IJST (2013) 37A3: 231-236
Iranian Journal of Science & Technology
http://ijsts.shirazu.ac.ir

$\frac{\lambda}{2}$ –Legendre curves in 3-dimensional Heisenberg group IN^3

S. Tul* and A. Sarioglugil

Department of Mathematics, Faculty of Sciences and Arts, Ondokuz Mayıs University, 55139, Kurupelit, Samsun, Turkey
E-mails: tul_sidiqa@hotmail.com & ayhans@omu.edu.tr

Abstract

In this study, we focused on $\frac{\lambda}{2}$- Legendre curves and non-$\frac{\lambda}{2}$- Legendre curves in 3-dimensional Heisenberg group IN^3. Also, we gave some characterizations of these curves.

Keywords: Heisenberg group; Sasakian manifold; Legendre curve

1. Introduction

In mathematics , the Heisenberg group, named after Werner Heisenberg, is the group of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

or its generalizations under the operation of matrix multiplication. In 1987, L. Bianchi classified the homogeneous metrics. L. Bianchi, E. Cartan and G. Vranceanu found the following 2-parameter family of homogeneous Riemannian metrics on the cartesian 3- space $IR^3(x,y,z)$:

$$g_{\lambda,\mu}=\frac{dx^2+dy^2}{\{1+\mu(x^2+y^2)\}}+\left\{dz+\frac{\lambda}{2}\frac{ydx-xdy}{\{1+\mu(x^2+y^2)\}}\right\}^2, \forall\, \lambda,\mu \in IR.$$

In this family, if $\lambda=\mu=0$, the Euclidean metric is obtained, and if $\lambda\neq 0, \mu=0$, the Heisenberg metric is obtained. Inoguchi studied the differential geometry of Heisenberg metric.

The Legendre curves play an important role in the study of contact manifolds. In a 3- dimensional Sasakian manifold, the Legendre curves are studied by Baikousis and Blair who gave the Frenet 3-frame in this space [1]. Yıldırım gave some characterizations of Legendre curves in Homogeneous space [2]. İlarslan gave a characterization of curves on non-Euclidean manifolds [3]. On the other hand, Baikosis and Hirica studied Legendre curves in Riemannian and Lorentzian Sasaki spaces [4]. Also, Legendre curves in α- Sasakian spaces are studied by Özgür and Tripathi [5]. In this study, we focused on $\frac{\lambda}{2}$–Legendre curves in 3-dimensional Heisenberg group in $IN^3$and gave a characterization of these curves. Also, we gave similar results for non-$\frac{\lambda}{2}$–Legendre curves in 3-dimensional Heisenberg group in IN^3.

*Corresponding author
Received: 13 February 2012 / Accepted: 8 April 2012

2. Preliminaries

In this section, we will give some basic concepts related to Sasakian geometry for later use.

The Heisenberg group IN^3 can be seen as the Euclidean space with the multiplication

$$(x,y,z)(x',y',z')=\left(x+x',y+y',z+z'+\frac{\lambda}{2}(xy'-yx')\right)$$

and with the Riemannian metric

$$g_\lambda=dx^2+dy^2+\left\{dz+\frac{\lambda}{2}\frac{ydx-xdy}{\{1+\mu(x^2+y^2)\}}\right\}^2, \forall\, \lambda,\mu\in IR. \qquad (1)$$

IN^3 is a three dimensional, connected, simply connected and 2-step nilpotent Lie group. The Lie algebra of IN^3 has a basis

$$\begin{cases} e_1=\frac{\partial}{\partial x}-y\frac{\partial}{\partial z} \\ e_2=\frac{\partial}{\partial x}-x\frac{\partial}{\partial z} \\ e_3=\frac{\partial}{\partial z} \end{cases} \qquad (2)$$

which is dual to

$$\begin{cases} \theta^1=dx \\ \theta^2=dy \\ \theta^3=dz+\frac{\lambda}{2}(ydx-xdy). \end{cases} \qquad (3)$$

For this basis Lie brackets are
$[e_1, e_2] = e_3, [e_3, e_1] = [e_2, e_3] = 0$,[6].

