of these curves.

Keywords: Heisenberg group; Sasakian manifold; Legendre curve

1. Introduction

In mathematics, the Heisenberg group, named after Werner Heisenberg, is the group of \( 3 \times 3 \) upper triangular matrices of the form

\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]

or its generalizations under the operation of matrix multiplication. In 1987, L. Bianchi classified the homogeneous metrics. L. Bianchi, E. Cartan and G. Vranceanu found the following 2-parameter family of homogeneous Riemannian metrics on the cartesian 3-space \( \mathbb{R}^3(\mathbf{x},\mathbf{y},\mathbf{z}) \):

\[
\theta_{\lambda,\mu} = \frac{dx^2 + dy^2}{(1 + \mu(x^2 + y^2))} + \frac{dz + \frac{\lambda}{2} (ydx - xdy)}{(1 + \mu(x^2 + y^2))}, \forall \lambda, \mu \in \mathbb{R}.
\]

In this family, if \( \lambda = \mu = 0 \), the Euclidean metric is obtained, and if \( \lambda \neq 0, \mu = 0 \), the Heisenberg metric is obtained. Inoguchi studied the differential geometry of Heisenberg metric.

The Legendre curves play an important role in the study of contact manifolds. In a 3-dimensional Sasakian manifold, the Legendre curves are studied by Baikoussis and Blair who gave the Frenet 3-frame in this space [1]. Yıldırım gave some characterizations of Legendre curves in Homogeneous space [2]. İlarslan gave a characterization of curves on non-Euclidean manifolds [3]. On the other hand, Baikoussis and Hircia studied Legendre curves in Riemannian and Lorentzian Sasakian spaces [4]. Also, Legendre curves in \( \alpha \)-Sasakian spaces are studied by Özgür and Tripathi [5]. In this study, we focused on \( \frac{\lambda}{2} \)-Legendre curves in 3-dimensional Heisenberg group \( IN^3 \) and gave a characterization of these curves. Also, we gave similar results for non-\( \frac{\lambda}{2} \)-Legendre curves in 3-dimensional Heisenberg group \( IN^3 \).

2. Preliminaries

In this section, we will give some basic concepts related to Sasakian geometry for later use.

The Heisenberg group \( IN^3 \) can be seen as the Euclidean space with the multiplication

\[
(x, y, z)(x', y', z') = \left( x + x', y + y', z + z' + \frac{\lambda}{2}(xy' - yx') \right)
\]

and with the Riemannian metric

\[
\theta_1 = dx^2 + dy^2 + \left( \frac{dx + \frac{\lambda}{2} ydx - xdy}{(1 + \mu(x^2 + y^2))} \right)^2, \forall \lambda, \mu \in \mathbb{R}.
\]

\( IN^3 \) is a three dimensional, connected, simply connected and 2-step nilpotent Lie group. The Lie algebra of \( IN^3 \) has a basis

\[
\begin{cases}
\theta_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z} \\
\theta_2 = \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \\
\theta_3 = \frac{\partial}{\partial z}
\end{cases}
\]

which is dual to

\[
\begin{cases}
\theta_1 = dx \\
\theta_2 = dy \\
\theta_3 = dz + \frac{\lambda}{2}(ydx - xdy)
\end{cases}
\]

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For this basis Lie brackets are 
\[ [e_1, e_2] = e_3, [e_2, e_3] = e_1, e_1] = 0.\] [6]

To study curves in \( \mathbb{R}^3 \), we shall use their Frenet vector fields and equations. Let \( \gamma: I \rightarrow \mathbb{R}^3 \) be a differentiable curve parameterized by arc length and let \( \{V_1, V_2, V_3\} \) be the orthonormal frame field tangent defined as follows: By \( V_1 \) we denote \( \gamma \) tangent to \( \gamma \), by \( V_2 \) the unit vector field in the direction \( D_{V_2}V_1 \) normal to \( \gamma \) and we choose \( V_3 = V_1 \times V_2 \), so that \( \{V_1, V_2, V_3\} \) is a positive oriented orthonormal basis. Thus, we have the following Frenet equations [7]:

\[
\begin{bmatrix}
D_{V_1}V_1 \\
D_{V_1}V_2 \\
D_{V_1}V_3
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & -\tau \\
0 & \tau & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix},
\]