To study curves in IN^3, we shall use their Frenet vector fields and equations. Let $\gamma: I \to IN^3$ be a differentiable curve parametrized by arc length and let $\{V_1, V_2, V_3\}$ be the orthonormal frame field tangent defined as follows: by V_1 we denote $\dot{\gamma}$ tangent to γ, by V_2 the unit vector field in the direction $D_{V_1}V_1$ normal to γ and we choose $V_3 = V_1 \times V_2$, so that $\{V_1, V_2, V_3\}$ is a positive oriented orthonormal basis. Thus, we have the following Frenet equations [7]:

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}V_2 \\ D_{V_1}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \quad (4)$$

Now, let us consider the odd-dimensional Riemannian manifold (M, g). So, the Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$tensor field φ, a vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi$$

and

$$g(\varphi X, \varphi X) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta o \varphi = 0$.

Such a manifold is said to be contact metric manifold, if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \Phi Y)$ is called the fundamental 2-form of M. If ξ is a Killing vector field, then M is said to be a K −contact manifold, we have
$(D_X\varphi)Y = R(\xi, X)Y$
for any $X, Y \in M$.

On the other hand, the almost contact metric structure of M is said to be normal if
$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$, [8, 9].

A normal contact metric manifold is called a Sasakian Manifold. It can be proved that a Sasakian manifold is K −contact, and that an almost contact metric manifold is Sasakian if and only if

$$(D_X\varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for any X, Y. Furthermore, assuming that $\eta = \theta^3$, $\xi = e_3$ and defining

$$\varphi: \chi(IN^3) \to \chi(IN^3), \varphi(X) = -a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + \frac{\lambda}{2}(x_1a_1 + x_2a_2)\frac{\partial}{\partial x_3}$$

where $\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} \in \chi(IN^3)$, it can be easily seen that IN^3 is a Sasakian space. Since all computings have $\frac{\lambda}{2}$ coefficients, we have denoted IN^3 as $\frac{\lambda}{2}$−Sasakian space. We need the following Lemma for later use:

Lemma: Let X and Y be two vector fields in $\chi(IN^3)$, D and $\widetilde{D}$ be Riemannian connections on $IN^3$and IE^3, respectively. Thus,

$$D_XY = \frac{\lambda}{2}X \wedge Y - g_\lambda([e_1, e_2], X)\varphi Y + \widetilde{D}_X^Y. \quad (5)$$

On the other hand, if D is the contact distribution in a contact manifold (M, φ, ξ, η), defined by the subspaces $D_m = \{X \in T_mM | \eta(X) = 0\}$, then a one-dimensional integral submanifold of D will be called a Legendre curve. A curve $\gamma: I \to M$, parametrized by its arc length is a Legendre curve if and only if $\eta(\dot{\gamma}) = 0$, [8, 9].

3. $\frac{\lambda}{2}$ − Legendre Curves in IN^3

Theorem 3.1. Let $\gamma: I \to IN^3$ be a non-geodesic $\frac{\lambda}{2}$− Legendre curve. The Frenet frame of γ is $\{V_1, \varphi V_1, \xi\}$ and the Frenet formulas are

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}\varphi V_1 \\ D_{V_1}\xi \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ \varphi V_1 \\ \xi \end{bmatrix}. \quad (6)$$

Proof: Let $\gamma: I \to IN^3$ be a curve with arc length parameter and the Frenet frame of γ be $\{V_1, V_2, V_3\}$. Assume that $\eta(\dot{\gamma}) = \sigma \neq 0$. In this case, an orthonormal basis of $\frac{\lambda}{2}$− Sasakian space is $\left\{V_1, \frac{\varphi V_1}{\sqrt{1-\sigma^2}}, \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\right\}$. From here, we get

$$D_{V_1}V_1 = \alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \ \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R}).$$

On the other hand, derivating σ we obtain

$$\begin{aligned} \dot{\sigma} &= D_{V_1}\sigma \\ &= D_{V_1}g_\lambda(V_1, \xi) \\ &= g_\lambda(D_{V_1}V_1, \xi) + g_\lambda(V_1, D_{V_1}\xi) \\ &= g_\lambda\left(\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \xi\right) + g_\lambda(V_1, -\frac{\lambda}{2}\varphi V_1) \\ &= \beta\sqrt{1 - \sigma^2}. \end{aligned}$$

From here, we say that

$$\beta = \dot{\sigma}\frac{1}{\sqrt{1-\sigma^2}}.$$

Since γ is a $\frac{\lambda}{2}$− Legendre curve, we can easily see that β =0. Moreover, from (4) we get $\alpha = \kappa$,

$V_2 = \varphi V_1$, $D_{V_1}V_1 = \kappa\varphi V_1$ and

$$D_{V_1}V_2 = \varphi D_{V_1}V_1 + (D_{V_1}\varphi)V_1$$
$$= \varphi(\kappa\varphi V_1) + \frac{\lambda}{2}\{g_\lambda(V_1,V_1)\xi - \eta(V_1)V_1\}$$
$$= -\kappa V_1 + \frac{\lambda}{2}\xi.$$

From (4), we get $V_3 = \xi$, $\tau = -\frac{\lambda}{2}$. Hence, the Serret-Frenet formulas are

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}\varphi V_1 \\ D_{V_1}\xi \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ \varphi V_1 \\ \xi \end{bmatrix}.$$

Theorem 3.2: Let $\gamma: I \to IN^3$ be a non-geodesic $\frac{\lambda}{2}-$ Legendre curve and $0 < |\eta(\dot{\gamma})| < 1$. The curvature and the torsion of γ are

$$\kappa = \sqrt{\alpha^2 + \beta^2}, \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R}) \tag{7}$$

and

$$\tau = \frac{\lambda}{2} + \frac{\alpha\dot{\beta} - \dot{\alpha}\beta}{\alpha^2 + \beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}, \tag{8}$$

respectively.

Proof: Let $\gamma: I \to IN^3$ be a curve with arc length parameter and the Frenet frame of γ be $\{V_1, V_2, V_3\}$. Assume that $\eta(\dot{\gamma}) = \sigma \neq 0$.In this case, an orthonormal basis of $\frac{\lambda}{2}-$ Sasakian space is $\left\{V_1, \frac{\varphi V_1}{\sqrt{1-\sigma^2}}, \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\right\}$. From here we get

$$D_{V_1}V_1 = \alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}},\ \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R}).$$

So, we obtain

$$\kappa = \|D_{V_1}V_1\| = \sqrt{\alpha^2 + \beta^2},\quad \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R})$$

and

$$V_2 = \frac{1}{\kappa}D_{V_1}V_1.$$

On the other hand, derivating φV_1, we have

$$D_{V_1}\varphi V_1 = \varphi D_{V_1}V_1 + (D_{V_1}\varphi)V_1$$
$$= \varphi\left(\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\right) + \frac{\lambda}{2}(\xi - \sigma V_1)$$
$$= -\frac{\alpha}{\sqrt{1-\sigma^2}}V_1 + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}\xi - \frac{\beta\sigma}{\sqrt{1-\sigma^2}}\varphi V_1 + \frac{\lambda}{2}(\xi - \sigma V_1). \tag{9}$$

Similaly, derivating $\xi - \sigma V_1$ we get,

$$D_{V_1}(\xi - \sigma V_1) = D_{V_1}\xi - \dot{\sigma}V_1 - \sigma D_{V_1}V_1$$
$$= -\frac{\lambda}{2}\varphi V_1 - \dot{\sigma}V_1 - \sigma\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} - \sigma\beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}. \tag{10}$$