(4)

Now, let us consider the odd-dimensional Riemannian manifold \((M,g)\). So, the Riemannian manifold \((M,g)\) is said to be almost contact metric manifold if there exist on \( M \) a \((1,1)\) tensor field \( \varphi \), a vector field \( \xi \) (called the Reeb vector field) and a 1-form \( \eta \) such that

\[ \eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi \]

and

\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \]

for any vector fields \( X, Y \) on \( M \). In particular, in an almost contact metric manifold we also have \( \varphi^2 = 0 \) and \( \eta \varphi \eta = 0 \).

Such a manifold is said to be contact metric manifold, if \( d\eta = \Phi \), where \( \Phi(X, Y) = g(\varphi X, \varphi Y) \) is called the fundamental 2-form of \( M \). If \( \xi \) is a Killing vector field, then \( M \) is said to be a \( K \)-contact manifold, we have

\[ (D_{\xi}\varphi)Y = R(\xi, X)Y \]

for any \( X, Y \) on \( M \).

On the other hand, the almost contact metric structure of \( M \) is said to be normal if

\[ [\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \]

[8, 9].

A normal contact metric manifold is called a Sasakian Manifold. It can be proved that a Sasakian manifold is \( K \)-contact, and that an almost contact metric manifold is Sasakian if and only if

\[ (D_{\xi}\varphi)Y = g(X, Y)\xi - \eta(Y)X \]

for any \( X, Y \). Furthermore, assuming that \( \eta = \theta^3 \), \( \xi = e_3 \) and defining \( \varphi: \chi(\mathbb{R}^3) \rightarrow \chi(\mathbb{R}^3) \)

\[ \varphi(X) = -a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + \frac{\lambda}{2}(x_1a_1 + x_2a_2) \frac{\partial}{\partial x_3} \]

where \( \sum_{i=1}^{2} a_i x_i \in \chi(\mathbb{R}^3) \), it can be easily seen that \( \mathbb{R}^3 \) is a Sasakian space. Since all computations have \( \frac{\lambda}{2} \) coefficients, we have denoted \( \mathbb{R}^3 \) as \( \frac{\lambda}{2} \)-Sasakian space. We need the following Lemma for later use:

**Lemma:** Let \( X \) and \( Y \) be two vector fields in \( \chi(\mathbb{R}^3) \), \( D \) be a Riemannian connection on \( \mathbb{R}^3 \) and \( \mathbb{R}^3 \) be a Sasakian space. The following equation holds:

\[ D_{V_1}V_1 = \kappa V_1 \]

where \( \kappa \) is the torsion of \( \mathbb{R}^3 \).

(5)

On the other hand, derivating \( \eta(\gamma) = 0 \) if and only if \( \eta(\gamma) = 0 \), [8, 9].

3. \( \frac{\lambda}{2} \)-Legendre Curves in \( \mathbb{R}^3 \)

**Theorem 3.1.** Let \( \gamma: I \rightarrow \mathbb{R}^3 \) be a non-geodesic \( \frac{\lambda}{2} \)-Legendre curve. The Frenet frame of \( \gamma \) is \( \{V_1, \varphi V_1, \xi\} \) and the Frenet formulas are

\[
\begin{bmatrix}
D_{V_1}V_1 \\
D_{V_1}\varphi V_1 \\
D_{V_1}\xi
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & -\tau \\
0 & \tau & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
\varphi V_1 \\
\xi
\end{bmatrix}.
\]

(6)

**Proof:** Let \( \gamma: I \rightarrow \mathbb{R}^3 \) be a curve with arc length parameter and the Frenet frame of \( \gamma \) be \( \{V_1, V_2, V_3\} \).