On the other hand, derivating σ we have

$$\dot{\sigma} = D_{V_1}\sigma$$
$$= D_{V_1}g_\lambda(V_1, \xi)$$
$$= g_\lambda(D_{V_1}V_1, \xi) + g_\lambda(V_1, D_{V_1}\xi)$$
$$= g_\lambda\left(\alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \xi\right) + g_\lambda(V_1, -\frac{\lambda}{2}\varphi V_1)$$
$$= \beta\sqrt{1-\sigma^2}.$$

From here, we see that

$$\beta = \dot{\sigma}\frac{1}{\sqrt{1-\sigma^2}}.$$

Similarly, derivating $\frac{\alpha}{\sqrt{1-\sigma^2}}$ and $\frac{\beta}{\sqrt{1-\sigma^2}}$ we obtain

$$D_{V_1}\left(\frac{\alpha}{\sqrt{1-\sigma^2}}\right) = \dot{\alpha}\frac{1}{\sqrt{1-\sigma^2}} + \alpha\beta\sigma\frac{1}{1-\sigma^2} \tag{11}$$

and

$$D_{V_1}\left(\frac{\beta}{\sqrt{1-\sigma^2}}\right) = \dot{\alpha}\frac{1}{\sqrt{1-\sigma^2}} + \beta^2\sigma\frac{1}{1-\sigma^2} \tag{12}$$

respectively. Furthermore,

$$D_{V_1}V_2 = D_{V_1}\left(\frac{1}{\kappa}D_{V_1}V_1\right)$$
$$= -\frac{\dot{\kappa}}{\kappa^2}D_{V_1}V_1 + \frac{1}{\kappa}D_{V_1}D_{V_1}V_1$$
$$= -\frac{\dot{\kappa}}{\kappa^2}D_{V_1}V_1 + \frac{1}{\kappa}D_{V_1}\left(\frac{\alpha}{\sqrt{1-\sigma^2}}\right)\varphi V_1 + \frac{1}{\kappa}\left(\frac{\alpha}{\sqrt{1-\sigma^2}}\right)D_{V_1}\varphi V_1 + \frac{1}{\kappa}D_{V_1}\left(\frac{\beta}{\sqrt{1-\sigma^2}}\right)(\xi - \sigma V_1) + \frac{1}{\kappa}\left(\frac{\beta}{\sqrt{1-\sigma^2}}\right)D_{V_1}(\xi - \sigma V_1).$$

Using (9), (10), (11) and (12), we get

$$D_{V_1}V_2 = -\kappa V_1 - \left(-\frac{\alpha\dot{\kappa}}{\kappa^2} + \frac{\dot{\alpha}}{\kappa} - \frac{\lambda\beta}{2\kappa} - \frac{\alpha\beta\sigma}{\kappa\sqrt{1-\sigma^2}}\right)\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \left(-\frac{\beta\dot{\kappa}}{\kappa^2} + \frac{\dot{\beta}}{\kappa} - \frac{\lambda\alpha}{2\kappa} - \frac{\alpha^2\sigma}{\kappa\sqrt{1-\sigma^2}}\right)\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}.$$

From (6), it can be easily seen that

$$\tau V_3 = \left(-\frac{\alpha\dot{\kappa}}{\kappa^2} + \frac{\dot{\alpha}}{\kappa} - \frac{\lambda\beta}{2\kappa} - \frac{\alpha\beta\sigma}{\kappa\sqrt{1-\sigma^2}}\right)\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \left(-\frac{\beta\dot{\kappa}}{\kappa^2} + \frac{\dot{\beta}}{\kappa} - \frac{\lambda\alpha}{2\kappa} - \frac{\alpha^2\sigma}{\kappa\sqrt{1-\sigma^2}}\right)\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}.$$

Taking the norm of the last equation, we have

$$\tau = \frac{\lambda}{2} + \frac{\alpha\dot{\beta} - \dot{\alpha}\beta}{\alpha^2+\beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}.$$

Lemma 3.1. Let $\gamma: I \to IN^3$ be a curve with arc length parameter and $\{V_1, V_2, V_3\}$ be the Frenet frame of γ. Then, the following equation is obtained:

$$D^3_{V_1}V_1 - 2\frac{\dot{\kappa}}{\kappa}D^2_{V_1}V_1 + \left(2\frac{\dot{\kappa}}{\kappa} - \frac{\ddot{\kappa}}{\kappa} + \kappa^2 + \frac{\lambda^2}{4}\right)D_{V_1}V_1 + \kappa\dot{\kappa}V_1 = 0. \quad (13)$$

Proof: From (6), we know that

$$D_{V_1}\varphi V_1 = -\kappa V_1 + \frac{\lambda}{2}\xi$$

and

$$D_{V_1}V_1 = -\kappa\varphi V_1.$$

From here,

$$D_{V_1}\frac{1}{\kappa}D_{V_1}V_1 = -\kappa V_1 + \frac{\lambda}{2}\xi$$
$$\Rightarrow \left(\frac{1}{\kappa}\right)' D_{V_1}V_1 + \frac{1}{\kappa}D^2_{V_1}V_1 = -\kappa V_1 + \frac{\lambda}{2}\xi.$$

Differentiating the last equation, we have

$$\frac{1}{\kappa}D^3_{V_1}V_1 + 2\left(\frac{1}{\kappa}\right)' D^2_{V_1}V_1 + \left(\left(\frac{1}{\kappa}\right)'' + \kappa + \frac{\lambda^2}{4}\frac{1}{\kappa}\right)D_{V_1}V_1 + \dot{\kappa}V_1 = 0.$$

Considering the last equation, we get

$$D^3_{V_1}V_1 - 2\frac{\dot{\kappa}}{\kappa}D^2_{V_1}V_1 + \left(2\frac{\dot{\kappa}}{\kappa} - \frac{\ddot{\kappa}}{\kappa} + \kappa^2 + \frac{\lambda^2}{4}\right)D_{V_1}V_1 + \kappa\dot{\kappa}V_1 = 0.$$

Theorem 3.3. Let $\gamma: I \to IN^3$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, be a $\frac{\lambda}{2}-$Legendre curve in IN^3 and α be the projection curve of γ on $z = 0$ plane. Then, the curvature of γ is the curvature of α.

Proof: The tangent vector field of γ is
$\dot{\gamma}(t) = \dot{\gamma_1}(t)e_1 + \dot{\gamma_2}(t)e_2 + \dot{\gamma_3}(t)e_3$.
We can choose the parameter of γ as $\dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2 = 1$. Then, if we choose $\gamma_1(t)$ and $\gamma_2(t)$ as $\dot{\gamma_1}(t) = -\sin\theta(t)$, $\dot{\gamma_2}(t) = \cos\theta(t)$, respectively, we obtain

$$D_{\dot{\gamma}(t)}\dot{\gamma}(t) = \ddot{\gamma_1}(t)e_1 + \ddot{\gamma_2}(t)e_2$$

and

$$\|D_{\dot{\gamma}(t)}\dot{\gamma}(t)\| = \frac{1}{2}\sqrt{\ddot{\gamma_1}(t)^2 + \ddot{\gamma_2}(t)^2}$$
$$\kappa = \dot{\theta}(t).$$

On the other hand, the projection curve α of γ on $z = 0$ plane is $\alpha(t) = (\gamma_1(t), \gamma_2(t))$. Thus, it can be easily seen that α is a unit speed curve. The curvature of α is

$$\kappa_\alpha = \frac{|\dot{\gamma_1}(t)\ddot{\gamma_2}(t) - \ddot{\gamma_1}(t)\dot{\gamma_2}(t)|}{\sqrt[3]{(\dot{\gamma_1}(t)^2 + \dot{\gamma_2}(t)^2)^2}}.$$

From here,

$$\kappa = \kappa_\alpha.$$

Corollary 3.1. Let γ be a non-geodesic Legendre curve in IN^3. Then,
i) γ is not a circle.
ii) If γ is a helix, it satisfies the following equation:
$\Delta H = \left(\kappa^2 + \frac{\lambda^2}{4}\right)H.$
iii) If γ is a line,
$g_\lambda(D_{V_1}V_1, \varphi V_1) = 0.$
iv) γ is not a planar curve.