Assume that \( \eta(\gamma) = 0 \). In this case, an orthonormal basis of \( \mathbb{R}^3 \) is

\[ \left\{ V_1, \frac{\varphi V_1}{\sqrt{1 - \sigma^2}}, \frac{V_1}{\sqrt{1 - \sigma^2}} \right\} \]

From here, we obtain

\[ D_{V_1}V_1 = \alpha \frac{\varphi V_1}{\sqrt{1 - \sigma^2}} + \beta \frac{V_1}{\sqrt{1 - \sigma^2}}, \alpha, \beta \in C^0(\mathbb{R}^3, \mathbb{R}). \]

On the other hand, derivating \( \sigma \) we obtain

\[ \dot{\sigma} = D_{V_1}\sigma = D_{V_1}g(1, \xi) = g_3(V_1, \xi) \]

where \( g_3(V_1, \xi) = g_3(\varphi V_1, \xi) + g_3(V_1, D_{V_1}\xi) \)

\[ g_3 \left( \frac{\varphi V_1}{\sqrt{1 - \sigma^2}}, \frac{V_1}{\sqrt{1 - \sigma^2}}, \frac{\varphi V_1}{\sqrt{1 - \sigma^2}}, \frac{V_1}{\sqrt{1 - \sigma^2}} \right) \]

\[ = \beta \sqrt{1 - \sigma^2}. \]

From here, we say that

\[ \beta = \sigma \frac{1}{\sqrt{1 - \sigma^2}}. \]

Since \( \gamma \) is a \( \frac{\lambda}{2} \)-Legendre curve, we can easily see that \( \beta = 0 \). Moreover, from (4) we get \( \alpha = \kappa, \)

\[ \]
\[ V_2 = \varphi V_1, \quad D_{v_2} V_1 = \kappa \varphi V_1 \quad \text{and} \]
\[ D_{v_1} V_2 = \varphi D_{v_1} V_1 + (D_{v_1} \varphi) V_1 \]
\[ = \varphi (\kappa \varphi V_1) + \frac{\lambda}{2} (g_2(V_1, V_1) \xi - \eta(V_1)V_1) \]
\[ = -\kappa V_1 + \frac{\lambda}{2} \xi. \]

From (4), we get \( V_3 = \xi, \quad \tau = -\frac{\lambda}{2} \). Hence, the Serret-Frenet formulas are
\[
\begin{bmatrix}
D_{v_1} V_1 \\
D_{v_1} \varphi V_1 \\
D_{v_1} V_3
\end{bmatrix} = \begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \frac{\lambda}{2} \\
0 & -\frac{\lambda}{2} & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
\varphi V_1 \\
\xi
\end{bmatrix}.
\]

**Theorem 3.2:** Let \( \gamma : I \rightarrow \mathbb{N}^3 \) be a non-geodesic \( \frac{\lambda}{2} \)-Legendre curve and \( 0 < |\eta(\dot{\gamma})| < 1 \). The curvature and the torsion of \( \gamma \) are
\[
\kappa = \sqrt{\alpha^2 + \beta^2}, \quad \alpha, \beta \in C^\infty([\mathbb{N}^3, \mathbb{R}])
\]
and
\[
\tau = \frac{\lambda}{2} + \frac{\alpha \beta - \alpha^2 + \beta^2}{2 \sqrt{1 - \sigma^2}}, \quad \text{for} \quad \sigma \neq 0
\]
respectively.