Proof: i) Since γ is a $\frac{\lambda}{2}-$Legendre curve, the torsion of γ is $-\frac{\lambda}{2}$. So, it can be easily seen that γ is not a circle.
ii) If γ is helix, $\frac{\kappa}{\tau}$ is constant. Also, on the ground that the torsion of γ is $-\frac{\lambda}{2}$, κ must be constant. So, $\dot{\kappa}, \ddot{\kappa} = 0$.
From (13), we obtain

$$D^3_{V_1}V_1 = -\left(\kappa^2 + \frac{\lambda^2}{4}\right)D_{V_1}V_1.$$

Using $V_1 = \dot{\gamma}$, $\Delta = -D_{V_1}D_{V_1}V_1$ and $H = D_{V_1}V_1$ we have

$$\Delta H = \left(\kappa^2 + \frac{\lambda^2}{4}\right)H.$$

iii) If γ is a line, the curvature of γ is zero. Also,
$D_{V_1}V_1 = \kappa\varphi V_1$.
From here, we get
$g_\lambda(D_{V_1}V_1, \varphi V_1) = 0.$
iv) Since γ is a $\frac{\lambda}{2}-$Legendre curve, the torsion of γ is not zero. So, it is said that γ is not a planar curve.

Example 3.1.
$\gamma: I \to \mathbb{N}^3, \gamma(t) = \left(r\cos t, r\sin t, \frac{\lambda}{2}r^2 t\right)$ is a curve in IN^3. If we assume that
$x = r\cos t$
$y = r\sin t$
$z = \frac{\lambda}{2}r^2$
we get

$$\dot{\gamma}(t) = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)_{\gamma(t)}.$$

Thus, using (1.3), we get

$$\begin{cases} \theta^1(\dot{\gamma}(t)) = -y \\ \theta^2((\dot{\gamma}(t)) = x \\ \theta^3(\dot{\gamma}(t)) = 0. \end{cases} \quad (14)$$

From (14), we can say that γ is a $\frac{\lambda}{2}-$Legendre curve. On the other hand, we obtain

$\|\dot{\gamma}(t)\| = \sqrt{[\theta^1(\dot{\gamma}(t))]^2 + [\theta^2(\dot{\gamma}(t))]^2 + [\theta^3(\dot{\gamma}(t))]^2}$
$= |r|$,
$V_1 = \mp\frac{y}{r}e_1 \mp \frac{x}{r}e_2$

and

$$\varphi V_1 = \mp\frac{x}{r}e_1 \mp \frac{y}{r}e_2.$$

Moreover, from (5) we have

$$\begin{aligned} D_{V_1}V_1 &= \frac{\lambda}{2}V_1 \wedge V_1 - g_\lambda([e_1,e_2],V_1)\varphi V_1 + \widetilde{D}_{V_1}^{V_1} \\ &= -g_\lambda([e_1,e_2],V_1)\varphi V_1 + \widetilde{D}_{V_1}^{V_1} \\ &= \mp\frac{1}{r}\varphi V_1. \end{aligned}$$

Namely, we see that

$$\kappa = \mp\frac{1}{r}$$

where κ is the curvature of γ. Also, we know that $\tau = -\frac{\lambda}{2}$ for a non-geodesic $\frac{\lambda}{2}-$ Legendre curve in $\mathbb{N}^3$. As a result, κ and τ are non-zero constants. So, γ is a helix.