**Proof:** Let \( \gamma : I \rightarrow \mathbb{N}^3 \) be a curve with arc length parameter and the Frenet frame of \( \gamma \) be \( \{V_1, V_2, V_3\} \). Assume that \( \eta(\dot{\gamma}) = \varphi \neq 0 \). In this case, an orthonormal basis of \( \frac{\lambda}{2} \)-Sasakian space is
\[
\left\{ V_1, \varphi V_1 \right\} = \left\{ V_1, \varphi V_1 \right\}.
\]
From here we get
\[
D_{v_1} V_1 = \alpha \frac{\varphi V_1}{\sqrt{1 - \sigma^2}} + \beta \frac{\xi - \alpha V_1}{\sqrt{1 - \sigma^2}}, \quad \alpha, \beta \in C^\infty([\mathbb{N}^3, \mathbb{R}]).
\]
So, we obtain
\[
\kappa = \|D_{v_1} V_1\| = \sqrt{\alpha^2 + \beta^2}, \quad \alpha, \beta \in C^\infty([\mathbb{N}^3, \mathbb{R}])
\]
and
\[
V_2 = \frac{1}{\kappa} D_{v_1} V_1.
\]
On the other hand, derivating \( \varphi V_1 \), we have
\[
D_{v_1} \varphi V_1 = \varphi D_{v_1} V_1 + (D_{v_1} \varphi) V_1
\]
\[= \varphi \left( \alpha \frac{\varphi V_1}{\sqrt{1 - \sigma^2}} + \beta \frac{\xi - \alpha V_1}{\sqrt{1 - \sigma^2}} \right) + \frac{\lambda}{2} (\xi - \sigma V_1)\]
\[= -\frac{\alpha}{\sqrt{1 - \sigma^2}} V_1 + \frac{\alpha \kappa}{\sqrt{1 - \sigma^2}} \xi - \frac{\beta \sigma}{\sqrt{1 - \sigma^2}} \varphi V_1 + \frac{\lambda}{2} (\xi - \sigma V_1). \quad (9)
\]
Similaly, derivating \( \xi - \sigma V_1 \) we get,
\[
D_{v_1} (\xi - \sigma V_1) = D_{v_1} \xi - \sigma V_1 - \sigma D_{v_1} V_1
\]
\[= -\frac{\lambda}{2} \varphi V_1 - \sigma V_1 - \sigma \alpha \frac{\varphi V_1}{\sqrt{1 - \sigma^2}} - \sigma \beta \frac{\xi - \alpha V_1}{\sqrt{1 - \sigma^2}}.
\]
(10)

On the other hand, derivating \( \sigma \) we have
\[
\dot{\sigma} = D_{v_1} \sigma
\]
\[= D_{v_1} g_3(V_1, \xi)
\]
\[= g_3(D_{v_1} V_1, \xi) + g_3(V_1, D_{v_1} \xi)
\]
\[= g_3 \left( \alpha \frac{\varphi V_1}{\sqrt{1 - \sigma^2}} + \beta \frac{\xi - \alpha V_1}{\sqrt{1 - \sigma^2}}, \xi \right) + g_3(V_1, -\frac{\lambda}{2} \varphi V_1)
\]
\[= \beta \sqrt{1 - \sigma^2}.
\]
From here, we see that
\[
\beta = \dot{\sigma} \frac{1}{\sqrt{1 - \sigma^2}}.
\]

Similarly, derivating \( \frac{\alpha \beta}{\sqrt{1 - \sigma^2}} \) and \( \frac{\beta \sigma}{\sqrt{1 - \sigma^2}} \) we obtain
\[
D_{v_1} \left( \frac{\alpha}{\sqrt{1 - \sigma^2}} \right) = \frac{\dot{\alpha}}{\sqrt{1 - \sigma^2}} + \frac{\alpha \beta \sigma}{1 - \sigma^2}
\]
(11)