Result 3.1. Helix in Euclidean space is a helix in $\frac{\lambda}{2}-$ Sasakian space, too. Also, it is a $\frac{\lambda}{2}-$ Legendre curve.

Corollary 3.2. γ: $I \rightarrow \mathbb{N}^3$ be a $\frac{\lambda}{2}-$ non-Legendre curve. Then,

i) If γ is a geodesic, it satisfies the following equation:

$$\widetilde{D}_{V_1}^{V_1} = g_\lambda([e_1,e_2],V_1)\varphi V_1.$$

ii) If γ is a circle,

$$\lambda = \frac{2\alpha\sigma}{\sqrt{1-\sigma^2}}$$

or

$$\lambda = -\frac{2\alpha\sigma}{\sqrt{1-\sigma^2}} + \dot{\theta}(t)r^2.$$

where $\alpha = r\cos\theta(t)$ and $\beta = r\sin\theta(t)$.

iii) If γ is a circular helix,

$$\tau = -\frac{\lambda}{2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}.$$

iv) If γ is a helix,

$$\alpha^2 + \beta^2 = c^2\left(\frac{\lambda}{2} + \frac{\alpha\dot{\beta}-\dot{\alpha}\beta}{\alpha^2+\beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}\right)^2.$$

Proof: i) If γ is a geodesic, $\kappa = \tau = 0$. So, from (7) we say that $\alpha = \beta = 0$ and τ is indefinite.
On the other hand, if γ is a geodesic, $D_{V_1}V_1 = 0$. So, from (5) we get

$$\widetilde{D}_{V_1}^{V_1} = g_\lambda([e_1,e_2],V_1)\varphi V_1.$$

ii) If γ is a circle, κ is a non-zero constant. In which case there are two situations:

a) We assume that α and β are constants. Thus,

$$\tau = -\frac{\lambda}{2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}} = 0$$

or

$$\lambda = \frac{2\alpha\sigma}{\sqrt{1-\sigma^2}}.$$

b) We assume that κ is a non-zero constant and α and β are not constants. Hence, if α and β are chosen as $r\cos\theta(t)$ and $r\sin\theta(t)$, respectively, it is found that
$\alpha^2 + \beta^2 = r^2$
and

$$\alpha\dot{\beta} - \dot{\alpha}\beta = \dot{\theta}(t)r^2.$$

Since $\tau = 0$, from (12) we get

$$\lambda = -\frac{2\alpha\sigma}{\sqrt{1-\sigma^2}} + \dot{\theta}(t)r^2.$$

iv) If γ is a helix, $\frac{\kappa}{\tau} = c, c \neq 0 = const$ and from (7) and (8)

$$\alpha^2 + \beta^2 = c^2\left(\frac{\lambda}{2} + \frac{\alpha\dot{\beta}-\dot{\alpha}\beta}{\alpha^2+\beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}\right)^2.$$

Example 3.2.
γ: $I \rightarrow \mathbb{N}^3, \gamma(t) = (r\cos t, r\sin t, c)$ is a curve in $\mathbb{N}^3$. If we assume that,
$x = r\cos t$
$y = r\sin t$
$z = c$
we get

$$\dot{\gamma}(t) = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)_{\gamma(t)}.$$

Thus using (3), we obtain

$$\begin{cases} \theta^1(\dot{\gamma}(t)) = -y \\ \theta^2((\dot{\gamma}(t)) = x \\ \theta^3(\dot{\gamma}(t)) = -r^2. \end{cases}$$

So, we can say that γ is not a $\frac{\lambda}{2}-$Legendre curve. On the other hand, we have

$$\dot{\gamma}(t) = (-ye_1 + xe_2 - r^2 e_3)_{\gamma(t)}$$

and

$$\|\dot{\gamma}(t)\| = \sqrt{[\theta^1(\dot{\gamma}(t))]^2 + [\theta^2(\dot{\gamma}(t))]^2 + [\theta^3(\dot{\gamma}(t))]^2}.$$