and
\[
D_{v_1} \left( \frac{\beta}{\sqrt{1 - \sigma^2}} \right) = \frac{\dot{\beta}}{\sqrt{1 - \sigma^2}} + \frac{\beta^2 \sigma}{1 - \sigma^2}
\]
(12)
respectively. Furthermore,
\[
D_{v_1} V_2 = D_{v_1} \left( \frac{1}{\kappa} D_{v_1} V_1 \right)
\]
\[= -\frac{\lambda}{\kappa} D_{v_1} V_1 + \frac{1}{\kappa} D_{v_1} \varphi V_1 + \frac{1}{\kappa} \frac{\alpha}{\sqrt{1 - \sigma^2}} D_{v_1} V_1 + \frac{1}{\kappa} \frac{\beta}{\sqrt{1 - \sigma^2}} (\xi - \sigma V_1) + \frac{1}{\kappa} \frac{\beta}{\sqrt{1 - \sigma^2}} D_{v_1} (\xi - \sigma V_1).
\]
Using (9), (10), (11) and (12), we get
\[
D_{v_1} V_2 = -\frac{\lambda}{\kappa} \varphi V_1 + \frac{1}{\kappa} \left( \frac{\alpha}{\sqrt{1 - \sigma^2}} \right) V_1 + \frac{1}{\kappa} \left( \frac{\beta}{\sqrt{1 - \sigma^2}} \right) \frac{\xi - \sigma V_1}{\sqrt{1 - \sigma^2}}.
\]
From (6), it can be easily seen that
\[
\tau V_3 = \left( -\frac{\alpha \kappa}{\kappa^2} + \frac{\lambda}{\kappa} - \frac{\alpha \beta}{\kappa^2} \right) \frac{\varphi V_1}{\sqrt{1 - \sigma^2}}
\]
\[+ \left( -\frac{\beta \kappa}{\kappa^2} + \frac{\lambda}{\kappa} - \frac{\beta \sigma}{\kappa^2} \right) \frac{\xi - \alpha V_1}{\sqrt{1 - \sigma^2}}.
\]
Taking the norm of the last equation, we have
Lemma 3.1. Let \( \gamma: I \to \mathbb{R}^3 \) be a curve with arc length parameter and \( \{V_1, V_2, V_3\} \) be the Frenet frame of \( \gamma \). Then, the following equation is obtained:

\[
D^2_{\gamma} V_1 - 2 \frac{\kappa}{\kappa} D^2_{\gamma} V_1' + 2 \frac{\kappa}{\kappa} (2 \frac{\kappa}{\kappa} - \frac{\kappa}{\kappa} + \kappa^2 + \frac{\lambda^2}{4}) D_{\gamma} V_1 + \kappa \kappa V_1 = 0.
\]

(13)

Proof: From (6), we know that

\[ D_{\gamma} \phi V_1 = -\kappa V_1 + \frac{\lambda}{2} \xi \]

and

\[ D_{\gamma} V_1 = -\kappa \phi V_1. \]

From here,

\[
D_{\gamma} V_1 = -\kappa V_1 + \frac{\lambda}{2} \xi
\]

\[ \Rightarrow \left( \frac{1}{\kappa} \right)' D_{\gamma} V_1 + \frac{1}{\kappa} D^2_{\gamma} V_1 = -\kappa V_1 + \frac{\lambda}{2} \xi. \]

Differentiating the last equation, we have

\[ \frac{1}{\kappa} D^2_{\gamma} V_1 + 2 \left( \frac{1}{\kappa} \right)' D^2_{\gamma} V_1 + \left( \frac{1}{\kappa} \right)'' + \kappa + \frac{\lambda^2}{4} \right) D_{\gamma} V_1 + \kappa \kappa V_1 = 0. \]

Considering the last equation, we get

\[ D^2_{\gamma} V_1 - 2 \frac{\kappa}{\kappa} D^2_{\gamma} V_1' + 2 \frac{\kappa}{\kappa} (2 \frac{\kappa}{\kappa} - \frac{\kappa}{\kappa} + \kappa^2 + \frac{\lambda^2}{4}) D_{\gamma} V_1 + \kappa \kappa V_1 = 0. \]

Theorem 3.3. Let \( \gamma: I \to \mathbb{R}^3 \), \( \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \), be a \( \frac{\lambda}{2} \)-Legendre curve in \( \mathbb{R}^3 \) and \( \alpha \) be the projection curve of \( \gamma \) on \( z = 0 \) plane. Then, the curvature of \( \gamma \) is the curvature of \( \alpha \).

Proof: The tangent vector field of \( \gamma \) is

\[ \hat{\gamma}(t) = \gamma_1(t)e_1 + \gamma_2(t)e_2 + \gamma_3(t)e_3. \]

We can choose the parameter of \( \gamma \) as \( \gamma_1(t)^2 + \gamma_2(t)^2 = 1 \). Then, if we choose \( \gamma_1(t) \) and \( \gamma_2(t) \) as \( \hat{\gamma}_1(t) = -\sin \theta(t) \), \( \gamma_2(t) = \cos \theta(t) \), respectively, we obtain

\[ D_{\gamma(t)} \gamma = \gamma_1(t)e_1 + \gamma_2(t)e_2 \]

and

\[ \|D_{\gamma(t)} \gamma\| = \frac{1}{2} \sqrt{\gamma_1(t)^2 + \gamma_2(t)^2} \]

\[ \kappa = \hat{\theta}(t). \]

On the other hand, the projection curve \( \alpha \) of \( \gamma \) on \( z = 0 \) plane is \( \alpha(t) = (\alpha_1(t), \alpha_2(t)) \). Thus, it can be easily seen that \( \alpha \) is a unit speed curve. The curvature of \( \alpha \) is

\[ \kappa_{\alpha} = \frac{\|\gamma_1(t)\gamma_2(t) - \gamma_2(t)\gamma_3(t)\|}{\sqrt{(\gamma_1(t)^2 + \gamma_2(t)^2)^2}}. \]

From here,

\[ \kappa = \kappa_{\alpha}. \]

Corollary 3.1. Let \( \gamma \) be a non-geodesic Legendre curve in \( \mathbb{R}^3 \). Then,

i) \( \gamma \) is not a circle.

ii) If \( \gamma \) is a helix, it satisfies the following equation:

\[ \Delta H = \left( \kappa^2 + \frac{\lambda^2}{4} \right) H. \]

iii) If \( \gamma \) is a line, \( g_3(D_{\gamma} \gamma, \phi V_1) = 0 \).

iv) \( \gamma \) is not a planar curve.

Proof: i) Since \( \gamma \) is a \( \frac{\lambda}{2} \)-Legendre curve, the torsion of \( \gamma \) is \( \frac{\lambda}{2} \). So, it can be easily seen that \( \gamma \) is not a circle.

ii) If \( \gamma \) is helix, \( \kappa_{\alpha} \) is constant. Also, on the ground that the torsion of \( \gamma \) is \( \frac{\lambda}{2} \), \( \kappa_{\alpha} \) must be constant. So, \( \hat{\kappa}, \hat{\kappa} = 0 \).

From (13), we obtain

\[ D^3_{\gamma} V_1 = -\left( \kappa^2 + \frac{\lambda^2}{4} \right) D_{\gamma} V_1. \]

Using \( V_1 = \hat{\gamma} \), \( \Delta = -D_{\gamma} D_{\gamma} V_1 \) and \( H = D_{\gamma} V_1 \) we have

\[ \Delta H = \left( \kappa^2 + \frac{\lambda^2}{4} \right) H. \]

iii) If \( \gamma \) is a line, the curvature of \( \gamma \) is zero. Also, \( D_{\gamma} V_1 = \kappa \phi V_1 \).

From here, we get \( g_3(D_{\gamma} \gamma, \phi V_1) = 0 \).

iv) Since \( \gamma \) is a \( \frac{\lambda}{2} \)-Legendre curve, the torsion of \( \gamma \) is not zero. So, it is said that \( \gamma \) is not a planar curve.

Example 3.1. \( \gamma: I \to \mathbb{R}^3 \), \( \gamma(t) = (r \cos t, r \sin t, \frac{\lambda}{2} r^2 t) \) is a curve in \( \mathbb{R}^3 \). If we assume that

\[ x = r \cos t \]

\[ y = r \sin t \]

\[ z = \frac{\lambda}{2} r^2 t \]

we get

\[ \hat{\gamma}(t) = \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \gamma(t). \]

Thus, using (1.3), we get
\begin{equation}
\begin{aligned}
\theta^1(\dot{y}(t)) &= -y \\
\theta^2(\dot{y}(t)) &= x \\
\theta^3(\dot{y}(t)) &= 0.
\end{aligned}
\end{equation}