Thus, we get

$$V_1 = -\frac{y}{r\sqrt{r^2+1}}e_1 + \frac{x}{r\sqrt{r^2+1}}e_2 - \frac{r}{\sqrt{r^2+1}}e_3$$

and

$$\varphi V_1 = -\frac{x}{r\sqrt{r^2+1}}e_1 - \frac{y}{r\sqrt{r^2+1}}e_2.$$

Moreover, from (5) we have

$$\begin{aligned} D_{V_1}V_1 &= \frac{\lambda}{2}V_1 \wedge V_1 - g_\lambda([e_1,e_2],V_1)\varphi V_1 + \widetilde{D}_{V_1}^{V_1} \\ &= -g_\lambda([e_1,e_2],V_1)\varphi V_1 + \widetilde{D_{V_1}V_1} \\ &= \left(\frac{1}{r\sqrt{r^2+1}} + \frac{\lambda}{2\sqrt{r^2+1}}\right)\varphi V_1. \end{aligned}$$

Since,

$$D_{V_1}V_1 = \alpha\frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta\frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\alpha,\beta \in \mathbb{R}$$

we obtain

$$\alpha = \frac{1}{r^3+r} + \frac{\lambda r}{2r^2+2}$$

and $\beta = 0$. On the other hand, we get

$$\kappa = \left|\frac{1}{r^3+r} + \frac{\lambda r}{2r^2+2}\right|$$

And

$$\tau = -\frac{\lambda}{2}\left(\frac{1}{r^2+1}\right) - \left(\frac{1}{r^2+1}\right)$$

where κ and τ are the curvature and the torsion of γ, respectively. As a result, we say that κ and τ are non-zero constants. Namely, γ is a circular helix.

Result 3.2. Circle in Euclidean space IE^3 is a circular helix in $\frac{\lambda}{2}-$Sasakian space.

Acknowledgments

The authors thank the referee for the useful suggestions and remarks for the revised version. This study is supported by the University of Ondokuz Mayıs Project no PYO. FEN. 1904. 11. 006.

References

[1] Baikoussis, C. & Blair, D.E. (1994). On Legendre Curves in Contact 3-Manifolds. *Geometriae Dedicata, 49,* 135-142.

[2] Yıldırım, A. (2005). *Differential Geometry of Curves in Homogeneous Space*. Doctorate Dissertation, Ankara Üniveristei Fen Bilimleri Enstitüsü, Ankara.

[3] İlarsalan, K. (2002). *Some Special Curves on Non-Euclidean Manifolds*, Doctorate Dissertation, Ankara Üniveristei Fen Bilimleri Enstitüsü, Ankara.

[4] Belkhelfa, M., Hirica, I. E., Rosca, R. & Verstlraelen, L. (2002). On Le-gendre Curves in Riemannian and Lorenztian Sasaki Spaces. *Soochow J. Math, 28*(1), 81-91.

[5] Özgür, C. & Tripathi, M. M. (2008). On Legendre curves in -Sasakian manifolds. *Bull. Malays. Math. Sci. Soc. 31*(1), 91-96.

[6] Inoguchı, J. I., Kumamoto, T., Ohsugi, N. & Suyama, Y. (1999). Differential Geometry of Curves and Surfaces in 3-Dimensionan Homogeneous Spaces I. *Fukuoka Univ. Sci. Reports, 29*(2), 155-182.

[7] Capogna, L., Danielli, D., Pauls, S. D. & Tyson, J. T. (2007). *An Introduction to Heisenberg Group and Isoperimetric Problem*, Boston, Basel, Birkhauser.

[8] Blair, D. E. (1976). *Contact 3-Manifolds in Riemannian Geometry*. New York, Springer 509.

[9] Blair, D. E. (2001). *Riemannian Geometry of Contact and Symplectic Manifolds*. Boston, Birkhauser.