From (14), we can say that $\gamma$ is a $\frac{1}{2}$-Legendre curve. On the other hand, we obtain
\[
\|\dot{\gamma}(t)\| = \sqrt{[\theta^1(\dot{y}(t))]^2 + [\theta^2(\dot{y}(t))]^2 + [\theta^3(\dot{y}(t))]^2} = |r|,
\]

\[V_1 = \mp \frac{y}{r} e_1 \mp \frac{x}{r} e_2\]

and
\[\varphi V_1 = \mp \frac{x}{r} e_1 \mp \frac{y}{r} e_2.\]

Moreover, from (5) we have
\[
D_{V_1} V_1 = \frac{\lambda}{2} V_1 \wedge V_1 - g_A([e_1, e_2], V_1) \varphi V_1 + \bar{D}_{V_1} V_1
= - g_A([e_1, e_2], V_1) \varphi V_1 + \bar{D}_{V_1} V_1
= \mp \frac{1}{r} \varphi V_1.
\]

Namely, we see that
\[\kappa = \mp \frac{1}{r}\]

where $\kappa$ is the curvature of $\gamma$. Also, we know that $\tau = -\frac{\lambda}{2}$ for a non-geodesic $\frac{1}{2}$-Legendre curve in $\mathbb{N}^3$. As a result, $\kappa$ and $\tau$ are non-zero constants. So, $\gamma$ is a helix.

**Result 3.1.** Helix in Euclidean space is a helix in $\frac{1}{2}$-Sasakian space, too. Also, it is a $\frac{1}{2}$-Legendre curve.

**Corollary 3.2.** $\gamma$: $I \rightarrow \mathbb{N}^3$ be a $\frac{1}{2}$-non-Legendre curve. Then,
\[\text{i) If } \gamma \text{ is a geodesic, it satisfies the following equation:}
D_{V_1} V_1 = g_A([e_1, e_2], V_1) \varphi V_1.
\]
\[\text{ii) If } \gamma \text{ is a circle,}
\]
\[\lambda = \frac{2\alpha \sigma}{\sqrt{1 - \sigma^2}}\]
or
\[\lambda = -\frac{2\alpha \sigma}{\sqrt{1 - \sigma^2}} + \dot{\theta}(t)r^2.
\]

where $\alpha = r \cos \theta(t)$ and $\beta = r \sin \theta(t)$.

\[\text{iii) If } \gamma \text{ is a circular helix,}
\]
\[\tau = -\frac{\lambda}{2} + \frac{\alpha \sigma}{\sqrt{1 - \sigma^2}}\]

\[\text{iv) If } \gamma \text{ is a helix,}
\]
\[\alpha^2 + \beta^2 = c^2 \left( \frac{1}{2} + \frac{\alpha \beta - \alpha \beta}{a^2 + \beta^2} + \frac{\alpha \sigma}{\sqrt{1 - \sigma^2}} \right)^2.
\]

**Proof:**

**i) If** $\gamma$ **is a geodesic,** $\kappa = \tau = 0$. **So, from (7) we say that** $\alpha = \beta = 0$ **and** $\tau$ **is indefinite.**

On the other hand, if $\gamma$ is a geodesic, $D_{V_1} V_1 = 0$. So, from (5) we get
\[\bar{D}_{V_1} V_1 = g_A([e_1, e_2], V_1) \varphi V_1.
\]

**ii) If** $\gamma$ **is a circle,** $\kappa$ **is a non-zero constant. In which case there are two situations:**

\[\text{a) We assume that } \alpha \text{ and } \beta \text{ are constants. Thus,}
\tau = -\frac{\lambda}{2} + \frac{\alpha \sigma}{\sqrt{1 - \sigma^2}} = 0
\]

or
\[\lambda = \frac{2\alpha \sigma}{\sqrt{1 - \sigma^2}}.
\]

\[\text{b) We assume that } \kappa \text{ is a non-zero constant and } \alpha \text{ and } \beta \text{ are not constants. Hence, if } \alpha \text{ and } \beta \text{ are chosen as } r \cos \theta(t) \text{ and } r \sin \theta(t), \text{ respectively, it is found that}
\alpha^2 + \beta^2 = r^2
\]

\[\text{and}
\alpha \beta - a \beta = \dot{\theta}(t)r^2.
\]

Since $\tau = 0$, from (12) we get
\[\lambda = -\frac{2\alpha \sigma}{\sqrt{1 - \sigma^2}} + \dot{\theta}(t)r^2.
\]

**iv) If** $\gamma$ **is a helix, $\kappa = c, c \neq 0 = const$ and from (7) and (8)**
\[\alpha^2 + \beta^2 = c^2 \left( \frac{1}{2} + \frac{\alpha \beta - \alpha \beta}{a^2 + \beta^2} + \frac{\alpha \sigma}{\sqrt{1 - \sigma^2}} \right)^2.
\]

**Example 3.2.**
\[\gamma: I \rightarrow \mathbb{N}^3, \gamma(t) = (r \cos t, r \sin t, c) \text{ is a curve in } \mathbb{N}^3. \text{ If we assume that,}
\]
\[x = r \cos t
\]
\[y = r \sin t
\]
\[z = c
\]

we get
\[\dot{\gamma}(t) = \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) \gamma(t).
\]

Thus using (3), we obtain
\[
\begin{aligned}
\theta^1(\dot{y}(t)) &= -y \\
\theta^2(\dot{y}(t)) &= x \\
\theta^3(\dot{y}(t)) &= -r^2.
\end{aligned}
\]
So, we can say that \( \gamma \) is not a \( \frac{\lambda}{2} \)-Legendre curve. On the other hand, we have
\[
\dot{\gamma}(t) = (-ye_1 + xe_2 - r^2e_3)_{\gamma(t)}
\]
and
\[
\|\dot{\gamma}(t)\| = \sqrt{[\theta^2(\dot{\gamma}(t))]^2 + [\theta^3(\dot{\gamma}(t))]^2 + [\theta^4(\dot{\gamma}(t))]^2}.
\]
Thus, we get
\[
V_1 = -\frac{y}{r\sqrt{r^2 + 1}}e_1 + \frac{x}{r\sqrt{r^2 + 1}}e_2 - \frac{r}{\sqrt{r^2 + 1}}e_3
\]
and
\[
\phi V_1 = -\frac{x}{r\sqrt{r^2 + 1}}e_1 - \frac{y}{r\sqrt{r^2 + 1}}e_2.
\]
Moreover, from (5) we have
\[
D_\nu V_1 = \frac{\lambda}{2} V_1 \wedge V_1 - g_\lambda([e_1, e_2], V_1)\phi V_1 + \overline{D}_\nu V_1
\]
\[
= -g_\lambda([e_1, e_2], V_1)\phi V_1 + \overline{D}_\nu V_1
\]
\[
= \left(\frac{1}{r\sqrt{r^2 + 1}} + \frac{\lambda}{2\sqrt{r^2 + 1}}\right)\phi V_1.
\]
Since,
\[
D_\nu V_1 = \alpha \frac{-\phi V_1}{\sqrt{1 - \sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1 - \sigma^2}}, \alpha, \beta \in \mathbb{R}
\]
we obtain
\[
\alpha = \frac{1}{r^3 + r} + \frac{\lambda r}{2r^2 + 2}
\]
and \( \beta = 0 \). On the other hand, we get
\[
\kappa = \frac{1}{r^3 + r} + \frac{\lambda r}{2r^2 + 2}
\]
And
\[
\tau = -\frac{\lambda}{2}\left(\frac{1}{r^2 + 1}\right) - \frac{1}{r^2 + 1}
\]
where \( \kappa \) and \( \tau \) are the curvature and the torsion of \( \gamma \), respectively. As a result, we say that \( \kappa \) and \( \tau \) are non-zero constants. Namely, \( \gamma \) is a circular helix.

**Result 3.2.** Circle in Euclidean space \( IE^3 \) is a circular helix in \( \frac{\lambda}{2} \)-Sasakian space.

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